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PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCLII.

PART I.

LONDON:

PRINTED BY RICHARD TAYLOR AND WILLIAM FRANCIS, RED LION COURT, FLEET STREET.

MDCCCLII.

ADVERTISEMENT.

The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body,

upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

The Meteorological Journal hitherto kept by the Assistant Secretary at the Apartments of the Royal Society, by order of the President and Council, and published in the Philosophical Transactions, has been discontinued. The Government, on the recommendation of the President and Council, has established at the Royal Observatory at Greenwich, under the superintendence of the Astronomer Royal, a Magnetical and Meteorological Observatory, where observations are made on an extended scale, which are regularly published. These, which correspond with the grand scheme of observations now carrying out in different parts of the globe, supersede the necessity of a continuance of the observations made at the Apartments of the Royal Society, which could not be rendered so perfect as was desirable, on account of the imperfections of the locality and the multiplied duties of the observer.

A List of Public Institutions and Individuals, entitled to receive a copy of the Philosophical Transactions of each year, on making application for the same directly or through their respective agents, within five years of the date of publication.

Observatories.

Armagh.
Cape of Good Hope.
Dublin.
Edinburgh.
Greenwich.
Madras.

Institutions.

Barbadoes ....Library and Museum.
Calcutta ....Asiatic Society.
Cambridge ....Philosophical Society.
Dublin..................Royal Dublin Society.
Royal Irish Academy.
Edinburgh ....Royal Society.
London ....Admiralty Library.
Chemical Society.
Entomological Society.
Geological Society.
Geological Survey of Great Britain.
Horticultural Society.
Institute of British Architects.
Institution of Civil Engineers.
Linnean Society.
London Institution.
Medical and Chirurgical Society.
Queen's Library.
Royal Asiatic Society.
Royal Astronomical Society.
Royal College of Physicians.
Royal Geographical Society.
Royal Institution of Great Britain.
Royal Society of Literature.
Society of Antiquaries.
Society for the Encouragement of Arts.
The Treasury.
United Service Museum.
Zoological Society.
Malta ....Public Library.
Manchester ....Literary and Philosophical Society.
Oxford..................Ashmolean Society.
Radcliffe Library.
Swansea ....Royal Institution.
Woolwich ....Royal Artillery Library.

Belgium.

Brussels ....Royal Academy of Sciences.

Denmark.

Altona ....Royal Observatory.
Copenhagen............Royal Society of Sciences.

France.

Paris ....Academy of Sciences.
Dépôt de la Marine.
Ecôle des Mines.
Paris ....Entomological Society.
Geographical Society.
Geological Society.
Jardin des Plantes.
Toulouse ....Academy of Sciences.

Germany.

Bonn ....Cæsarean Acad. of Naturalists.
Göttingen ....University.
Mannheim ....Observatory.
Munich ....Royal Academy of Sciences.

Italy.

Naples ....Institute of Sciences.
Milan ....Institute of Sciences, Letters and Arts.
Modena ....Italian Society of Sciences.
Turin ....Royal Academy of Sciences.

Netherlands.

Amsterdam ....Royal Institute.
Rotterdam ....Batavian Society of Experimental Philosophy.

Prussia.

Berlin ....Royal Academy of Sciences.
Society of Experimental Philosophy.

Portugal.

Lisbon ....Royal Academy of Sciences.

Russia.

Pulkowa ....Observatory.
St. Petersburgh ......Imperial Academy of Sciences.

Spain.

Cadiz ....Observatory.

Sweden and Norway.

Drontheim ....Royal Society of Sciences.
Stockholm ....Royal Academy of Sciences

Switzerland.

Geneva ....Société de Phys. et d'Hist. Naturelle.

United States.

Boston..................American Academy of Sciences.
Cambridge ....Harvard University.
Philadelphia ....American Philosophical Society.
Washington............Smithsonian Institution.
Observatory.

The *fifty* Foreign Members of the Royal Society.

A List of Public Institutions and Individuals, entitled to receive a copy of the Astronomical Observations (including Magnetism and Meteorology) made at the Royal Observatory at Greenwich, on making application for the same directly or through their respective agents, within two years of the date of publication.

Observatories.

Altona.
Armagh.
Berlin.
Breslau.
Brussels.
Cadiz.
Cambridge.
Cape of Good Hope.
Coimbra.
Copenhagen.
Dorpat.
Dublin.
Edinburgh.
Helsingfors.
Königsberg.
Madras.
Mannheim.
Marseilles.
Milan.
Munich.
Oxford.
Palermo.
Paris.
Seeberg.
Trevandrum.
Tübingen.
Turin.
Vienna.
Wilna.

Institutions.

Aberdeen............University.
Berlin ....Academy.
Bologna ....Academy.
Boston ....American Academy of Sciences.
Brunswick, U.S. ...Bowdoin College.
Cambridge ....Trinity College Library.
Cambridge, U.S....Harvard University.
Dublin ....University.
Edinburgh ....University.
Royal Society.
Glasgow ....University.
Göttingen ....University.
Leyden ....University.
London ....Board of Ordnance.
Queen's Library.
Royal Institution.
Royal Society.
Oxford................Savilian Library.
Paris ....Academy of Sciences.
Board of Longitude.
Dépôt de la Marine.
Philadelphia ......American Philosophical Society.
St. Andrews ......University.
St. Petersburgh ...Imperial Academy.
Stockholm ....Royal Academy of Sciences.
Upsal ....Royal Society.
Waterville (U.S.)...College.

Individuals.

Christie, S. H., Esq. .... Woolwich.
Lubbock, Sir John William, Bart. ... London.
Lowndes Professor of Astronomy ... Cambridge.
Plumian Professor of Astronomy ......
President of the Royal Society......... London.
Smyth, Captain W. H., R N. .... Aylesbury.
South, Sir James .... Kensington.

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Observatories.

Bombay	
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Cambridge, United States	Professor Lovering.
Catherineburgh	M. Rochkoff.
Christiania	M. Hansteen.
Cincinnati	Dr. Locke.
Gotha	
Hammerfest	
Heidelberg	M. Tiedemann.
Hobarton	Commander Kay, R.N.
Kasan	M. Simonoff.
Kew	F. Ronalds, Esq.
Kremsmünster	Professor Koller.
Leipsic	
Marburg	Professor Gerling.
Nertchinsk	M. Prang, 2nd.
Nikolaieff	Dr. Knorre.
Pekin	M. Gachkévitche.
Prague	M. Kreil.
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St. Petersburgh	M. Kupffer.
Sitka	Messrs. Homann and Ivanoff.
Stockholm	Professor Selander.
Tiflis	M. Philadelphine.
Toronto	Captain Lefroy, R.A.
Upsal	Professor Svanberg.
Warsaw	Col. G. Du Plat (British Consul).
Washington	Lt. Maury, U.S. Navy.

Institutions.

Bombay	Geographical Society.
Bonn	University.
Bowditch Library	United States.
Cambridge	Philosophical Society.
Cherkow	University.
Falmouth	Royal Cornwall Polytechnic Society.
Kiew	University.
London	House of Lords, Library.
London	House of Commons, Library.
	King's College.
	Royal Society.
	University.
Moscow	University.
St. Bernard	Convent.
St. Petersburgh	Geographical Society.
Washington	Smithsonian Institution.
Woolwich	Office of Mag. and Met. Publication.

Individuals.

Bache, Dr. A. D.	Washington.
Barlow, P. W., Esq.	Woolwich.
Colebrooke, Sir W.	
Demidoff, Prince Anatole de	Florence.
Dove, Professor	Berlin.
Elliot, Capt. C. M.	
Erman, Dr. Adolph	Berlin.
Fox, R. W., Esq.	Falmouth.
Gauss, Professor	Göttingen.
Gilliss, Lt. J. M., U.S. Navy	Washington.
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Kupffer, A. T.	St. Petersburgh.
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Loomis, Professor	New York University.
Melvill, J. C., Esq.	East India House.
Mentchikoff, Prince	St. Petersburgh.
Phillips, John, Esq.	York.
Quetelet, A.	Brussels.
Redfield, W. C., Esq.	New York.
Reid, Col. Sir W., R.E.	Malta.
Riddell, Capt., R.A.	Woolwich.
Roget, P. M., M.D.	London.
Sabine, Colonel, R.A.	Woolwich.
Senftenberg, Baron von	Prague.
Wartmann, Professor Elie	Geneva.
Younghusband, Capt., R.A.	Woolwich.

ROYAL MEDALS.

HER MAJESTY QUEEN VICTORIA, in restoring the Foundation of the Royal Medals, has been graciously pleased to approve the following regulations for the award of them:

That the Royal Medals be given for such papers only as have been presented to the Royal Society, and inserted in their Transactions.

That the triennial Cycle of subjects be the same as that hitherto in operation: viz.
1. Astronomy; Physiology, including the Natural History of Organized Beings.
2. Physics; Geology or Mineralogy.
3. Mathematics; Chemistry.

That, in case no paper, coming within these stipulations, should be considered deserving of the Royal Medal, in any given year, the Council have the power of awarding such Medal to the author of any other paper on either of the several subjects forming the Cycle, that may have been presented to the Society and inserted in their Transactions; preference being given to the subjects of the year immediately preceding: the award being, in such case, subject to the approbation of Her Majesty.

The Council propose to give one of the Royal Medals in the year 1852 for the most important paper in Physics, communicated to the Royal Society after the

termination of the Session in June 1848, and prior to the termination of the Session in June 1851, and printed in the Philosophical Transactions.

The Council propose also to give one of the Royal Medals in the year 1852 for the most important paper in Geology or Mineralogy, communicated to the Royal Society after the termination of the Session in June 1848, and prior to the termination of the Session in June 1851, and printed in the Philosophical Transactions.

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ERRATA.—Part II. 1851.

In equations 11, 12, 13, 15, 16, 17, 18, 19, 23, 24, *for* $H_1 \mp H_2$ *read* $H_1 - H_2$.

In equation 19, *for* $\frac{1}{12}\mu$ *read* $\frac{1}{24}\mu$.

In page 620, *omit* the third, fourth and fifth lines

CONTENTS.

APPENDIX.

Adjudication of the Medals of the Royal Society for the year 1852 by the President and Council.

The Copley Medal to Baron Alexander von Humboldt, for his eminent services in Terrestrial Physics during a series of years.

The Royal Medal in the department of Physics, to J. P. Joule, Esq., F.R.S., for his paper "On the Mechanical Equivalent of Heat," printed in the Philosophical Transactions for the year 1850.

The second Royal Medal to T. H. Huxley, Esq., F.R.S., for his paper "On the Anatomy and the Affinities of the Family of the Medusæ," printed in the Philosophical Transactions for the year 1849.

The Rumford Medal to G. G. Stokes, Esq., F.R.S., for his "Discovery of the Change in the Refrangibility of Light."

PRESENTS

RECEIVED BY

THE ROYAL SOCIETY,

WITH THE

NAMES OF THE DONORS.

From November 1851 to June 1852.

PRESENTS.	DONORS.
ACADEMIES and SOCIETIES.	
Amsterdam:—	
Verhandelingen der eerste Klasse van het Koninklijk-Nederlandsche Instituut. Derde Reeks. 4de Deel. 4to. *Amsterdam* 1851.	The Institute.
Tijdschrift. Afl. 1 to 4. Vol. IV. 8vo. *Amsterdam* 1851.	———
Genootschap Natura Artis Magistra:—Bijdragen tot de Dierkunde. 2de en 3de Aflevering. 4to. *Amsterdam* 1851.	The Scientific Commission of the Zoological Gardens, Amsterdam.
Kort Verslag der Wetenschappelijke Werkzaamheden, &c. 8vo. *Amsterdam*.	———
Basel:—Bericht ueber die Verhandlungen der Naturforschenden Gesellschaft. Aug. 1848 to June 1850. 8vo. *Basel* 1851.	The Society.
Batavia, Natural History Society:— Natuurkundig Tijdschrift voor Nederlandsch Indië. Eerste Jaargang. Afl. 1 to 6. 8vo. *Batavia* 1850–51.	The Society.
Berlin:—	
Abhandlungen der Königlichen Akademie der Wissenschaften. Aus dem Jahre 1849. 4to. *Berlin* 1851.	The Academy.
Monatsbericht. March 1851 to April 1852. 8vo.	———
Verzeichniss der .. beobachteten Sterne, &c. Akademische Sternkarten. Tomes I. XI. XX.	———
Liste der Akademie 1852.	———
Berwickshire:—Naturalists' Club. Proceedings. No. 2. Vol. III.	The Club.
Bologna:—	
Novi Commentarii Academiæ Scientiarum Instituti Bononiensis. Tomus decimus.	The Academy.
Memorie della Accademia delle Scienze dell' Istituto di Bologna. Tomo 1.	———
Breslau:—	
Novorum Actorum Academiæ Cæsareæ Leopoldino-carolinæ Naturæ Curiosorum Tom. XXIII. *Breslau* and *Bonn* 1851. 4to.	The Academy.
Achtundzwanzigster Jahresbericht der Schlesischen Gesellschaft für vaterländische Kultur im Jahre 1850. 4to. *Breslau*.	The Society.

Presents.	Donors.
ACADEMIES and SOCIETIES *(continued)*.	
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Cambridge:—Transactions of the Philosophical Society. Part 2. Vol. IX. 4to. *Cambridge* 1851.	The Society.
Copenhagen:—Forhandlinger ved de skandinaviske Naturforskeres. Femte Mode. 8vo. *Kjöbenhavn* 1849.	The Society.
Cornwall:—Eighteenth and Nineteenth Annual Report of the Royal Cornwall Polytechnic Society, 1851. 8vo. *Falmouth.*	The Society.
Dublin:—	
Royal Irish Academy. Proceedings. Part 1. Vol. V. 8vo.	The Academy.
Geological Society. Journal. Part 1. Vol. V. 8vo. *Dublin* 1851.	The Society.
Edinburgh:—	
Royal Society. Transactions. Part 2. Vol. XX. 4to.	The Society.
Proceedings. Nos. 40, 41. Vol. III. 8vo.	———
Transactions of the Royal Scottish Society of Arts. Part 5. Vol. III. 8vo.	The Society.
Transactions of the Architectural Institute of Scotland, Session 1850–51, and Parts 1 to 4. Vol. II. 8vo.	The Institute.
Glasgow:—Proceedings of the Philosophical Society. No. 3. Vol. III.	The Society.
Göttingen:—	
Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen vom Jahre 1851. 8vo.	The Academy.
Zur Erinnerrung an Albrecht von Haller und zur Geschichte der Societäten der Wissenschaften. 4to. *Göttingen* 1852.	———
Halle:—Jahresbericht des naturwissenschaftlichen Vereines, Dritter Jahrgang, 1850. 8vo. *Berlin* 1851.	The Society.
Lausanne:—	
Société Vaudoise des Sciences Naturelles. Bulletin. Nos. 20 to 22. Tom. III. 8vo.	The Society.
Observations Météorologiques faites à Morges, Nov. 1849 to Nov. 1850.	———
Catalogue des Ouvrages et Brochures appartenant à la Société. 8vo. *Lausanne* 1850.	———
Leeds:—Philosophical and Literary Society. Thirteenth Report. 8vo. *Leeds* 1850.	The Society.
Leipzig:—Königlich Sächsischen Gesellschaft der Wissenchaften. Math. Phys. Classe. Nos. 2 and 3. 8vo. *Leipzig* 1851.	The Society.
Liverpool:—Literary and Philosophical Society. Proceedings. No. 6. 8vo. *Liverpool* 1851.	The Society.
London:—	
Astronomical Society. Memoirs. Vol. XIX. and XX. 4to. 1851–52.	The Society.
——— Proceedings. Vol. X. and XI. 8vo. 1850–51.	———
British Association. Report of the Twentieth Meeting. 8vo. *London* 1851.	The Association.
Chemical Society. Quarterly Journal. Nos. 2 and 3. Vol. IV. 8vo. 1851.	The Society.
Entomological Society. Transactions. Parts 6 to 8. Vol. I. 8vo. *London* 1851.	The Society.
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Geological Society. Quarterly Journal. Nos. 27 and 28. Vol. VII. Parts 1 and 2. Vol. VIII. 8vo.	The Society.

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Horticultural Society. Journal. Nos. 3 and 4. Vol. VI. and 1 and 2. Vol. VII. 8vo.	The Society.
———— List of Fellows, Sept. 1851.	————
Institute of British Architects. Proceedings. Nov. 1850 to May 31, 1852.	The Institute.
Institution of Civil Engineers. Proceedings. Sessions 1849–50, 1850–51.	The Institution.
———— List of Members for 1851–52.	————
Microscopical Society. Transactions. Parts 1 and 2. Vol. III. 8vo. 1850–51.	The Society.
Pathological Society. Report of Proceedings, Fifth Session, 1850–51. 8vo.	The Society.
Royal Agricultural Society. Journal. Part 1. Vol. XII. 8vo.	The Society.
Journal of the Royal Asiatic Society. Part 1. Vol. XIV. Part 1. Vol. XIII. 8vo.	The Society.
Royal Institution. Proceedings. Nos. 6 to 10. 8vo.	The Institution.
Royal Medical and Chirurgical Society. Transactions. Vol. XXXIV. 8vo. 1851.	The Society.
Society of Antiquaries. Archæologia. Vols. XXXIII. and XXXIV. 4to. 1849–52.	The Society.
Statistical Society. Journal. Parts 3 and 4. Vol. XIV. and Parts 1 and 2. Vol. XV. 8vo.	The Society.
Zoological Society. Transactions. Vol. IV. Part 2. 4to. *London* 1852.	The Society.
Manchester:—Memoirs of the Literary and Philosophical Society. Vol. IX. 8vo. *London* and *Manchester* 1851.	The Society.
Madrid:—	
Memorias de la Real Academia de Ciencias. Part 1. Tomo I. 4to. *Madrid* 1850.	The Academy.
Resumen de las Actas. 1849–50. 8vo. *Madrid* 1850.	————
Moscow:—	
Nouveaux Mémoires de la Société Impériale des Naturalistes de Moscou. Tome IX. 4to. *Moscou* 1851.	The Society.
Bulletin. Année 1849, No. 4. 1850, No. 4. 1851, Nos. 1 and 2. 8vo. *Moscou* 1849, 1851.	————
Munich:—	
Abhandlungen der Königlich Bayerischen Akademie der Wissenschaften. Historischen Classe. 6ᵉⁿ Band. 2ᵉ Abt. Philosoph.-Philolog. Classe. 6ᵉⁿ Band. 2ᵉ Abt. Mathemat. Physikal. Classe. 6ᵉⁿ Band. 1ᵉ Abt. 4to. *München* 1851.	The Society.
Gelehrte Anzeigen. Bänd. XXXII. XXXIII. 4to.	————
Bulletin. Nos. 1 to 43, 1851. 4to.	————
Schilderung der Naturverhältnisse in Süd-Abyssinien. 4to. *München* 1851. Festrede von Dr. J. R. Roth.	————
Die Germanen und die Römer in ihren Wechselverhältnisse vor dem Falle des Westreiches. Festrede von Dr. Wittmann. 4to.	————
Naples:—Rendiconto della Reale Accademia delle Scienze. No. 51. 4to.	The Academy.
New York:—Annals of the Lyceum of Natural History of New York. Nos. 1 to 4. Vol. IV. and Nos. 3, 4, and 6. Vol. V. 8vo.	The Lyceum.
Paris:—	
Annuaire de l'Institut Nationale de France pour l'Année 1852.	The Institut.
Archives du Muséum d'Histoire Naturelle. Liv. 1, 2, 3, 4. Tom. V. 1 et 2. Tom. VI. 4to. *Paris*.	The Museum.
Catalogue Méthodique de la Collection des Reptiles. 8vo. *Paris* 1851.	————

Presents.	Donors.
ACADEMIES and SOCIETIES (*continued*).	
Paris:—	
Catalogue de la Collection Entomologique. Liv. 1, 2. 8vo. *Paris* 1850.	The Museum.
Catalogue Méthodique de la Collection des Mammifères. 8vo. *Paris* 1851.	———
Bulletin de la Société de Géographie. Tomes I. et II. 4me Série. 8vo.	The Society.
Bulletin de la Société Géologique de France. Feuilles 10 to 34. Tom. VIII. and 1 to 4. Tom. IX. 8vo.	The Society.
Mémoires. Part 1. Tom. IV. 4to. *Paris* 1851.	———
Comptes Rendus des Séances et Mémoires de la Société de Biologie. Tomes I. et II. 8vo. *Paris* 1850–51.	The Society.
Mémoires de l'Institut Nationale de France. Académie des Inscriptions et Belles Lettres. Tom. XIX. 4to. *Paris* 1851.	The Institute.
Mémoires présentés par divers Savants à l'Académie des Sciences. Tom. XIII. 4to. *Paris* 1852.	———
Notices et Extraits des Manuscrits de la Bibliothèque Nationale. Tom. XVII. 4to. *Paris* 1851.	———
Comptes Rendus de l'Académie des Sciences. Nos. 22 to 26. Tom. XXXIII. and Nos. 1 to 22. Tom. XXXIV.	———
Catalogue des Ouvrages et Mémoires Scientifiques de Siméon-Denis Poisson. 4to. *Paris* 1851.	———
Philadelphia:—	
American Philosophical Society. Proceedings. No. 6.	The Society.
Academy of Natural Sciences. Proceedings. No. 11. Vol. V. 8vo.	The Academy.
Franklin Institute. Journal. Nos. 1 to 4. Vol. XX. Third Series. Nos. 1 to 5. Vols. XXII. XXIII.	The Institute.
Rome:—Atti dell' Accademia Pontificia de' Nuovi Lincei. Anno 4. Sessione 5 and 6. 4to. *Roma* 1851.	The Academy.
St. Petersburgh:—	
Mémoires de l'Académie Impériale des Sciences. Sciences Math. et Phys. Liv. 3 et 4. Tom. IV. Divers Savants. Liv. 5 et 6. Tom. VI. 4to. *Pétersbourg* 1850–51.	The Academy.
Bulletin de la Classe Historico-Philologique. Tom. VIII. 4to. 1851.	———
Stockholm:—	
Kongl. Vetenskaps-Akademiens Handlingar för år 1849. 8vo. *Stockholm* 1851.	The Academy.
Öfversigt, &c. Sjunde årgången 1850.	———
Årsberättelser om Botaniska Arbeten och Upptäckter 1845–1848. Förra delen. *Stockholm* 1850.	———
Årsberättelse om Technologiens Framsteg. 8vo. *Stockholm* 1851.	———
Årsberättelse om Framstegen i Insekternas, Myriapodernas och Arachnidernas Naturalhistoria för 1847–48. 8vo. *Stockholm* 1851.	———
Berättelse om Framstegen i Fysik under år 1849. 8vo. *Stockholm* 1851.	———
Landtbruket förr och nu, &c. Tal, hållet i Kongl. Vetenskaps-Akademien. 8vo. *Stockholm* 1851.	———
Turin:—	
Memorie della Reale Accademia delle Scienze. Serie Seconde. Tomo XI. 4to. *Torino* 1851.	The Academy.
Utrecht:—	
Aanteekeningen van het verhandelde in de Sectievergadering van het Provinciaal Utrechtsch Genootschap, 1849–50. 8vo. *Utrecht* 1850–51.	The Society.

PRESENTS.	DONORS.
ACADEMIES and SOCIETIES (*continued*).	
Utrecht:—Verslag van het verhandelde, &c. 1849–50. 8vo.	The Society.
Van Diemen's Land:—Papers and Proceedings of the Royal Society. Part 3. Vol. I. 8vo. *Tasmania* 1851.	The Society.
Vienna:—	
Denkschriften der kaiserlichen Akademie der Wissenschaften.	The Academy.
Mathematisch-naturwissenschaftliche Classe. Band. I. II. Lief 1, 2, 3. Band III. Lief 1. Tafeln Abt. 1. 1850–51. fol.	———
Philosophisch-historische Classe. Band. I. II. Abt. 1, 2. fol. 1850–51.	———
Sitzungsberichte, Phil.-hist. Classe. Heft 1849–50 and 1–5. Bd. VI. and 1–5. Bd. VII. Jahrgang 1849, 1851. 8vo.	———
Math-natur. Classe 1849–50. Heft 1–5. Bd. VI. 1–5. Bd. VII. Jahrgang 1849, 1851. 8vo.	———
Archæologische Analecten. Tafeln zu den Sitzungsberichten.	———
Die Alterthümer vom Hallstätter Salzberg und dessen Umgebung. Beilage zu den Sitzungsberichten.	———
Jahrbuch der K. K. Geologischen Reichsanstalt, Zweiter Jahrgang. No. 1. 4to. *Wien.*	———
Fontes Rerum Austriacarum. Österreichische Geschichtsquellen. Band III. Abt. 2. 8vo. *Wien* 1851.	
Naturwissenschaftliche Abhandlungen. 4ter Band. 4to. *Wien* 1851.	Dr. Wilhelm Haidinger.
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PHILOSOPHICAL TRANSACTIONS.

I. The Bakerian Lecture.—*Contributions to the Physiology of Vision.—Part the Second. On some remarkable, and hitherto unobserved, Phenomena of Binocular Vision* (continued). *By* Charles Wheatstone, *F.R.S., Professor of Experimental Philosophy in King's College, London, Corresponding Member of the Academies of Science of Paris, Berlin, Brussels, Turin, Rome, Dublin, &c., of the Philosophical Society of Cambridge, the National Institute at Washington, &c.*

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§ 17.

In § 3. of the first part of my "Contributions to the Physiology of Vision," published in the Philosophical Transactions for 1838, speaking of the stereoscope, I stated, "The pictures will indeed coincide when the sliding pannels are in a variety of different positions, and consequently when viewed under different inclinations of the optic axes; but there is only one position in which the binocular image will be immediately seen single, of its proper magnitude, and without fatigue to the eyes, because in this position only the ordinary relations between the magnitude of the pictures on the retina, the inclination of the optic axes, and the adaptation of the eye to distinct vision at different distances, are preserved. The alteration in the apparent magnitude of the binocular images, when these usual relations are disturbed, will be discussed in another paper of this series, with a variety of remarkable phenomena depending thereon."

In 1833, five years before the publication of the memoir just mentioned, these yet unpublished investigations were announced in the third edition of Herbert Mayo's "Outlines of Human Physiology" in the following words:—" Mr. Wheatstone has shown, in a paper he is about to publish, that if by artificial means the usual relations which subsist between the degree of inclination of the optic axes and the visual angle which the object subtends on the retina be disturbed, some extraordinary illusions may be produced. Thus, the magnitude of the image remaining constant on the

retina, its apparent size may be made to vary with every alteration of the angular inclination of the optic axes."

I shall resume the consideration of the phenomena of binocular vision with this subject, because the facts I have ascertained regarding it are necessary to be understood before entering on the new experiments relating to stereoscopic appearances which I intend to bring forward on the present occasion.

Under the ordinary conditions of vision, when an object is placed at a certain distance before the eyes, several concurring circumstances remain constant, and they always vary in the same order when the distance of the object is changed. Thus, as we approach the object, or as it is brought nearer to us, the magnitude of the picture on the retina increases; the inclination of the optic axes, required to cause the pictures to fall on corresponding places of the retinæ, becomes greater; the divergence of the rays of light proceeding from each point of the object, and which determines the adaptation of the eyes to distinct vision of that point, increases; and the dissimilarity of the two pictures projected on the retinæ also becomes greater. It is important to ascertain in what manner our perception of the magnitude and distance of objects depends on these various circumstances, and to inquire which are the most, and which the least influential in the judgements we form. To advance this inquiry beyond the point to which it has hitherto been brought, it is not sufficient to content ourselves with drawing conclusions from observations on the circumstances under which vision naturally occurs, as preceding writers on this subject mostly have done, but it is necessary to have more extended recourse to the methods so successfully employed in experimental philosophy, and to endeavour, wherever it be possible, not only to analyse the elements of vision, but also to recombine them in unusual manners, so that they may be associated under circumstances that never naturally occur.

The instrument I shall proceed to describe enables these abnormal combinations to be made in a very simple and effectual manner. Its principal object is to cause the binocular pictures to coincide, with any inclination of the optic axes, while their magnitudes on the retinæ remain the same; or inversely, while the optic axes remain at the same angle, to cause the size of the pictures on the retinæ to vary in any manner.

Two plane mirrors inclined 90° to each other are placed together and fixed vertically upon a horizontal board. Two wooden arms move round a common centre situated on this board in the vertical plane which bisects the angle of the mirrors, and about 1½ inch beyond their line of junction. Upon each of these arms is placed an upright pannel, at right angles thereto, for the purpose of receiving its appropriate picture, and each pannel is made to slide to and from the opposite mirror. The eyes being placed before the mirrors, the right eye to the right mirror and the left eye to the left mirror, and the pannels being adjusted to the same distances, however the arms be moved round their centre, the distance of the reflected image of each picture from the eye will remain exactly the same, and consequently its retinal magnitude

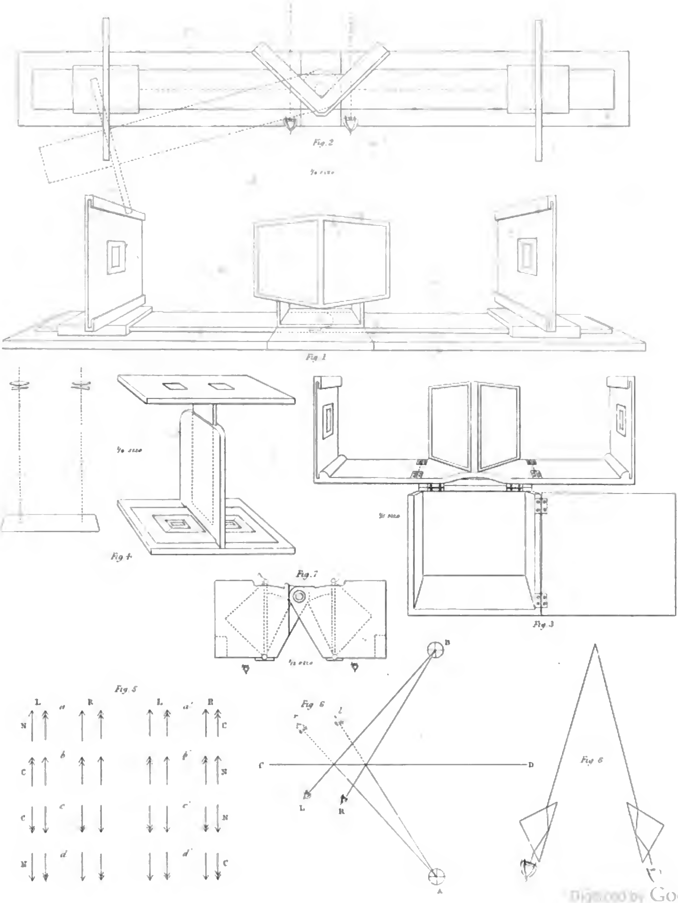

Fig. 2
Fig. 1
Fig. 4
Fig. 7
Fig. 3
Fig. 5
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Fig. 6
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Fig. 6

will be unchanged. But as the two reflected images do not occupy the same place when the pictures are in different positions, to cause the former to coincide the optic axes must converge differently. When the arms are in the same straight line, the images coincide while the optic axes are parallel; and as they form a less angle with each other, the optic axes converge more to occasion the coincidence. When the arms remain in the same positions, while the pannels slide towards or from the mirrors, the convergence of the optic axes remains the same, but the magnitude of the pictures on the retinæ increases as the distance decreases. By the arrangement described, and which is represented by figs. 1 and 2 Plate I., the reflected pictures are always perpendicular to the optic axes, and the corresponding points of the pictures, when they are exactly similar, fall upon corresponding points of the retinæ. The instrument has an adjustment for otherwise inclining them if it be required.

Let us now attend to the effects produced. The pictures being fixed at the same distance from the mirrors, there is a certain adjustment of the arms at which the binocular image will appear of its natural size, that is, the size we judge the picture itself to be when we look at it directly; in this case the magnitude of the pictures on the retinæ and the inclination of the optic axes preserve their usual relation to each other. If now the arms be moved back, so as to cause a less convergence of the axes, the image will appear to increase in magnitude until the arms are in a straight line and the optic axes are parallel; and, on the other hand, if the arms be moved forwards, so as to form a less angle, the optic axes will converge more, and the image will appear gradually smaller. In this manner, while the retinal magnitude remains the same, the perceived magnitude of the binocular object varies through a very considerable range.

The instrument being again adjusted so that the image shall be seen of its natural size; on sliding the pictures nearer the mirrors its perceived magnitude will be augmented, and on sliding them from the mirrors it will appear diminished in size. During these variations of magnitude the inclination of the optic axes remains the same.

The perceived magnitude of an object, therefore, diminishes as the inclination of the axes becomes greater, while the distance remains the same; and it increases, when the inclination of the axes remains the same, while the distance diminishes. When both these conditions vary inversely, as they do in ordinary vision when the distance of an object changes, the perceived magnitude remains the same*.

Before I proceed further it will be proper to explain the meaning of some of the terms I employ. I call the magnitude of the object itself, the real or objective mag-

* Several cases of the alteration of the perceived magnitude of objects are mentioned by Dr. R. SMITH (Complete System of Opticks, 1738, vol. ii. p. 388, and rem. 526 and 532); and Dr. R. DARWIN (Philosophical Transactions, vol. lxxvi. p. 313) observed that when an ocular spectrum was impressed on both eyes it appeared magnified when they were directed to a wall at a considerable distance. The facts noticed by these authors are satisfactorily explained by the above considerations.

nitude; the magnitude of the picture on the retina, the retinal magnitude; and the magnitude we estimate the object to be from its retinal magnitude and the inclination of the optic axes conjointly, I name the perceived magnitude. I do not use the term apparent magnitude, because, according to its ordinary acceptation, it sometimes means what I call retinal, and at other times what I name perceived magnitude.

We have seen in what manner our perception of magnitude is modified by the new associations which this instrument enables us to form; let us now examine how our perception of distance is affected by them. If we continue to observe the binocular picture whilst it apparently increases or decreases, in consequence of the inclination of the optic axes varying while the magnitude of the impressions on the retinæ remains the same, it does not appear either to approach or to recede; and yet if we attentively regard it in any fixed position, it is perceived to be at a different distance. On the other hand, if we continue to regard the binocular picture, enlarging and diminishing in consequence of the change of retinal magnitude while the convergence of the axes remains the same, we perceive it to approach or recede in the most evident manner; but on fixing the attention to it, when it is stationary, at any instant, it appears to be at the same distance at one time as it is at another.

Convergence of the optic axes therefore suggests fixed distance to the mind; variation of retinal magnitude suggests change of distance. We may, as I have above shown, perceive an object approach or recede without appearing to change its distance, and an object to be at a different distance, without appearing to approach or recede; these paradoxical effects render it difficult, until the phenomena are well apprehended, to know, or to express, what we actually do perceive.

It is the prevalent opinion that the sensation which accompanies the inclination of the optic axes immediately suggests distance, and that the perceived magnitude of an object is a judgement arising from our consciousness of its distance and of the magnitude of its picture on the retina. From the experiments I have brought forward, it rather appears to me that what the sensation which is connected with the convergence of the axes immediately suggests is a correction of the retinal magnitude to make it agree with the real magnitude of the object, and that distance, instead of being a simple perception, is a judgement arising from a comparison of the retinal and perceived magnitudes. However this may be, unless other signs accompany this sensation the notion of distance we thence derive is uncertain and obscure, whereas the perception of the change of magnitude it occasions is obvious and unmistakeable.

To see, in their full extent, the variations of magnitude exhibited by the instrument I have described, it is necessary to attend to the following observations.

As the inclination of the optic axes corresponding to a different distance is habitually, under ordinary circumstances, accompanied with the particular adaptation of the eyes required for distinct vision at that distance, it is difficult to disassociate these two conditions so as to see with equal distinctness the binocular picture when

the optic axes are parallel, and when they converge greatly, although the pictures remain, in both cases, at the same distance from the eyes. The adaptation is, therefore, not entirely dependent on the divergence of the rays of light which proceed from the object regarded, but also, in some degree, on the inclination of the optic axes. I have acquired by practice considerable power of adjustment, or rather dis-adjustment, of the eyes, and can, without having recourse to artificial means, see the binocular picture distinctly when its perceived magnitude is widely different. Those to whom such an effort is painful may employ short-sighted spectacles to see the binocular picture when the eyes converge within the limit of distinct vision for the distance at which the pictures are placed; and long-sighted spectacles when the eyes converge beyond that limit, or become parallel.

There is a means of avoiding to a very considerable extent the influence of the adjustment of the eyes, and thereby enabling the pictures to be seen distinctly within the entire range of the inclination of the optic axes. This is by looking at the reflected images in the mirrors through two very minute apertures, not larger than fine pin-holes, placed near each eye, and illuminating the pictures by a very strong light; sunshine in the middle of the day answers the purpose very well. By this expedient the divergence of the rays of light is greatly diminished, and the adaptation of the eyes does not materially influence the result.

§ 18.

Leaving this subject, I will now revert to the stereoscope and its effects.

Since 1838 numerous modifications of the stereoscope have occurred to me, and several ingenious arrangements have also been proposed by Sir DAVID BREWSTER and Prof. DOVE; but there is no form of the instrument which has so many advantages for investigating the phenomena of binocular vision as the original reflecting stereoscope. Pictures of any size may be placed in it, and it admits of every kind of adjustment.

I have constructed a very portable reflecting stereoscope which is represented at fig. 3. The sides fold over the mirrors, and the mirrors then fold into a box, which is not larger than 6 inches in any of its dimensions. To avoid the second feeble reflection from the anterior surface of the silvered glass, which has a bad effect when the attention is attracted to it, I have sometimes employed reflecting prisms. The reflecting surfaces of the prisms should be silvered in order to obviate the unequal brightness of the field of view on each side of the limit of total reflection; and as it would be too costly to employ very large prisms, they should have an adjustment to accommodate their distance to the width between the eyes of the observer.

I have, for many years past, employed also another means to occasion, without any straining of the eyes, the coincidence of the pictures so that the image in relief shall appear of the same magnitude and at the same distance as the object which they represent would do if it were itself directly regarded. In this apparatus, prisms

being employed to deflect the rays of light proceeding from the pictures, so as to make them appear to occupy the same place, I have called it the refracting stereoscope.

It is represented by fig. 4. It consists of a base 6 inches long and 4 inches broad, upon which stands an upright partition, 5 inches high, dividing it equally; this partition is capable of extension by means of a slide to double the length, and carries at its upper extremity a board placed parallel to the base, and of the same dimensions. In this upper board there are two apertures an inch square, one on each side of the partition, the centres of which are 2½ inches from each other; in these apertures are fixed a pair of glass prisms having their faces inclined 15°, and their refractive angles turned towards each other. The stereoscope pictures are to be placed on the base, and their centres ought not to exceed the distance of 2½ inches.

A pair of plate-glass prisms, their faces making with each other an angle of 12°, will bring two pictures, the corresponding points of which are 2½ inches apart, to coincidence at a distance of 12 inches, and a pair with an angle of 15° will occasion coincidence at 8 inches.

The refracting stereoscope has the advantage of portability, but it is limited to pictures of small dimensions. It is well suited for Daguerreotypes, which are usually of small size, and, on account of the nature of their reflecting surface, must be viewed in a particular direction with respect to the light which falls upon them; whereas in the reflecting stereoscope it is somewhat difficult to render the two Daguerreotypes equally visible. For drawings and Talbotypes it however offers no advantages, though it is equally well suited for them when their dimensions are small.

Stereoscopic drawings afford a means of illustrating works with figures of three dimensions, instead of with mere plane representations. Works on crystallography, solid geometry, spherical trigonometry, architecture, machinery, &c., might be thus rendered more instructive, from the perfect counterpart of the solid figure seen from a single point of view being represented, instead of merely one of its plane projections. For this purpose the corresponding binocular figures must be engraved in parallel vertical columns, and their coalescence may be effected by viewing them through a pair of prisms, similar to those employed in the refracting stereoscope, placed in a frame at the proper distance from each other. If the engravings should be less than 2½ inches apart, the prisms may be dispensed with by persons who have command over the adaptation of their eyes, particularly if they be short-sighted.

§ 19.

At the date of the publication of my experiments on binocular vision, the brilliant photographic discoveries of Talbot, Niepce and Daguerre, had not been announced to the world. To illustrate the phenomena of the stereoscope I could therefore, at that time, only employ drawings made by the hands of an artist. Mere outline figures, or even shaded perspective drawings of simple objects, do not present much

difficulty; but it is evidently impossible for the most accurate and accomplished artist to delineate, by the sole aid of his eye, the two projections necessary to form the stereoscopic relief of objects as they exist in nature with their delicate differences of outline, light and shade. What the hand of the artist was unable to accomplish, the chemical action of light, directed by the camera, has enabled us to effect.

It was at the beginning of 1839, about six months after the appearance of my memoir in the Philosophical Transactions, that the photographic art became known, and soon after, at my request, Mr. Talbot, the inventor, and Mr. Collen (one of the first cultivators of the art) obligingly prepared for me stereoscopic Talbotypes of full-sized statues, buildings, and even portraits of living persons. M. Quetelet, to whom I communicated this application and sent specimens, made mention of it in the Bulletins of the Brussels Academy of October 1841. To M. Fizeau and M. Claudet I was indebted for the first Daguerreotypes executed for the stereoscope. The beautiful stereoscopic representations of statuary, architecture, machinery, natural history specimens, portraits of living persons, single and in groups, &c., which have recently been produced by M. Soleil and M. Claudet, are now too well known to the public to need more than a slight reference to them.

With respect to the means of preparing the binocular photographs (and in this general term I include both Talbotypes and Daguerreotypes), little requires to be said beyond a few directions as to the proper positions in which it is necessary to place the camera in order to obtain the two required projections.

We will suppose that the binocular pictures are required to be seen in the stereoscope at a distance of 8 inches before the eyes, in which case the convergence of the optic axes is about 18°. To obtain the proper projections for this distance, the camera must be placed, with its lens accurately directed towards the object, successively in two points of the circumference of a circle of which the object is the centre, and the points at which the camera is so placed must have the angular distance of 18° from each other, exactly that of the optic axes in the stereoscope. The distance of the camera from the object may be taken arbitrarily, for, so long as the same angle is employed, whatever that distance may be, the pictures will exhibit in the stereoscope the same relief, and be seen at the same distance of 8 inches, only the magnitude of the picture will appear different. Miniature stereoscopic representations of buildings and full-sized statues are therefore obtained merely by taking the two projections of the object from a considerable distance, but at the same angle as if the object were only 8 inches distant, that is, at an angle of 18°.

To produce the best effect, it is necessary that the pictures be so placed in the stereoscope that each eye shall see its respective picture at the proper point of sight: if this condition be not attended to, the binocular perspective will be incorrect.

For obtaining binocular photographic portraits, it has been found advantageous to employ, simultaneously, two cameras fixed at the proper angular positions.

I subjoin a Table of the inclinations of the optic axes which correspond to different

distances; it also shows the angular positions of the camera required to obtain binocular pictures which shall appear at a given distance in the stereoscope in their true relief.

Inclination of the optic axes	2°	4°	6°	8°	10°	12°	14°	16°	18°	20°	22°	24°	26°	28°	30°
Distance in inches..........................	71·5	35·7	23·8	17·8	13·2	11·8	10·1	8·8	7·8	7·0	6·4	5·8	5·4	5·0	4·6

The distance is equal to $\frac{a}{2}$ cotang $\frac{\theta}{2}$; a denoting the distance between the two eyes, and θ the inclination of the optic axes.

§ 20.

As the inclination of the optic axes diminishes by the removal of an object to which they are directed to a greater distance, not only does the magnitude of the pictures projected by it on the retinæ proportionately diminish, but the dissimilarity of the pictures becomes less. The difference of distance between any two points of each of the pictures will diminish until the projections become sensibly similar. Under the usual circumstances attending the vision of a solid object placed at a given distance, a particular inclination of the axes is invariably accompanied by a specific pair of dissimilar projections; and if the distance be changed, a different inclination of the axes is accompanied by another pair of projections; but, by means of the stereoscope, we have it within our power to associate these circumstances abnormally, and to cause any degree of inclination of the axes to coexist with any dissimilarity of the two pictures. To ascertain experimentally what takes place under these circumstances M. Claudet prepared for me a number of Daguerreotypes of the same bust, taken at a variety of different angles, so that I was enabled to place in the stereoscope two pictures taken at any angular distance from 2° to 18°, the former corresponding with a distance of about 6 feet, and the latter with a distance of about 8 inches. The effect of a pair of near projections seen with a distant convergence of the optic axes, is to give an undue elongation to lines joining two unequally distant points, so that all the features of a bust appear to be exaggerated in depth. The effect, on the contrary, of a pair of distant projections, seen with a *near* convergence of the axes, is to give an undue shortening to the same lines, so that the appearance of a bas-relief is obtained from the two projections of the bust. The apparent dimensions in breadth and height remain in both cases the same.

§ 21.

To reproduce the conditions of the binocular vision of a solid object as completely as possible by means of its two plane projections, it is necessary, as I have before stated, that the projections shall be such as correspond exactly with the inclination of the optic axes under which they are viewed. I have already shown in § 20 what takes place when this condition is not strictly observed, and I may add that the

mind is not unpleasantly affected by a considerable incongruity in this respect; on the contrary, the effect in many cases seems heightened by viewing the solid appearance, intended for a determinate degree of inclination of the axes, under an angle several degrees less; the reality is as it were exaggerated. When the optic axes are parallel, in strictness there should be no difference between the pictures presented to each eye, and in this case there would be no binocular relief; but I find that an excellent effect is produced when the axes are nearly parallel by pictures taken at an inclination of 7° or 8°, and even a difference of 16° or 17° has no decidedly bad effect.

This circumstance enables us to combine the ideal amplification arising from viewing pictures placed near the eyes under a small inclination, or even parallelism, of the optic axes mentioned in § 17, with the perception of solidity arising from the dissimilarity of the projections; for this purpose, the pictures in the refracting stereoscope, or their reflected images in the reflecting instrument, must be viewed through lenses the focal distance of which is equal to the distance between them and the pictures; the perceived magnitude of the binocular image will increase with the nearness of the pictures, and depends almost entirely on the disassociation of the retinal magnitude from its usually accompanying inclination of the optic axes, the actual magnifying power of the lenses having a very small influence.

The sole use of the lenses is to render the rays of light parallel, which it is necessary they should be for distinct vision when the optic axes are parallel. When the reflecting stereoscope is employed, this means of magnifying the effect is not of much utility, as pictures of any size may be adapted to that instrument. But in the case of the refracting stereoscope it may be advantageously made use of. By combining lenses with the refracting stereoscope, described in § 18, Daguerreotypes somewhat wider than the width between the eyes may be employed. Sir David Brewster has used, to effect the same purpose, semi-lenses with their edges directed towards each other, which serve at the same time to render the rays less convergent and slightly to displace the pictures towards each other. Two corresponding Daguerreotypes, each not exceeding in breadth the width between the eyes, being placed close to each other, and viewed with lenses of short focal distance, will even without the aid of the prisms give an apparently highly magnified binocular image in bold relief.

There is a peculiarity in such images worthy of remark; although the optic axes are parallel, or nearly so, the image does not appear to be referred to the distance we should, from this circumstance, suppose it to be, but it is perceived to be much nearer, and indeed more so, as the pictures are nearer the eyes, though the inclination of the optic axes remains the same, and should therefore suggest the same distance; it seems as if the dissimilarity of the projections, corresponding as they do to a nearer distance than that which would be suggested by the former circumstance alone, alters in some degree the perception of distance.

I recommend, as a convenient arrangement of a refracting stereoscope for viewing

Daguerreotypes of small dimensions, the instrument represented, fig. 4, shortened in its length from 8 inches to 5, and lenses of 5 inches focal distance placed before and close to the prisms.

§ 22.

I now proceed to another subject—to the consideration of those phenomena which I have termed Conversions of Relief.

In § 5 of my first memoir I noticed the remarkable circumstance, that when the drawing intended to be seen by the right eye is presented to the left eye in the stereoscope, and *vice versâ*, a totally different solid figure is perceived to that seen before the transposition. I called this the converse figure, and showed that it differs from the normal figure in the circumstance, that those points which appear the most distant in the latter, appear the nearest in the former.

The pictures being, in the first place, presented directly to their corresponding eyes, as in the refracting stereoscope, and exhibiting therefore the resultant image in its normal relief, the conversion of the relief may be effected in three different ways,—1st, by transposing the pictures from one eye to the other, as mentioned above; 2ndly, by reflecting the pictures, while they remain presented to the same eye, as in the reflecting stereoscope; and 3rdly, by inverting the position of the pictures without transposing them.

The following considerations will explain the cause of the conversion of relief in the preceding cases.

If two different objects, or parts of an object (fig. 5 *a*), have a greater lateral distance between them on the right-hand picture than that which they have on the left-hand picture, the optic axes must converge more to make the left-hand than to make the right-hand objects coincide, and the left-hand object will appear the nearest.

If the pictures be now transposed from one eye to the other (fig. 5 *a'*), the greatest distance will be between the corresponding points of the picture presented to the left eye; the optic axes must therefore converge less to make the left-hand objects coincide, and the right-hand object will appear the nearest.

If the pictures, remaining untransposed, be each separately reflected (fig. 5 *b*), the relative distances of the corresponding objects remain the same to each eye, and the left-hand object will still appear nearest; but in consequence of the lateral inversion of the objects in each picture by reflexion, that which was previously on the left will now be on the right, and therefore, the object which before appeared nearest, will now appear farthest.

When the pictures are turned upside down, still remaining untransposed (fig 5 *c*), the objects are reversed with respect to the right and left, in the same manner as they are when reflected, and the lateral distances between the objects remaining the same to each eye, precisely the same conversion of relief is produced as in the preceding case, except that the resultant image is inverted. The diagram (fig. 5) repre-

sents all the possible changes of the two binocular pictures; those marked N show the normal relief, and those marked C the converse relief.

But it may be asked why, if the reflection or inversion of the binocular pictures of an object gives rise to the mental idea of the converse relief, the same converse relief is not observed when the object itself is reflected in a mirror, or inverted. The reason is this; that in the former cases the projections to each eye are separately reflected or inverted, still remaining presented to the same eye, whereas, by the reflection or inversion of the object itself, not only are the projections reflected or inverted, but they are also transposed from one eye to the other; and these circumstances occurring simultaneously reproduce the normal relief.

Fig. 6 will render this evident in the case of reflexion: A is the object, B its reflexion in the mirror CD; RB and LB are the directions in which the right and left eyes view the reflected image respectively, and *l*A and *r*A the directions in which the eyes would view the corresponding face of the object directly.

In the case of an inverted object, it is obvious that that projection which was before seen by the right eye must be seen by the left eye, and the contrary.

It is possible to make this normal or converse relief appear while one of the pictures remains constantly presented to the same eye. This result may be thus obtained. Having taken a photograph of the object, which should be one the converse of which has a meaning, take two others at the same angular distance (say 18°), one on the right side, the other on the left side of the original. Of the three pictures thus taken, if the middle one be presented to the right eye, and the left picture to the left eye, a normal relief will be seen; but if the right picture be presented to the left eye, the other remaining unchanged, a converse relief will be seen. In like manner, if the middle picture be presented to the left eye, and the right picture to the right eye, a normal relief will appear; but if the left picture be presented to the right eye, the converse relief will present itself. It must be observed, that the normal and converse reliefs, when the same picture remains presented to the same eye, belong to two different positions of the object.

§ 23.

Hitherto I have taken into consideration only those cases of the conversion of relief which are exhibited by binocular pictures in the stereoscope, when they are transposed, reflected or inverted; I shall now proceed to show how phenomena of the same kind may be elicited by regarding objects themselves, by means of an instrument adapted for the purpose. As this instrument conveys to the mind false perceptions of all external objects, I have called it the Pseudoscope. It is represented by fig. 7, and is thus constructed: two rectangular prisms of flint glass, the faces of which are 1·2 inch square, are placed in a frame with their hypothenuses parallel, and 2·1 inches from each other; each prism has a motion on an axis corresponding with the angle nearest the eyes, so that they may be adjusted that their bases may

have any inclination towards each other; and the frame itself is adjustable by a hinge at *a*, in order to bring the prisms nearer each other to suit the eyes of the observer.

The instrument being held to the eyes, and adjusted to an object, so that it shall appear single, each eye will see a reflected image of that projection of the object which would be seen by the same eye without the pseudoscope. This is exactly the contrary of what occurs when the eyes regard the reflected image of an object in a looking-glass; the left eye then sees the reflected image of the right-hand projection, and the right eye the reflected image of the left projection, as shown by fig. 6.

Plane mirrors cannot be substituted for the reflecting prisms, for this reason; the refraction of the rays of light at the incident and emergent surfaces of the prisms enables the reflexion of an object to be seen when the object is even behind the prolongation of the reflecting surface, as shown at fig. 8, and thus the reflected binocular image may be seen in the same place as the object itself, whereas the images cannot be made by means of plane mirrors thus to coincide.

When the pseudoscope is so adjusted as to see a near object while the optic axes are parallel, to view a more distant object with the same adjustment, the axes must converge, and the more so as the object is more distant; all nearer objects than that seen when the axes are parallel, will appear double, because the optic axes can never be simultaneously directed to them. If this instrument be so adjusted that very distant objects are seen single when the eyes are parallel, *all* nearer objects will appear double, because the optic axes can never converge to make their binocular images coincide. If the attention is required to be directed to an object at a particular distance, the best mode of viewing it with the pseudoscope is to adjust the instrument so that the object shall appear at the proper distance and of its natural size. In this case the more distant objects will appear nearer and smaller, and the nearer objects will appear more distant and larger.

In ordinary vision, whenever the distance of an object varies, the magnitude of the picture on the retina, and the degree of convergence of the optic axes, always maintain a constant relation to each other, both increasing or decreasing together; and the perceived magnitude, suggesting to the mind the real magnitude of the object, in consequence thereof remains the same. The instrument I described in § 17 shows what illusions arise when the usual relations of these elements of our perceptions are disturbed, by causing one to remain constant while the other varies. The pseudoscope exhibits the still more curious illusions, which result from combining these elements inversely, so that as an object becomes nearer, its larger picture on the retina is accompanied by a less convergence of the optic axes. With the pseudoscope we have a glance, as it were, into another visible world, in which external objects and our internal perceptions have no longer their habitual relation with each other.

I will now proceed to describe some of the illusions produced by the aid of this

instrument. Those which may be strictly designated conversions of relief, in which the illusive appearance has the same relation to that of the real object as a cast to a mould, or a mould to a cast, are very readily perceived. I must however remark, that it is necessary to illuminate the object equally, so as to allow no lights or shades to appear upon them, for their presence has a considerable influence on the judgement, and is one of the principal causes of the perception of the proper relief when a single eye is employed.

The inside of a tea-cup appears as a solid convex body; the effect is more striking if there are painted figures within the cup.

A china vase, ornamented with coloured flowers in relief, presents a very remarkable appearance; we apparently see a vertical section of the interior of the vase, with painted hollow impressions of the flowers.

A small terrestrial globe appears as a concave hemisphere; on turning it round on its axis, it was curious to see different portions of the spherical map appear and disappear in a manner that nothing in external nature can imitate.

A bust regarded in front becomes a deep hollow mask; the appearance when regarded in profile is equally striking.

A framed picture hanging against a wall, appears as if imbedded in a cavity made in the wall.

A medal, or the impression of a seal, is perfectly converted into a representation of the die from which it has been struck; and, on the other hand, the mould or die of a medal, or an engraved seal, becomes a *fac-simile* of the medal or raised impression. It will also be observed, that if the medal be placed on a flat surface, as a sheet of paper, it will appear sunk beneath the surface; and if it be placed in a hollow of the same size, it will appear to stand above the surface as much as it actually is below it.

These appearances are not always immediately perceived; and some much more readily present themselves than others. Those converse forms which have a meaning, and resemble real forms we have been accustomed to see, are those which are the most easily apprehended. Viewed with the pseudoscope, notwithstanding the inversion of the pictures on the retina, the natural appearance of the object continues to intrude itself, when sometimes suddenly, and at other times gradually, the converse occupies its place. The reason of this is, that the relief and distance of objects is not suggested to the mind solely by the binocular pictures and the convergence of the optic axes, but also by other signs, which are perceived by means of each eye singly; among which the most effective are the distributions of light and shade and the perspective forms which we have been accustomed to see accompany these appearances. One idea being therefore suggested to the mind by one set of signs, and another totally incompatible idea by another set, according as the mental attention is directed to the one and abstracted from the other, the normal

form or its converse is perceived. This mental attention is involuntary; no immediate effort of the will can call up one idea while the other continues to present itself, though the transition may be facilitated by intentionally removing some of the signs which suggest the preponderating idea; thus the converse form being perceived, closing either eye will most frequently cause an instant reversion to the normal form; and always, if the monocular signs of relief are sufficiently suggestive.

I know of nothing more wonderful, among the phenomena of perception, than the spontaneous successive occurrence of these two very different ideas in the mind, while all external circumstances remain precisely the same. Thus a small statuary group, an elegant and beautiful object, without any apparent cause becomes converted into another totally dissimilar object uncouth in appearance, and which gives rise to no agreeable emotions in the mind; yet in both cases all the sensations that intervene between objective reality and ideal conception continue unchanged.

The effects of the pseudoscope I have already mentioned, may be strictly called conversions of relief, because the illusive appearance is in each case the converse impression of the relief of the real object. If, however, the object consists of parts detached from and behind each other, the preceding term is inappropriate to denote the effects which result, but the more general expression conversion or inversion of distance may be employed to designate them. I proceed to call attention to a few such effects.

Skeleton figures of geometrical solids, as cubes, pyramids, &c., readily show their converse.

Two objects at different distances, being simultaneously regarded, the most remote will appear the nearest and the nearest the most remote.

An ivory foot rule, held immediately before the eyes a little inclined to the horizon with its remote end elevated, appears inclined in the opposite way, its nearer end elevated, and as if the observer were looking at its lower surface. Its form also undergoes a change. Since the nearest end, the retinal magnitude of which is the largest, appears farthest from the eyes, and the nearest end, the retinal magnitude of which is greatest, appears near the eyes, the rule will no longer be perceived to be rectangular, but trapezoidal. If the rule be placed horizontally, and it be regarded with the pseudoscope at an angle of 45°, it will appear with the form just described standing vertically.

Any object placed before the wall of a room will appear behind the wall, and as if an aperture of the proper dimensions had been made in the wall to allow it to be seen; if the object be illuminated by a candle, its shadow will appear as far before the object as in reality it is behind.

The appearance of a plant is very remarkable; as the branches which are farthest from the eye are perceived to be the nearest, those parts which are actually obscured by the branches before them, appear broken away and allow the parts apparently be-

hind them to be seen. A flowering shrub before a hedge appears to be transferred behind it; and a tree standing outside a window may be brought visibly within the room in which the observer is standing.

I have before observed that the transition from the normal to the converse perception is often gradual; I will give one instance of this as an illustration. The object was a page of medallions embossed on card-board, and the raised impressions were protected from injury by a thick piece of mill-board having apertures in it made to correspond to each medallion. The page was placed horizontally, illuminated by a candle placed beyond it, and looked at through the pseudoscope at an angle of 45°; for the first moment the page appeared as it would have done without the instrument; soon after the medallions appeared level with the upper surface, and the shadows on the upper parts of the circular apertures were converted into deep depressions as if cut out with a tool; they next, from horizontal, became vertical, each standing erect on the horizontal plane, and immediately afterwards the reliefs were all changed into hollows; finally, the page itself stood vertical, but with that change of form which I indicated in the case of the rule, the upper edge appearing much shorter than the lower edge: the series of changes being now complete, the final form remained constant as long as the object was regarded.

In endeavouring to analyse the phenomena of converse perception, it must be borne in mind that the transposition of distances has reference only to distances from the retinæ, not to absolute horizontal distances in space. Thus, if a straight ruler be held in the vertical plane perpendicular to the optic base, and also inclined 45° to the horizon so that its upper end shall be the most distant, when the eyes are directed horizontally towards it, the rule will appear exactly in the converse position. If the rule be now removed lower down in the same vertical plane, its inclination remaining unchanged, so that to look upon it the plane of the optic axes must be inclined 45°, it will appear unaltered in position, because its two pictures are parallel on the retinæ, and the optic axes would require the same convergence to make the upper and lower ends coalesce. The rule being removed still lower down, instead of its position being apparently reversed, it will appear to have a greater inclination on the same side than the object itself has. In the first case the more distant end is actually farthest from the eyes; in the second the near and remote ends are equally distant; and in the third the nearest end is most distant.

Attention to what I have just stated will explain many anomalous circumstances which occur when the eyes are differently directed towards the same object. It may also be necessary to remark, that the conversion of distance takes place only within those limits in which the optic axes sensibly converge, or the pictures projected on the retinæ are sensibly dissimilar. Beyond this range there is no mutual transposition of the apparent distances of objects with the pseudoscope; a distant view therefore appears unchanged.

Some very paradoxical results are obtained when objects in motion are viewed

through the pseudoscope. When an object approaches, the magnitude of its picture on the retinæ increases as in ordinary vision, but the inclination of the optic axes, instead of increasing, becomes less, as I have already explained. Now an enlargement of the picture on the retina invariably suggests approach, and a less convergence of the optic axes indicates that the object is at a greater distance; and we have thus two contradictory suggestions. Hence, if two objects be placed side by side at a certain distance before the eyes, and one of them be moved forwards, so as to vary its distance from the other, its continually enlarging picture on the retina makes it appear to come towards the eyes, as it actually does, while at the same time it appears at every step at a greater distance beyond the fixed object; from one suggestion the object appears to approach, from the other to have receded. I again observe that retinal magnitude does not itself suggest distance, but from its changes we infer changes of distance.

I have hitherto only described the pseudoscope constructed with two reflecting prisms. This is the most convenient apparatus for effecting the conversion of distance and relief that has occurred to me; but other means may be employed, which I will briefly mention.

1st. Two plane mirrors are placed together so as to form a very obtuse angle towards the eyes of the observer; immediately before them the object is to be placed at such distance that a reflected image shall appear in each mirror. The eyes being placed before and a little above the object, must be caused to converge to a point between the object and the mirrors; the right-hand image of the left eye will then unite with the left-hand image of the right eye, and the converse relief will be perceived. The disadvantages of this method are that only particular objects can be examined, and it requires a painful adaptation of the eye to distinct vision.

2ndly. Place between the object and each eye a lens of small focal distance, and adjust the distances of the object and the lenses so that distinct inverted images of the object shall be seen by each eye; on directing the eyes to the place of the object the two images will unite, and the converse relief be perceived. As the rays of light proceeding from the images have a greater divergence than those which would proceed from the point to which the optic axes are directed, long-sighted persons will see the binocular image more distinctly by wearing a pair of short-sighted spectacles. In this experiment the field of view is very small on account of the distance at which it is necessary to place the lenses from the eyes; but I have been enabled in this manner to see beautifully the converse relief of a small ivory bust and of other small objects, which however should be inverted in order to see them direct.

3rdly. The inverted images of the lenses, instead of being received immediately by the eyes as just described, may be thrown on a plate of ground glass as in the case of the ordinary camera-obscura, and may be then caused to unite by the means employed in any form of the refracting stereoscope.

§ 24.

The cases of the conversion of relief when the object is regarded with one eye only, some of which were known more than a century ago, were taken into consideration and endeavoured to be explained by me in § 11 of the first part of this memoir, and Sir David Brewster* has published some interesting and instructive observations on the same subject; I will therefore not revert to this matter here, but only to say that I have myself never observed the conversion of relief when looking with both eyes immediately on a solid object, and if it has been observed by others under such circumstances, I should be inclined to attribute the effect to an inequality in the impressions on the two eyes so that one only is attended to. But the plane shaded representation of a solid object, the relief of which is not very deep, may easily be made to appear at will either as the solid which it is intended to represent or as its converse, even when both eyes are employed. This effect is strikingly observed in the glyptographic engravings of medals of low relief, and depends entirely on whether the light is so placed that it would cast the same shadows on the real object as are represented in the picture, or that it would cast shadows in the opposite direction. In the former case the picture appears with the relief it was intended to suggest; in the latter with the converse relief. I have observed similar effects with Daguerreotypes of medallions and cameos, and with carefully shaded drawings of simple objects.

* Transactions of the Royal Society of Edinburgh, vol. xv. p. 365 and 657.

II. *On the Automatic Registration of Magnetometers, and Meteorological Instruments, by Photography.—No. IV.* By Charles Brooke, *M.B., F.R.S.*

Received May 8,—Read June 19, 1851.

On the Automatic Temperature Compensation of the Force Magnetometers.

A PORTION of the funds liberally contributed by the Government for the advancement of science, and placed at the disposal of the President and Council of the Royal Society, having been entrusted to the author for the accomplishment of the above object on a plan which was submitted to the Astronomer Royal and Colonel Sabine in the spring of last year*, and by them considered feasible, he considers that he cannot better fulfil the obligation of reporting progress at the present period, than by laying before the Royal Society a description of the instruments now constructed.

So long as the results of the variations of magnetic force were deduced from eye-observation only, at the periods of which the temperature as well as the position of the magnets was recorded, a correction for the influence of change of temperature on the instruments themselves could be readily estimated and applied; but in deducing mean values from the photographic registers, especially those for intervals involving considerable changes of temperature, it is manifest that the greatest degree of accuracy cannot be attained, unless either the apparent values were individually corrected by means of a separate register of the thermometer enclosed in the box with the magnet, or the instrument possessed within itself an approximate automatic correction for the effects of change of temperature.

The object would not be unattainable by the former means, but the process would be both difficult and laborious; it therefore appeared more desirable to attempt its accomplishment by the latter. Referring therefore to the equation of equilibrium of the bifilar magnet, viz.

$$mX = W\frac{ab}{l}\sin\theta,$$

in which m is the magnetic moment of the bar, X the horizontal component of the earth's magnetic force, W the weight of the suspended bar and its appurtenances, l the length of the suspension skeins, a and b the upper and lower intervals of their centres, and θ the angle of torsion, it is evident that the object in view would be

* The author feels bound to express his belief that a somewhat similar plan of compensating the force magnetometers subsequently proposed by Mr. Brown at the last meeting of the British Association was entirely original.

accomplished, if by any mechanical agency either of the quantities W, a, or b could be *simultaneously* influenced by change of temperature, and *proportionally* to the altered value of m.

As it would be desirable that the correction should involve the small coefficient of the second power of the temperature, a very definite value of which has been determined*, it was at first proposed to act on W by means of a hollow glass globe attached to the suspension frame of the magnet, with a vertical tubular stem dipping into a cup of mercury. It is clear that as the elastic force of the air contained in the bulb is diminished by heat, a column of mercury in the stem would fall, and the diminution of the suspended weight thus occasioned, might by a due adjustment of the capacity of the bulb be rendered equivalent to the constant part of the loss of power in the bar, and would thus represent the coefficient of the first power of the temperature, while an equivalent to the coefficient of the second power might be obtained by a due adjustment of the *diameter* of the stem. It is evident that the smaller the diameter of the stem, or in other words, the longer the space occupied by the mercury depressed during a large interval of temperature, the greater would be the difference of the spaces corresponding to successive small intervals. But this arrangement would be liable to several small sources of error; first, an alteration of the suspended weight by hygrometric changes in the length of the suspension skein; secondly, a change in the bulk of the air in the bulb corresponding to barometric changes; and thirdly, a small and uncertain secular variation, from the evaporation of mercury in the cistern in which the stem would be immersed.

From these considerations, it has been deemed advisable to abandon the attempt to introduce in the compensation an equivalent for the small coefficient of t^2, and to rest satisfied with acting on a or b by means which would effect an equal change for equal intervals of temperature, and would therefore represent the coefficient of t only. Metallic expansion naturally suggested itself as the means of accomplishing this object, and b the lower interval of the skeins as the most appropriate point of application, inasmuch as the compensating apparatus would then be in close proximity with the magnet, and being isolated from the surrounding atmosphere by enclosure in the same case or cases, might be presumed to vary in temperature *slowly* and *simultaneously* with the magnet; and if the changes in temperature were not sufficiently simultaneous, the radiating capacity of the magnet might be increased by partially coating it with dead-black varnish, or diminished by gilding and burnishing it, to any required extent.

The compensating apparatus consists of a glass rod clamped at its middle point to the centre of magnet, the axes of the rod and bar being parallel: the free ends of the rod are enclosed in two zinc tubes, at the inner ends of which, where they nearly meet in the centre, and to their upper surface, two hooks are attached: two loops at the ends of the suspension skein are attached to these hooks, the skein passing over

* See a paper, No. III. by the Author in the Phil. Trans., Part I. for 1850.

a pulley at the point of suspension. Towards either end, the rod and tubes are connected by a moveable clamp, by which the two may be clamped together at any required distance from the centre. It is evident that by elevation of temperature the free ends of the zinc tubes will be approximated to each other by a quantity equal to the difference of the expansion of the lengths of zinc and glass that intervene between the sliding clamps and the free ends of the tubes, and consequently that a diminution to the same extent of the distance between the lower ends of the suspension skein will take place. The variation of the interval between the lower ends of the skein, corresponding with any given variation of temperature, may be made to bear any required ratio to the whole interval, first by a due adjustment of the upper and lower intervals of the skein, and secondly by varying the position of the sliding clamps, that is, of the acting lengths of the expanding tubes: the former may be considered as a coarse, the latter as a fine adjustment. The glass rod rests on rollers attached to the under surface of the tubes opposite to the hooks, in order that no jerking may be occasioned by the expansion or contraction of the zinc tubes. By these means the quantity b in the preceding formula may be made to vary by change of temperature, proportionably to the change of the quantity m with any required degree of exactness; so far, at least, as the variation of m is directly proportional to the variation of temperature.

In the adjustment of the instrument the following steps are necessary. First, let the temperature coefficient of the bar be determined by the method described in the Paper No. III. previously mentioned; secondly, let the lower interval of the skeins be taken, such that the ratio of the difference of linear expansion of the whole length of the tubes and glass rod (which for convenience is made the same length as the bar), between 32° F. and any given higher temperature, say 92° F., to the distance between the threads, may be considerably less than the whole correction for that interval of temperature; ample scope will thus be allowed for determining by experiment the requisite amount of compensation, as is the case in the adjustment of chronometers.

The pulley over which the skein passes at the upper point of suspension being made of brass, there will be an increase of the upper interval between the threads, with increase of temperature: in order to compensate for this, it will be necessary to take the acting lengths of the compensator somewhat greater than the calculated length, expressing the value of the temperature coefficient.

The instrument having been thus approximately adjusted, the magnet and its appurtenances are now to be enclosed in a rectangular jacketed zinc box. The water in the jacket may be raised to any required temperature, and the temperature maintained nearly constant for any required period, by heating a pipe connecting the inlet and outlet of the jacket by a jet or jets of gas.

A uniformity of the temperature of the jacket is obtained by the introduction of suitable diaphragms to ensure a complete circulation throughout its entire extent. A

photographic register of the variations of the instrument is now to be obtained by the means described in the two first papers of this series*, the temperature of a thermometer enclosed in the zinc box, and placed as near as possible to the magnet being simultaneously recorded at convenient intervals of time, corresponding to temperature changes of not more than two degrees. If the change of temperature is slow, that is, if a variation of 40° Fahr. is spread over a space of at least six or eight hours, the temperature of the mercury may fairly be assumed to be the same as that of the bar. If, in order to avoid the consumption of time, or for any other reason, it should be found desirable to observe during more rapid changes of temperature, some such means as those before mentioned must be employed to ensure a uniformity of radiating capacity in the magnet, the compensator, and the thermometer.

A photographic register of the bifilar variations during known changes of temperature having been thus obtained, should now be compared with that of another bifilar instrument of which the temperature has undergone comparatively little variation, as will be the case under ordinary circumstances: a comparison of these will readily show whether the instrument under trial is over- or under-compensated, and the requisite correction may be made by shifting the sliding clamps. A few repetitions of the same process will suffice to give the requisite accuracy of correction.

It may be remarked that the most satisfactory method would be to register simultaneously a carefully compensated instrument on the same sheet of paper; the means of doing this are not however at present in existence.

Compensation of the Balanced Magnetometer.

The horizontal displacement of the centre of gravity by metallic expansion is the most palpable means of effecting a compensation for the changes of temperature in this instrument; the only question is the precise method by which the numerical value of the temperature coefficients can be most nearly represented. The plan adopted has been that of attaching a small thermometer parallel to the axis of the magnet, and as nearly as possible in the same horizontal plane with the centre of gravity of the magnet and its appendages. The statical moment of the mercury displaced from the bulb of the thermometer by any given elevation of temperature, as x° above 32° Fahr., may be represented by the same formula which expresses the temperature coefficients, namely,

$$cx+ex^2.$$

For let w be the weight of mercury contained in one degree of the tube, and let the tube be taken such that the distance from the centre of the bulb (which is presumed to be a symmetrical figure of revolution) to the point 32° Fahr. may be kc, and length of one degree $2ke$; then at any temperature $32^\circ+x^\circ$ the statical moment of the mercury displaced by a small change of temperature, dx, will be $w\,(kc+2kex)\,dx$,

* Published in the Philosophical Transactions, Part I., for 1847.

and consequently the statical moment of the mercury displaced between the temperatures of 32° and $32°+x°$ will be $(kcx+kex^2)\ w$; let now v be the weight which, placed at a unit of distance from the point of support, will represent the temperature correction for 1° above 32° FAHR.; it will only remain to obtain a bulb of such size that

$$kw=v;$$

we shall then clearly have the statical moment of the mercury displaced from the bulb by $x°$ of elevation of temperature above 32° FAHR., and transferred to the vacant portion of the tube represented by

$$(cx+ex^2)\ v,$$

and consequently a correction in weight equivalent to the temperature coefficient will have been applied to the bar.

As the change of density of the thread of mercury in the stem has not been taken into the account, it will be better to place the point of the tube which corresponds to a distance $\frac{1}{2}kc$ below the point 32° opposite the knife-edge; the error from this cause will then be quite negligible.

The value of v is to be determined experimentally by observing the displacement of the register line occasioned by a small known weight placed on the bar at a known distance from the point of support, and comparing this with the scale coefficient obtained in the usual manner.

Owing to the short period of time that has elapsed since the completion of the instruments, and the difficulty of making accurate magnetic observations in a locality subject to the constant tremors and vibrations of a London thoroughfare, the constants have not been determined with a sufficient degree of accuracy for publication; when satisfactorily determined, they will be communicated to the Society in the form of a Supplement to this paper.

Description of the Plate.

Figs. 1 and 2. Plan and elevation of the bifilar compensator, half the actual size. *a a* the magnet, *b* the clamp which attaches the glass rod to the magnet. *c c* the zinc tubes enclosing the glass rod. *d d* the adjusting clamps, consisting of two parts; the outer encircles the zinc tube, the inner passes, and nearly fills the interval, between the tube and glass rod. They are capable of sliding for adjustment when the screws are loosened; when tightened, the rod and tubes are held together. *e e* screws for adjusting the distance between the hooks *h h*; these should be withdrawn, when the clamps *d* are fixed. *o o*, fig. 2, are the ends of the clamping pieces interposed between the tubes and the rod.

Fig. 3. The brass collar to which the hook is attached seen in section, full size. The glass rod *f* rests on a roller *g*, that there may be no jerking in the expansion and contraction of the tube; *i i* two screws for fixing the collar while the scale and temperature coefficients are determined; when these are tightened the clamps *d* and screws *e* should be relaxed.

Fig. 4. Plan of the balanced magnet compensator. *k k* the magnet. *l l* the thermometer tube held in two Ys by a spring *m* tightened by a screw. *n n* the clamp that attaches the magnet to the frame in which the agate knife-edges are fixed.

Fig. 1.

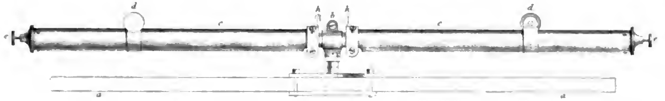


Fig. 2.

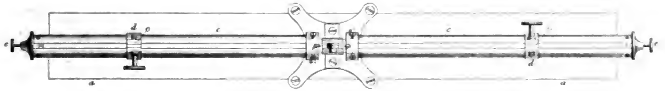
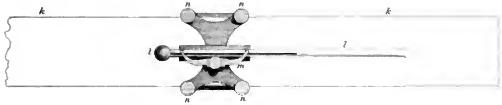


Fig. 3.

Fig. 4.



III. *Experimental Researches in Electricity.—Twenty-eighth Series.* By MICHAEL FARADAY, *Esq., D.C.L., F.R.S., Fullerian Prof. Chem. Royal Institution, Foreign Associate of the Acad. Sciences, Paris, Ord. Boruss. Pour le Mérite, Eq., Memb. Royal and Imp. Acadd. of Sciences, Petersburgh, Florence, Copenhagen, Berlin, Göttingen, Modena, Stockholm, Munich, Bruxelles, Vienna, Bologna, &c. &c.*

Received October 22,—Read November 27 and December 11, 1851.

§ 36. *On Lines of Magnetic Force; their definite character; and their distribution within a Magnet and through Space.*

3070. FROM my earliest experiments on the relation of electricity and magnetism (114. note), I have had to think and speak of lines of magnetic force as representations of the magnetic power; not merely in the points of quality and direction, but also in quantity. The necessity I was under of a more frequent use of the term in some recent researches (2149. &c.), has led me to believe that the time has arrived, when the idea conveyed by the phrase should be stated very clearly, and should also be carefully examined, that it may be ascertained how far it may be truly applied in representing magnetic conditions and phenomena; how far it may be useful in their elucidation; and, also, how far it may assist in leading the mind correctly on to further conceptions of the physical nature of the force, and the recognition of the possible effects, either new or old, which may be produced by it.

3071. A line of magnetic force may be defined as that line which is described by a very small magnetic needle, when it is so moved in either direction correspondent to its length, that the needle is constantly a tangent to the line of motion; or it is that line along which, if a transverse wire be moved in either direction, there is no tendency to the formation of any current in the wire, whilst if moved in any other direction there is such a tendency; or it is that line which coincides with the direction of the magnecrystallic axis of a crystal of bismuth, which is carried in either direction along it. The direction of these lines about and amongst magnets and electric currents, is easily represented and understood, in a general manner, by the ordinary use of iron filings.

3072. These lines have not merely a determinate direction, recognizable as above (3071.), but, because they are related to a polar or antithetical power, have opposite qualities or conditions in opposite directions; these qualities, which have to be distinguished and identified, are made manifest to us, either by the position of the ends of the magnetic needle, or by the direction of the current induced in the moving wire.

3073. A point equally important to the definition of these lines is, that they represent a determinate and unchanging amount of force. Though, therefore, their forms, as they exist between two or more centres or sources of magnetic power, may vary very greatly, and also the space through which they may be traced, yet the sum of power contained in any one section of a given portion of the lines is exactly equal to the sum of power in any other section of the same lines, however altered in form, or however convergent or divergent they may be at the second place. The experimental proof of this character of the lines will be given hereafter (3109. &c.).

3074. Now it appears to me that these lines may be employed with great advantage to represent the nature, condition, direction and comparative amount of the magnetic forces; and that in many cases they have, to the physical reasoner at least, a superiority over that method which represents the forces as concentrated in centres of action, such as the poles of magnets or needles; or some other methods, as, for instance, that which considers north or south magnetisms as fluids diffused over the ends or amongst the particles of a bar. No doubt, any of these methods which does not assume too much, will, with a faithful application, give true results; and so they all ought to give the same results as far as they can respectively be applied. But some may, by their very nature, be applicable to a far greater extent, and give far more varied results, than others. For just as either geometry or analysis may be employed to solve correctly a particular problem, though one has far more power and capability, generally speaking, than the other; or just as either the idea of the reflexion of images, or that of the reverberation of sounds may be used to represent certain physical forces and conditions; so may the idea of the attractions and repulsions of centres, or that of the disposition of magnetic fluids, or that of lines of force, be applied in the consideration of magnetic phenomena. It is the occasional and more frequent use of the latter which I at present wish to advocate.

3075. I desire to restrict the meaning of the term *line of force*, so that it shall imply no more than the condition of the force in any given place, as to strength and direction; and not to include (at present) any idea of the nature of the physical cause of the phenomena; or to be tied up with, or in any way dependent on, such an idea. Still, there is no impropriety in endeavouring to conceive the method in which the physical forces are either excited, or exist, or are transmitted; nor, when these by experiment and comparison are ascertained in any given degree, in representing them by any method which we adopt to represent the mere forces, provided no error is thereby introduced. On the contrary, when the natural truth and the conventional representation of it most closely agree, then are we most advanced in our knowledge. The emission and the ether theories present such cases in relation to light. The idea of a fluid or of two fluids is the same for electricity; and there the further idea of a current has been raised, which indeed has such hold on the mind as occasionally to embarrass the science as respects the true character of the physical agencies, and may be doing so, even now, to a degree which we at present little suspect. The

same is the case with the idea of a magnetic fluid or fluids, or with the assumption of magnetic centres of action of which the resultants are at the poles. How the magnetic force is transferred through bodies or through space we know not; whether the result is merely action at a distance, as in the case of gravity; or by some intermediate agency, as in the cases of light, heat, the electric current, and (as I believe) static electric action. The idea of magnetic fluids, as applied by some, or of magnetic centres of action, does not include that of the latter kind of transmission, but the idea of lines of force does. Nevertheless, because a particular method of representing the forces does not include such a mode of transmission, the latter is not therefore disproved; and that method of representation which harmonizes with it may be the most true to nature. The general conclusion of philosophers seems to be, that such cases are by far the most numerous, and for my own part, considering the relation of a vacuum to the magnetic force and the general character of magnetic phenomena external to the magnet, I am more inclined to the notion that in the transmission of the force there is such an action, external to the magnet, than that the effects are merely attraction and repulsion at a distance. Such an action may be a function of the ether; for it is not at all unlikely that, if there be an ether, it should have other uses than simply the conveyance of radiations (2591. 2787.). Perhaps when we are more clearly instructed in this matter, we shall see the source of the contradictions which are supposed to exist between the results of Coulomb, Harris and other philosophers, and find that they are not contradictions in reality, but mere differences in degree, dependent upon partial or imperfect views of the phenomena and their causes.

3076. Lines of magnetic force may be recognized, either by their action on a magnetic needle, or on a conducting body moving across them. Each of these actions may be employed also to indicate, either the direction of the line, or the force exerted at any given point in it; and this they do with advantages for the one method or the other under particular circumstances. The actions are however very different in their nature. The needle shows its results by attractions and repulsions; the moving conductor or wire shows it by the production of a current of electricity. The latter is an effect entirely unlike that produced on the needle, and due to a different action of the forces; so that it gives a view and a result of properties of the lines of force, such as the attractions and repulsions of the needle could never show. For this and other reasons I propose to develope and apply the method by a moving conductor on the present occasion.

3077. The general principles of the development of an electric current in a wire moving under the influence of magnetic forces, were given on a former occasion, in the First and Second Series of these Researches (36. &c.); it will therefore be unnecessary to do more than to call attention, at this time, to the special character of its indi-

cations as compared to those of a magnetic needle, and to show how it becomes a peculiar and important addition to it, in the illustration of magnetic action.

3078. The moving wire produces its greatest effect and indication, not when passing from stronger to weaker places, or the reverse, but when moving in places of equal action, *i. e.* transversely across the lines of force (217.).

3079. It determines the direction of the polarity by an effect entirely independent of pointing or such like results of attraction or repulsion; *i. e.* by the direction of the electric current produced in it during the motion*.

3080. The principle can be applied to the examination of the forces *within* numerous solid bodies, as the metals, as well as outside in the air. It is not often embarrassed by the difference of the surrounding media, and can be used in fluids, gases or a vacuum with equal facility. Hence it can penetrate and be employed where the needle is forbidden; and in other cases where the needle might be resorted to, though greatly embarrassed by the media around it, the moving wire may be used with an immediate result (3142.).

3081. The method can even be applied with equal facility to the interior of a magnet (3116.), a place utterly inaccessible to the magnetic needle.

3082. The moving wire can be made to sum up or give the resultant at once of the magnetic action at many different places, *i. e.* the action due to an area or section of the lines of force, and so supply experimental comparisons which the needle could not give, except with very great labour, and then imperfectly. Whether the wire moves directly or obliquely across the lines of force, in one direction or another, it sums up, with the same accuracy in principle, the amount of the forces represented by the lines it has crossed (3113.).

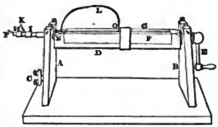
3083. So a moving wire may be accepted as a correct philosophical indication of the presence of magnetic force. Illustrations of the capabilities already referred to, will arise and be pointed out in the present paper; and though its sensibility does not as yet approach to that of the magnetic needle, still, there is no doubt that it may be very greatly increased. The diversity of its possible arrangements, and the great advantage of that diversity, is already very manifest to myself. Though both it and the needle depend for their results upon essential characters and qualities of the magnetic force, yet those which are influential, and, therefore indicated, in the one case, are very different from those which are active in the other; I mean, as far as we have been able as yet to refer directly the effects to essential characters: and this difference may, hereafter, enable the wire to a give a new insight into the nature of the magnetic force; and so it may, finally, bear upon inquiries, such as whether magnetic polarity is axial or dependent upon transverse lateral conditions; whether

* A natural standard of this polarity may be obtained, by referring to the lines of force of the earth, in the northern hemisphere, thus:—if a person with arms extended move forward in these latitudes, then the direction of the electric current, which would tend to be produced in a wire represented by the arms, would be from the right-hand through the arm and body towards the left.

the transmission of the force is after the manner of a vibration or current, or simply action at a distance; and the many other questions that arise in the minds of those who are pursuing this branch of knowledge.

3084. I will proceed to take the case of a simple bar magnet, employing it in illustration of what has been said respecting the lines of force and the moving conductor, and also for the purpose of ascertaining how these lines of force are disposed, both without and within the magnet itself, upon which they are dependent or to which they belong. For this purpose the following apparatus was employed. Let fig. 1 represent a wooden stand, of which the base is a board 17·5 inches in length, and 6 inches in breadth, and 0·8 of an inch in thickness: these dimensions will serve as a scale for the other parts. A B are two wooden uprights; D is an axis of wood having two long depressions cut into it, for the purpose of carrying the two bar magnets F and G. The wood is not cut away quite across the axis, but is left in the middle, so that the magnets are about $\frac{1}{15}$th of an inch apart. From O towards the supports A, it is removed, however, as low down as the axis of revolution, so as to form a notch between the two magnets when they are in their places; and by further removal of the wood, this notch is continued on to the end of the axis at P. This notch, or opening, is intended to receive a wire, which can be carried down the axis of rotation, and then passing out between the two magnets, anywhere between O and N, can be returned towards the end P on the outside. The magnets are so placed, that the central line of their compound system coincides with the axis of rotation; E being a handle by which rotation, when required, is given. H and I are two copper rings, slipping tightly on to the axis, by which communication is to be made between a wire adjusted so as to revolve with the magnets, and the fixed ends of wires proceeding from a galvanometer. Thus, let L represent a covered wire; which being led along the bottom of the notch in the axis of the apparatus, and passing out at the equatorial parts of the magnets, returns into the notch again near N, and terminates at P. When the form of the wire loop is determined and given to it, then a little piece of soft wood is placed between the wires in the notch at K, of such thickness, that when the ring I is put into its place, it shall press upon the upper wire, the piece of wood, and the lower wire, and keep all tightly fixed together, and at the same time leave the two wires effectually separated. The second ring, H, is then put into its place on the axis, and the introduction of a small wedge of wood, at the end of the axis, serves to press the end P into close and perfect contact with the ring H, and keep all in order. So the wire is free to revolve with the magnets, and the rings H and I are its virtual terminations.

Fig. 1.

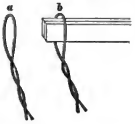


Two clips, as at C, hold the ends of the galvanometer wire (also of copper); and the latter are made to press against the rings by their elasticity, and give an effectual contact bearing, which generates no current, either by difference of nature or by friction, during the revolution of the axis.

3085. The two magnets are bars, each 12 inches long, 1 inch broad, and 0·4 of an inch thick. They weigh each 19 ounces, and are of such a strength as to lift each other end to end and no more. When the two are adjusted in their place, it is with the similar poles together, so that they shall act as one magnet, with a division down the middle: they are retained in their place by tying, or, at times, by a ring of copper which slips tightly over them and the axis.

3086. The galvanometer is a very delicate instrument made by Ruhmkorff (2651.). It was placed about 6 feet from the magnet apparatus, and was not affected by any revolution of the latter. The wires, connecting it with the magnets, were of copper, 0·04 of an inch in diameter, and in their whole length about 25 feet. The length of the wire in the galvanometer I do not know; its diameter was $\frac{1}{135}$th of an inch. The condition of the galvanometer, wires, and magnets, was such, that when the bend of the wires was formed into a loop, and that carried once over the pole of the united magnets, as from *a* to *b*, fig. 2, the galvanometer needle was deflected two degrees or more. The vibration of the needle was slow, and it was easy therefore to reiterate this action five or six times, or oftener, breaking and making contact with the galvanometer at right intervals, so as to combine the effect of like induced currents; and then a deflection of 10° or 15° on either side of zero could be readily obtained. The arrangement, therefore, was sufficiently sensible for first experiments; and though the resistance opposed by the thin long galvanometer wire to feeble currents was considerable, yet it would always be the same, and would not interfere with results, where the final effect was equal to 0°, nor in those where the consequences were shown, not by absolute measurement, but by comparative differences.

Fig. 2.

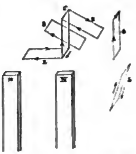


3087. The first practical result produced by the apparatus described, in respect of magneto-electric induction generally, is, that a piece of metal or conducting matter which moves across lines of magnetic force, has, or tends to have, a current of electricity produced in it. A more restricted and precise expression of the full effect is the following. If a continuous circuit of conducting matter be traced out, or conceived of, either in a solid or fluid mass of metal or conducting matter, or in wires or bars of metal arranged in non-conducting matter or space; which being moved, crosses lines of magnetic force, or being still, is by the translation of a magnet crossed by such lines of force; and further, if, by inequality of angular motion, or by contrary motion of different parts of the circuit, or by inequality of the motion in the same direction, one part crosses either more or fewer lines than the other; then a current will exist round it, due to the differential relation of the two or more inter-

secting parts during the time of the motion: the direction of which current will be determined (with lines having a given direction of polarity) by the direction of the intersection, combined with the relative amount of the intersection in the two or more efficient and determining (or intersecting) parts of the circuit.

3088. Thus, if fig. 3 represent a magnetic pole N, and over it a circuit, formed of metal, which may be of any shape, and which is at first in the position *c*; then if that circuit be moved in one direction into the position 1; or in the contrary direction into position 2; or by a double direction of motion into position 3; or by translation into position 4; or into position 5; or any position between the first and these or any resembling them; or, if the first position *c* being retained, the pole move to, or towards, the position *n*; then, an electric current will be produced in the circuit, having in every case the same direction, being that which is marked in the figure by arrows. Reverse motions will give currents in the reverse direction (256. &c.).

Fig. 3.



3089. The general principles of the production of electrical currents by magnetic induction have been formerly given (27. &c.)*, and the law of the direction of the current in relation to the lines of force, stated (114, 3079 note). But the full meaning of the above description can only be appreciated hereafter, when the experimental results, which supply a larger knowledge of the relations of the current to the *lines of force*, have been described.

3090. When *lines of force* are spoken of as crossing a conducting circuit (3087.), it must be considered as effected by the *translation* of a magnet. No mere rotation of a bar magnet on its axis, produces any induction effect on circuits exterior to it; for then, the conditions above described (3088.) are not fulfilled. The system of power about the magnet must not be considered as revolving with the magnet, any more than the rays of light which emanate from the sun are supposed to revolve with the sun. The magnet may even, in certain cases (3097.), be considered as revolving amongst its own forces, and producing a full electric effect, sensible at the galvanometer.

3091. In the first instance the wire was carried down the axis of the magnet to the middle distance, then led out at the equatorial part, and returned on the outside; fig. 4 will represent such a disposition. Supposing the magnet and wire to revolve once, it is evident that the wire *a* may be considered as passing in at the axis of the magnet, and returning from *b* across the lines of force external to the magnet, to the axis again at *c*; and that in one revolution,

Fig. 4.

* Philosophical Transactions, 1832, page 131, &c.

the wire from *b* to *c* has intersected once, all the lines of force emanating from the N end of the magnet. In other words, whatever course the wire may take from *b* to *c*, the whole system of lines belonging to the magnet has been *once* crossed by the wire. In order to have a correct notion of the relation of the result, we will suppose a person standing at the handle E, fig. 1 (3084.), and looking along the magnets, the magnets being fixed, and the wire loop from *b* to *c* turned over toward the left-hand into a horizontal plane; then, if that loop be moved over towards the right-hand, the magnet remaining stationary, it will be equivalent to a *direct* revolution (according to the hands of a watch or clock) of 180°, and will produce a feeble current in a given direction at the galvanometer. If it be carried back 180° in the reverse direction, it will produce a corresponding current in the reverse direction to the former. If the wire be held in a vertical, or any other plane, so that it may be considered as fixed, and the magnet be rotated through half a revolution, it will also produce a current; and if rotated in the contrary direction, will produce a contrary current; but as to the *direction* of the currents, that produced by the *direct* revolution of the wire is the same as that produced by the *reverse* revolution of the magnet; and that produced by the *reverse* revolution of the wire is the same as that produced by the *direct* revolution of the magnet. A more precise reference of the direction of the current to the particular pole employed, and the direction of the revolution of the wire or magnet, is not at present necessary; but if required is obtained at once by reference to fig. 3 (3088.), or to the general law (114. 3079. note).

3092. The magnet and loop being rotated together in either direction, no trace of an electric current was produced. In this case the effect, if any, could be greatly exalted, because the rotation could be continued for 10, 20, or any number of revolutions without derangement, and it was easy to make thirty revolutions or more within the time of the swing of the galvanometer needle in one direction. It was also easy, if any effect were produced, to accumulate it upon the galvanometer by reversing the rotation at the due time. But no amount of revolution of the magnet and wire together could produce any effect.

3093. The loop was then taken out of the axis of the magnet, but attached to it by a piece of pasteboard, so that all should be fixed together and revolve with the same angular velocity, fig. 5; but whatever the shape or disposition of the loop, whether large or small, near or distant, open or shut, in one plane, or contorted into various planes; whatever the shape or condition, or place, provided it moved altogether with the magnet, no current was produced.

Fig. 5.

3094. Furthermore, when the loop was out of the magnets, and by expedients of arrangement, was retained immoveable, whilst the magnet revolved, no amount of rotation of the magnet (unaccompanied by translation of place) produced any degree of current through the loop.

3095. The loop of wire was then made of two parts; the portion *c*, fig. 6, on the

outside of the magnet, was fixed at *b*, and the portion *a*, being a separate piece, was carried along the axis until it came in contact with the former at *d*; the revolution of one part was thus permitted either with or without the other, yet preserving always metallic contact and a complete circuit for the induced current. In this case, when the external wire and the magnet were fixed, no current was produced by any amount of revolution of the wire *a* on its axis. Neither was any current produced when the magnet and wire, *c d*, were revolved together, whether the wire *a* revolved with them or not. When the magnet was revolved without the external part of *c d*, or the latter revolved without the magnet, then currents were produced as before (3091.).

Fig. 6.

3096. The magnet was now included in the circuit, in the following manner. The wire *a*, fig. 7, was placed in metallic contact on both sides of the interval between the magnets at N (or the pole), and the part *c* was brought into contact with the centre at *d*. The result was in everything the same as when the wire *a* was continued up to *d*, *i. e.* no amount of revolution of the magnet and part *c* together could produce any electric current. When *c* was made to terminate at *e* or the equatorial part of the magnet, the result was precisely the same. Also, when *c* terminated at *e*, the part *a* of the wire was continued to the centre at *d*, and there the contact perfected, but the result was still the same. No difference, therefore, was produced, by the use between N and *d*, or *d* and *e*, of the parts of the magnet in place of an insulated copper wire, for the completion of the circuit in which the induced current was to travel. No rotation of the part *a* produced any effect, wherever it was made to terminate.

Fig. 7.

3097. In order to obtain the power of rotating the magnet without the external part of the wire, a copper ring was fixed round, and in contact with it at the equatorial part, and the wire *c*, fig. 8, made to bear by spring pressure against this ring, and also against the ring H on the axis, fig. 1 (3084.); the circuit was examined, and found complete. Now when the wire *c e* was fixed and the magnet rotated, a current was produced, and that to the same amount for the same number of revolutions, whether the part of the wire *a* terminated at N, or was continued on to the centre of the magnet, or was insulated from the magnet and continued up to the copper ring *e*. When the wire, by expedients, which though rough were sufficient, was made to revolve whilst the magnet was still, currents in the contrary direction were produced, in accordance with the effect before described (3091.); and the results when the wire and magnet rotated together (3092.), show that these are in amount exactly equal to the former. When the inner and the outer wires were both motionless, and the magnet only revolved, a current in the full proportion was produced, and that, whether the axial wire *a* made contact at the pole of the magnet or in the centre.

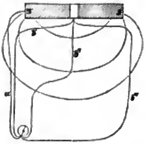
Fig. 8.

3098. Another arrangement of the magnet and wires was of the following kind. A radial insulated wire was fixed in the middle of the magnets, from the centre *d*, fig. 9, to the circumference *b*, being connected there with the equatorial ring (3097.); an axial wire touched this radial wire at the centre and passed out at the pole; the external part of the circuit, pressing on the ring at the equator, proceeded on the outside over the pole to form the communication as before. In the case where the magnet was revolved without the axial and the external wire, the full and proper current was produced; the small wire, *d b*, being, however, the only part in which this current could be generated by the motion; for it replaced, under these circumstances, the body of the magnet employed on the former occasion (3097.).

Fig. 9.

3099. The external part of the wire, instead of being carried back over that pole of the magnet at which the axial wire entered, was continued away over the other pole, and so round by a long circuit to the galvanometer; still the revolution of the magnet, under any of the described circumstances, produced exactly the same results as before. It will be evident by inspection of fig. 10, that, however the wires are carried away, the general result will, according to the assumed principles of action, be the same; for if *a* be the axial wire, and b', b'', b''' the equatorial wire, represented in three different positions, whatever magnetic lines of force pass across the latter wire in one position, will also pass across it in the other, or in any other position which can be given to it. The distance of the wire at the place of intersection with the lines of force, has been shown, by the experiments (3093.), to be unimportant.

Fig. 10.



3100. Whilst considering the condition of the forces of a magnet, it may be admitted, that the two magnets used in the experimental investigations described, act truly as one central magnet. We have only to conceive smaller similar magnets to be introduced to fill up the narrow space not occupied by the wire, and then the complete magnet would be realized:—or it may be viewed as a magnet once perfect, which has had certain parts removed; and we know that neither of these changes would disturb the general disposition of the forces. In and around the bar magnet the forces are distributed in the simplest and most regular manner. Supposing the bar removed from other magnetic influences, then its power must be considered as extending to any distance, according to the recognized law; but, adopting the representative idea of *lines of force* (3074.), any wire or line proceeding from a point in the magnetic equator of the bar, over one of the poles, so as to pass through the magnetic axis, and so on to a point on the opposite side of the magnetic equator, must intersect *all* the lines in the plane through which it passes, whether its course be over the one pole or the other. So also a wire proceeding from the end of the magnet at the magnetic axis, to a point at the magnetic equator, must intersect

curves equal to half those of a great plane, however small or great the length of the wire may be; and though by its tortuous course it may pass out of one plane into another on its way to the equator.

3101. Further, if such a wire as that last described be revolved once round the end of the magnet to which it is related, a slipping contact at the equator being permitted for the purpose, it will intersect *all* the lines of force during the revolution; and that, whether the polar contact is absolutely coincident with the magnetic axis, or is anywhere else at the end of the bar, provided it remain for the time unchanged. All this is true, though the magnet may be subject, by induction at a distance, to other magnets or bodies, and may be exerting part of its force on them, so as to make the distribution of its power very irregular as compared to the case of the independent bar (3084.), or may have an irregular or contorted shape, even up to the horseshoe form. It is evident, indeed, that if a wire have one of its ends applied to *any* point on the surface of a magnet, and the other end to a point in the magnetic equator, and the latter be slipped once round the magnetic equator, and the loop of wire be made to pass over either pole, so as at last to resume its first position, it will in the course of its journey have intersected *once* every line of force belonging to the magnet.

3102. A wire from pole to pole which passes close to the equator, of course intersects half the external lines of force in a great plane, twice, in opposite directions as regards the polarity; and, therefore, when revolved round the magnet, has no electric current induced in it. If it do not touch at the equator, still, whatever lines it intersects, are twice intersected, and so the same equilibrium is preserved. If the magnet rotate under the wire, it acts the part of the central rotating wire already referred to (3095.); or if any course for the electric current other than a right line is assumed in it, that course is subject to the law of neutrality above stated, as will be seen by reference to the internal condition of the magnet itself (3117.). Hence the reason why no currents are produced, under any circumstances of motion, by the application of such conducting circuits to the magnet. I may further observe, in reference to the intersection of the lines of force, that if a wire ring, a little larger in diameter than the magnet, be held edgeways at one of the poles, so that the lines of force there shall be in its plane, and be then turned 90° and carried over the pole to the equator (3088.), it will intersect *once* all the lines of the magnet, except the very few which will remain unintersected at the equator.

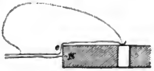
3103. Whilst endeavouring to establish experimentally the definite amount of the power represented by the *lines of force*, it is necessary to take certain precautions, or the results will be in error. For instance, ten revolutions of the wire about the magnet, or of the magnet within the fixed wire (3097.), ought to give a constant deflection at the galvanometer, and yet without any change in the position of the wire the results may at different times differ very much from each other; being at one time 9°, and at another only 4° or 5°. I found this to be due to difference of velocity within certain limits, and to be explained and guarded against as follows.

3104. If a wire move across lines of force slowly, a feeble electric current is produced in it, continuing for the time of the motion; if it move across the same lines quickly, a stronger current is produced for a shorter time. The effect of the current which deflects a galvanometer needle, is opposed by the action of the earth, which tends to return the needle to zero. A continuous weak current, therefore, cannot deflect it so far as a continuous stronger current. If the currents be limited in duration, the same effect will occur unless the time of the swing of the needle to one side be not considerably more than the time of either of the currents. If the time of the needle-swing be ten, and the time of ten quick rotations be six, then all the effect of the induced current is exerted in swinging the needle; but if the time of ten slow rotations be twelve or fifteen, then part of the current produced is not recognized by the extent of the vibration, but only by its holding the needle out awhile, at the extremity of a smaller arc of declination. Therefore, when quick and slow velocity was compared, and, indeed, in every case of comparative rotations of the wire and magnet, only that number of rotations was taken which could be well included within the time of the needle's journey to one side; when the needle, therefore, was seen to travel on to its extreme distance after the rotation and the inducing current had ceased. If the needle began to return the instant the motion was over, such an experiment was rejected for purposes of comparison. When these precautions were attended to, and velocities of revolution taken, which occupied times from one-third to three-fourths of that required for the swing of the needle, then the same number of revolutions (ten) gave the same amount of deflection, namely 9°·5, with my apparatus, though the time of revolution varied as 1 : 2, or even in a higher degree.

3105. Another cause of difference produced by varying velocity, is the diminution of the action of the current on the needle, as the angle which the latter forms with the convolutions of the coil increases. Hence a constant current produces more effect on the deflection of the needle for the first moments of time than afterwards. This effect, however, was scarcely sensible for swinging deflections of 9° or 10°, produced by currents which were over before the needle had moved through 4° or 5°.

3106. It has already been shown, that it is a matter of indifference whether the wire revolve in one direction or the magnet in the other (3091.); and this is still further proved by the cases where the magnet and the wire revolve together (3092.); for then the currents which tend to form are exactly equal and opposed to each other, whatever the position of the wire may be. As the immobility of the needle is a point more easily ascertained than the extent of an arc, indicated only for a moment, and as the rotations of the magnet and wire conjointly can be made rapid and continuous, the proof in such cases is very satisfactory.

Fig. 11.



3107. Proceeding to experiment upon the effect of the *distance* of the wire *c*, fig. 11, from the magnet, the wire was made to vary, so that sometimes it was not more than 8 inches long (being of copper and 0·04 of an inch in diameter), and only half an inch from the magnet, whilst at

other times it was 6 or 8 feet long and extended to a great distance. The deflection due to ten revolutions of the magnet was observed, and the average of several observations, for each position of the wire, taken: these were very close (with the precautions before described) for the same position; and the averages for different positions agreed perfectly together, being 9°·5. I endeavoured to repeat these experiments on distance by moving the wire and preserving the magnet stationary in the manner before described (3091.); they were not so striking because time would only allow of smaller deflections being obtained (3104.), but the same number of journeys through an arc of 180° gave the same deflection at the galvanometer, whether the course of the wire was close to the magnet or far off; and the deflection agreed with those obtained when the magnet was rotating and the wire at rest.

3108. As to *velocity* of motion; when the magnet was rotating and the wire placed at *different distances*, then ten revolutions of the magnet produced the same deflection of the needle, whether the motion was *quicker* or *slower*, and whatever the distance of the wire, provided the precautions before described were attended to (3104.). That the same would be true if the wire were moving and the magnet still, is shown by this; that whatever the velocity with which the wire and magnet revolve together, and whatever their distance apart, they exactly neutralize and equal each other (3096.).

3109. From these results the following conclusions may be drawn. The *amount* of magnetic force, as shown by its effect in evolving electric currents, is determinate for the same lines of force, whatever the distance of the point or plane, at which their power is exerted, is from the magnet. Or it is the same in any two, or more, sections of the same lines of force, whatever their form or their distance from the seat of the power may be. This is shown by the results with the magnet and the wire, when both are in the circuit (3108.); and also by the wire loop revolving with the magnet (3092.); where the tendencies of currents to form in the two parts oppose and exactly neutralize or compensate each other.

3110. In the latter case very varying sections outside of the magnet may be compared to each other; thus, the wire may be conceived of as passing (or be actually formed so as to intersect) lines of force near the pole, and then, being continued *along* a line of force until over the equator, may be directed so as to intersect the same lines of force in the contrary direction, and then return along a line of force to its commencement; and so two surface sections may be compared. It is manifest that every loop forming a complete circuit, which is in a great plane passing through the axis of the magnet, must have precisely the same lines of force passing into and passing out of it, though they may, so to say, be expanded in one part and compressed in another; or (speaking in the language of radiation) be more intense in one part and less intense in the other. It is also as manifest, that, if the loop be not in one plane, still, on making one complete revolution, either with or without the magnet, it will have intersected in its two opposite parts an exactly equal amount of lines of force.

Hence the comparison of any one section of a given amount of lines of force with any other section is rendered, experimentally, very extensive.

3111. Such results prove, that, under the circumstances, there is no loss, or destruction, or evanescence, or latent state of the magnetic power by distance.

3112. Also that convergence or divergence of the lines of force causes no difference in their amount.

3113. That obliquity of intersection causes no difference. It is easy so to shape the loop (3110.), that it shall intersect the lines of force directly across at both places of intersection, or directly at one and obliquely at the other, or obliquely in any degree at both; and yet the result is always the same (3093.).

3114. It is also evident, by the results of the rotation of the wire and magnet (3097. 3106.), that when a wire is moving amongst equal lines (or in a field of equal magnetic force), and with an uniform motion, then the current of electricity produced is proportionate to the *time*; and also to the *velocity* of motion.

3115. They also prove, generally, that the quantity of electricity thrown into a current is directly as the amount of curves intersected.

3116. In addition to these results, this method of investigation gives much insight into the internal condition of the magnet, and the manner in which the lines of force (which represent truly all that we are acquainted with of the peculiar action of the magnet) either terminate at its exterior, or at any assumed points, to be called poles; or are continued and disposed of within. For this purpose, let us consider the external loop (3093.) of fig. 5. When revolving with the magnet no current is produced, because the lines of force which are intersected on the one part, are again intersected in an opposing direction on the other (3110.). But if one part of the loop be taken down the axis of the magnet, and the wire then pass out at the equator (3091.), still the same absence of effect is produced; and yet it is evident that, external to the magnet, every part of the wire passes through lines of force, which conspire together to produce a current; for all the external lines of force are then intersected by that wire in one revolution (3101.). We must therefore look to the part of the wire *within* the magnet, for a power equal to that capable of being exerted externally, and we find it in that small portion which represents a radius at the central and equatorial parts. When, in fact, the axial part of the wire was rotated it produced no effect (3095.); when the axial, the inner radial, and the external parts were revolved together, they produced no effect; when the external wire alone was revolved, *directly*, it produced a current (3091.); and when the internal radius wire alone (being insulated from the magnet) revolved, *directly*, it also produced a current (3095. 3098.) in the contrary direction to the former; and the two were exactly equal in power; for when both portions of the wire moved together *directly*, they perfectly compensated each other (3095.). This radius wire may be replaced by the magnet itself (3096. 3118.).

3117. So, by this test there exist lines of force within the magnet, of the same *nature* as those without. What is more, they are exactly equal in *amount* to those without. They have a relation in *direction* to those without; and in fact are continuations of them, absolutely unchanged in their nature, so far as the experimental test can be applied to them. Every line of force therefore, at whatever distance it may be taken from the magnet, must be considered as a closed circuit, passing in some part of its course through the magnet, and having an equal amount of force in every part of its course.

3118. When the axial part of the wire is dismissed and the magnet employed in its place, so as to be included in the circuit, it is easy to see how it acts the part of the conductor. For suppose the wire itself to be continued from N to *b*, fig. 12, by any of the three paths indicated by dotted lines, the effect is the same in all the cases, both by experiment (3093.) and by principle (3100.). For whatever the form of the path, it will in one revolution intersect the same amount of lines of force within the magnet, as are intersected in the contrary direction by the part of the wire outside the magnet; and when the magnet is employed to complete the circuit in place of the internal wire, then its substance produces precisely the same result; for direction and every other circumstance which influences the result remains the same: one conductor has simply been substituted for another. The great mass of the magnet might be supposed able to do something more than the thin wire, but the reason why it only equals it in effect will be seen hereafter (3137.). And as the axial wire, in revolving, does nothing but conduct (3095.), all the effect being produced by that part which represents a radius between the axis and the equator (3098.); so the magnet, revolving as a cylinder, is as to its mass like the revolving wire; with the exception of so much of it as represents a radius connecting together the two points at the pole or axis and at the equator, where communication with the wire is completed. As was shown long ago (220.), if a cylinder magnet be revolved, and the ends of the galvanometer wires *a c* be applied to the extremities of its axis, no current is evolved; but if *a* be applied to one end, it matters not which, and *c* be applied at the equator or any other part on the surface of the cylinder, a current always in the same direction for the same rotation will be produced.

Fig. 12.

3119. Further to prove these points, the magnets were cut in half through the equatorial plane, and then, either a disc of copper placed there, or a wire radius only, or the magnets brought together again: and these three arrangements were used in succession to complete the circuit from the axial wire (3095.) to a fixed wire at the surface of the equator. Whichever was employed the current produced was the same, both in direction and amount. If the cylinder magnet above described (3118.) be terminated at the ends by attached discs of silver or copper, the wires applied to their surfaces, as they revolve with the magnet, produce precisely the same currents as to direction as if applied to the surface of the magnet itself (218. 219.).

3120. In this striking disposition of the forces of a magnet, as exhibited by the moving wire, it exactly resembles an electro-magnetic helix, both as to the direction of the lines of force in closed circuits, and in their equal sum within and without. No doubt, the magnet is the most heterogeneous in its nature, being composed, as we are well-aware, of parts which differ much in the degree of their magnetic development; so much so, that some of the internal portions appear frequently to act as keepers or submagnets to the parts which are further from the centre, and so, for the time, to form complete circuits, or something equivalent to them, within. But these make no part of the resultant of force externally, and it is only that resultant which is sensible to us in any way; either by the action on a needle, or other magnets, or soft iron, or the moving wire. So also the power which is manifest *within* the magnet by its effect on the moving mass, is still only that same resultant; being equal to, and by polarity and other qualities, identical with it. No doubt, there are cases, as upon the approach of a keeper to the poles, or the approximation of other magnets, either in favourable or adverse positions, when more external force is developed, or it may be a portion apparently thrown inwards and so the external force diminished. But in these cases, that which remains externally existent, corresponds precisely to that which is the resultant internally; for when either the same, or contrary poles, of a powerful horseshoe magnet were placed within an inch and a half of the poles of the bar magnets, prepared to rotate with the attached wires (3092.), as before described, still, upon their revolution, not the slightest action at the galvanometer was perceived; the forces within the magnet and those without perfectly compensating each other.

3121. The definite character of the forces of an invariable magnet, at whatever distance they are observed from the magnet, has been already insisted upon (3109.). How much more strikingly does that point come forth now, that, being able to observe within the magnet, we find the same definite character there; every section of the forces, whether within or without the magnet, being exactly of the same amount! The power of a magnet may therefore be easily represented by the effects of *any* section of its lines of force; and as the currents induced by two different magnets may easily be conducted through one wire, or be, in other ways, compared to each other, so facilities may thus arise for the establishment of a standard amongst magnets.

3122. On the other hand, the use of the idea of *lines of force*, which I recommend, to represent the true and real magnetic forces, makes it very desirable that we should find a unit of such force, if it can be attainable, by any experimental arrangement, just as one desires to have a unit for rays of light or heat. It does not seem to me improbable that further research will supply the means of establishing a standard of this kind. In the mean time, for the enlargement of the utility of the idea in relation to the magnetic force, and to indicate its conditions graphically, lines may be employed as representing these units in any given case. I have so employed them in former series of these Researches (2807. 2821. 2831. 2874. &c.), where the direction of the *line of force* is shown at once, and the relative amount of force, or of lines of

force in a given space, indicated by their concentration or separation, *i. e.* by their number in that space. Such a use of unit lines involves, I believe, no error either in the direction of the polarity or in the amount of force indicated at any given spot included in the diagrams.

3123. The currents produced in wires, when they cross lines of magnetic force, are so feeble in intensity (though abundant enough in quantity, as many results show), that a fine wire galvanometer must of necessity offer great obstruction to their passage. Therefore, before entering upon further experimental inquiries, I had another galvanometer constructed, in which the needles belonging to that made by Ruhmkorff were employed, but the coil was replaced by a single convolution of very stout wire. The wire was of copper, 0·2 of an inch in diameter. It passed horizontally under the lower needle, then, as nearly as might be, between that and the upper needle, over the upper, and then again between that and the lower needle, fig. 13, and was afterwards attached to the stand, and continued for 19 or 20 feet outside of the glass cover. Such a wire had abundant conducting power; and though it passed but once round each needle, gave a deflection many times greater than that belonging to the former galvanometer. Thus when the ends of the 19 feet of wire were soldered together, so as to form one loop or circuit, the passage of the wire once between the poles of a horseshoe magnet (3124.), caused a deflection, or rather swing of the needle of above 90°. I have had a more perfect instrument, of the same kind, constructed, in which the conducting coil was cut out of plates of copper, so as to form a square band 0·2 of an inch in thickness, which passed twice round the vibration plane of each needle, as represented, fig. 14. The length of metal around the needles was 24 inches, and the galvanometer was very sensitive, but the experiments to be described were made chiefly with the former instrument.

Fig. 13.

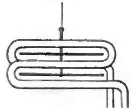
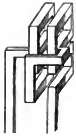


Fig. 14.



3124. It was necessary, first, to ascertain the effect of certain circumstances upon this simple galvanometer, as to their modification of its indications. The magnet to be used was a compound horseshoe instrument, weighing 16 lbs., and able to support 40 lbs. by the keeper or submagnet. It is some years since it was magnetized, and it is therefore, probably, in a nearly constant state as to power. The poles have the form delineated, fig. 15. Their distance apart is 1·375 inch, and the distance downwards, from their summit to the bottom or equator of the magnet, is 8·5 inches. The galvanometer stood in the prolongation of the magnetic axis, *i. e.* the line from pole to pole, and whether it were 6 or only 3 feet distant, was

Fig. 15.

hardly at all affected in the time of its vibration, being adjusted so nearly astatic as to require about ten seconds to swing to the right or to the left.

3125. On passing the wire across the magnetic field, as just described (3123.), but with different velocities, effects different in degree were obtained at the galvanometer, for the reasons formerly given (3104. 3106.). The quickest velocity gave the greatest result, equal at times to 140°, whilst a very slow motion gave only 30° or 40°. Still with moderately quick velocities the effects were nearly alike, and by operating with the same velocity, and taking the average of several observations, a very uniform result could be obtained.

3126. On cutting the wire across, and then putting the ends together in various ways, it was found that great care was requisite in making contact, in this or in similar cases. Thus, to press the ends lightly together was not sufficient; they required to be well and recently cleaned and pressed closely into contact. Junctions effected by soldering or dipping into cups of mercury were still better, when made with care, and were employed at the galvanometer and elsewhere as often as possible.

3127. To ascertain generally the obstruction caused by the interposition of thin wires, 28 inches of copper wire, 0·045 of an inch in diameter, were introduced into the circuit at a part away from the magnet, with excellent junctions. The oscillation or swing, which before was 140° or more, was now reduced to 40°. On taking out the wire and replacing it by another, also of copper, but only 19·5 inches in length, and 0·0135 in diameter, the deflection was reduced to 7° or 8°.

3128. For a rough comparison of the power of this magnet and the former bar magnets (3085.), by the present galvanometer, the thick wire was bent into a loop (3086.), and the two bar magnets, with like ends together, passed quickly through it up to the equatorial part; the deflection was about 30°. Such a passage intersected nearly all the lines of force of the bar magnets. A similar motion of the magnets close to, but outside of, the loop, produced no effect at the galvanometer.

3129. In respect of the alteration of the lines of force, either in position or in total amount, by bringing the poles of the horseshoe magnet (3124.) much nearer together, the following experiments were made. The distance between the poles is 1·375 inch; by placing a cube of soft iron, 0·8 of an inch in the side, within this space, it was diminished to 0·575, and thus, virtually, the distance apart much lessened, and, as was afterwards shown experimentally (3130.), the external power of the magnet concentrated there. Then, whilst the cube was in place, the thick wire of 0·2 of an inch in diameter, was arranged so as to pass across the magnetic axis or place of strongest action, and fixed; after which the iron cube was alternately removed and again restored, and the effects observed. Feeble electric currents were produced at these times; but whether the cube was put into its place from below, or above, or the sides, the current produced was always in the same direction; and when it was removed the current produced was in the reverse direction. If the cube were carried up to,

by, and away from, the magnetic axis in one motion, then there was no effect at the galvanometer. On the other hand, when the wire was carried across the magnetic field as described (3123.), so as to intersect all the lines of force in one movement, and sum up their power at the galvanometer, then there was no difference in the result, whether the iron cube was in its place or not; showing, as far as this apparatus could indicate, that the sum of power in the section of all the lines of force external to the magnet, was the same under both circumstances, though the distribution of it was different.

3130. The very action produced by the cube, when in and out of place (3129.), upon the forces which affected the stationary wire, was a proof of the difference of distribution at different times.

3131. A block of bismuth, employed in place of the iron cube, had no sensible effect upon the wire whether it were still or moving.

3132. This galvanometer was first employed for a repetition of all the former experiments with the bar magnets (3091. &c.). The results were absolutely the same, except that the amount of the deviation produced, when deviation was a result, was larger than in the former cases.

3133. For the comparison of different thicknesses of the same metal, I took copper wires in lengths of 10·5 inches, and different diameters, and bending them into loops of a form and size such as would admit them to pass with facility over a pole of the horseshoe magnet, soldered them to the ends of two conducting rods, made of copper wire 0·2 of an inch in diameter and 35 inches in length each, which were fixed on opposite sides of a narrow slip of wood. The whole arrangement is seen in fig. 16; the terminations *a b* dip into the mercurial cups of the galvanometer, the parts at *c* are brought so close together as to touch, except for the intervention of a piece of card, and thus the parts from *c* to *a b* are thrown out of action, except as mere conductors, whilst the loop, being made to descend over one magnetic pole, intersects very nearly the whole of the magnetic curves, and always the same proportion.

Fig. 16.

3134. The former magnet was too powerful for comparative experiments, therefore a smaller one was employed, consisting of five plates, weighing 8 lbs., and able to carry 21 lbs. easily at the keeper. The poles were 1·2 inch apart and an inch thick each, in the direction of the magnetic axis. If less magnetic power were required, an adjustment was easily made, by applying the keeper to the side upon both limbs, the magnetic communication being effected either nearer to the poles, or nearer to the equator or bend, as less or more power was required. The descent of the loop between the poles is then best regulated by causing the conductor wires to bear ultimately against a stopping-block.

3135. The effect of a quick and a slow motion was found to be the same as before (3104. 3105.). Such velocities as the hand could impart were very effectual, and gave results of very considerable uniformity when quick motions were employed.

3136. Three different loops were compared together, consisting of copper wire, the diameters of which were 0·2, 0·1 and 0·05 of an inch, or as 4, 2 and 1; their sectional areas or masses therefore were as 16, 4 and 1. Ten or twelve observations were made with each loop; the results were near together, and the average for each loop, being the extent of the swing declination on one side from zero, is as follows:—

Copper wire of $\frac{1}{20}$th of an inch in thickness	16°·00
Copper wire of $\frac{1}{10}$th of an inch in thickness	44°·40
Copper wire of $\frac{1}{5}$th of an inch in thickness	57°·37

Now though the thicker wires produced the largest effect, the results were evidently not at all in proportion to the masses of the wires; the smaller having greatly the advantage in that respect. On the other hand, when four of the smaller wires were placed side by side, so as to form one loop equal in mass to the second loop, it gave the same result as that loop, being of the same power.

3137. The disproportion of the *difference* of these three wires is evidently a consequence of the relative difference of the mere conducting part of the circuit. To compare accurately the effect of the lines of force on wires of different diameters moving across them, these diameters should continue to, and through the galvanometer (205.), otherwise the thin wire current has an advantage given to it in the conducting part, which the thick wire current has not. Hence the reason why a thin wire galvanometer, such as that before described (3086.), gives results which are alike, for thick or thin wire loops, or for fasciculi of few or many wires. To enlarge the comparison, I soldered on to two pairs of conductors, the dimensions of those described (3133.), two cylinders of copper, each 5·5 inches long, but one was only 0·2 of an inch thick and the other 0·7, or twelve times the mass of the first, fig. 17. They were then passed in succession between the poles of the magnet, and gave results very nearly alike.

Fig. 17.

If there was any difference, the effect was highest with the smallest cylinder; and this may very well be; for as the magnetic field was not equal in force, but most intense in the magnetic axis, so it is evident, that whilst one part of the large cylinder, in passing across, was at the axis, other parts were in places of less intense force and action, and so a return current may have existed in them, which could not occur to the same extent in a cylinder little more than a fourth of the diameter of the former, and which, at the same time, had an outlet for the currents equal to its own diameter, through the conducting wires. A similar relation of mass occurs in the case where the body of the magnet itself, in revolving, does no more than a small radial wire within it (3118.).

3138. The influence of this lateral conduction (3137.), in cases of magneto-electric conduction, must be well understood; otherwise, in the application of the principles to investigation, errors will frequently creep in. Their effect may be shown in the following instances:—a loop of four wires, 0·048 of an inch in diameter (3136.), was passed over the pole of the magnet, and produced a certain result of deflection or swing; when the wires were separated two and two, so as to be half or three-quarters of an inch apart, and when, therefore, in moving across the magnetic field, one pair went before the others, the effect was less, for the reason already given in the case of the copper cylinder (3137.). When three wires were allowed to go by together, but one taken aside a couple of inches, the effect fell very much; and when that fourth one was cut across to prevent the return current in it, the effect of the three rose at the galvanometer very greatly, almost equalling the effect of the four when together.

3139. A loop was constructed of seventy-six equal fine copper wires, each 10·5 inches long and 0·0125 of an inch in diameter, and its effect observed when more and more of the wires were cut away. As it is the comparison of the smaller numbers of wires, one with the other, that is of most value, I will give the averages of each number for several observations, in the reverse order in which they were obtained; and I introduce the results with larger numbers of wires only for the general purpose of showing how the effect passes into that with the cylinder of copper (3137.), the galvanometer conductors always being of the same length and thickness.

1 wire produced an average swing of	8°·3
2 wires produced an average swing of	15·3
3 wires produced an average swing of	21·8
4 wires produced an average swing of	27·9
5 wires produced an average swing of	34·4
6 wires produced an average swing of	37·8
8 wires produced an average swing of	50·1
12 wires produced an average swing of	65·1
16 wires produced an average swing of	80·5
26 wires produced an average swing of	118·0

36 almost swung the needle round.
46 stronger than the last.
56 swung the needle quite round.
66 a little stronger.
76 stronger: swung the needle freely round the circle.

Each time that the needle passed 180°, it was returned, that the torsion force might remain the same for every case.

3140. When the loop of four equal wires (3136.) was employed, so arranged that, in respect of the part which passed between the poles, they should be close together in one plane, it made no difference in the result, whether that plane was perpendicular

to the magnetic axis or parallel to it; *i. e.* whether the wires in moving, formed a band which moved edgeways or flat ways; the results were the same as with the four wires close together, so as to represent, as far as they could, a round or square wire.

3141. From all these results it may be concluded, that the current or amount of electricity evolved in the wire moving amongst the lines of force, is not, simply, as the space occupied by its breadth correspondent to the direction of the line of force, which has relation to the *polarity* of the power, nor by that width or dimension of it which includes the number or *amount* of the lines of force, and which, corresponding to the direction of the motion, has relation to the *equatorial* condition of the lines; but is jointly as the compound ratio of the two, or as the mass of the moving wire. The power acts just as well on the interior portions of the wire as on the exterior or superficial portions, and a central particle, surrounded on all sides by copper, is just in the same relation to the force as those which, being superficial, have air next them on one side.

3142. By immersing the poles of the magnet in different media, and then making comparative experiments with the same copper wire loop (3145.), it was found that the amount of the induced current was the same in air, water, alcohol and oil of turpentine. The experiments in air were repeated between those with the liquids, so as to give a very consistent and safe result as to the equality of action in all the cases.

3143. The effect of *variation of substance* was the next subject which seemed to me important to bring under investigation, because it has a direct relation to the amount of force exerted, or ready to be exerted, within solid bodies, at any distance from the magnet, in situations and under circumstances where it was absolutely impossible to apply the vibrations of a magnetic needle, or any other form of the effects of attractive and repulsive forces. The interior of such bodies as iron, copper, bismuth, mercury, &c., including the most paramagnetic and the most diamagnetic, seemed, in this way, open to experimental investigation, both as to the amount of lines of force traversing them under various circumstances, and also as to the direction of the lines or their polarity.

3144. In an early series of these Researches*, experiments bearing upon this subject are described (205–213.). Wires of different metals were moved across the lines of force of a magnet, and the result arrived at was, that the currents induced in these different bodies were proportional to their electro-conducting power (202. 213.).

3145. The thick wire galvanometer (3123.), with its good and short conducting communications, promised however better results, and therefore loops like those already described of copper wire (3133.), were prepared with wires of different metals, all of the same diameter, namely, 0·04 of an inch, being only $\frac{1}{25}$th of the substance of the conducting and galvanometer wire. The metals were copper, silver, iron, tin,

* Philosophical Transactions, 1832, pp. 179–182.

lead, platinum, zinc. Under these circumstances the substance concerned in the excitement of the current is made to vary, whilst the conducting part of the system is very good and remains the same. The results with these loops were as follows, being the average of from six to ten experiments for each loop :—

Copper	63°·0
Silver	61·9
Zinc	31·5
Tin	19·1
Iron	18·0
Platinum	16·9
Lead	12·1

3146. In order to dismiss, as much as possible, the obstruction caused by bad conducting power, and bring out any difference that might exist between paramagnetic and diamagnetic metals, three metals were selected, namely, tin, iron and lead in wires, as before, of 0·04 of an inch diameter; but the length was restricted to 3 inches, instead of extending to 10·5 inches, and the rest of the loop was made up of the conducting copper wire of 0·2 in diameter, as in fig. 18. Of course, the effect of the whole loop is a mixed effect, being partly due to the power represented by the lines intersected by the thick copper portion, and partly by those intersected by the three inches of special wire passing between the poles. But as the great amount of force is concentrated within a space not more than an inch and a half or 2 inches in extent (as is seen on carrying any of the loops across the magnetic axis), and as even that could be made still more concentrated by using the iron cube (3129.), and so bringing the poles virtually nearer to each other, it was hoped that the chief effect would be there, and so any peculiar difference existing between iron on the one hand and tin and lead on the other, be rendered manifest, especially as the resistance to conduction was greatly diminished by shortening the wires from 10·5 to 3 inches.

Fig. 18.

3147. The many experiments made with each metal were very close together. The average of the results for the three metals was as follows:—

Tin	37°·1
Iron	34·8
Lead	25·4

The proportions, and therefore the results, are almost identical with those obtained before (3145.).

3148. When lead and copper, arranged at the bar magnets (3084. 3085.), had been compared in former experiments with each other by the fine wire galvanometer, the results for both had been the *same*. But then the two wires used were short, and

far thicker than the wires of the galvanometer or of the conducting circuit, and were therefore limited in the production of their peculiar action, by those circumstances of mass already described (3137.). To show that that was the case, I now, with the thick wire galvanometer, employed two equal loops of copper and iron wire, 0·2 of an inch in thickness, fig. 16 (3133.), passing them equably over the pole of the small horseshoe magnet, reduced by the keeper (3134.). The results were very consistent, and the mean of them was, for

Copper	41°·7
Iron	33·7

3149. Here, therefore, the difference between copper and iron is not so great as that of 1 to 1·24; whilst when the conductors, not concerned in the excitement, were very good, and able, comparatively, to carry on to the galvanometer nearly all the effect of the excitement, it was as great as 1 to 3·5, the difference being in the latter case above tenfold what it is in the former.

3150. To raise the effect dependent upon the mass in relation to that of the conducting wires to a still higher degree, I had a cylinder of iron, 5·5 inches in length and 0·7 of an inch in diameter, soldered on to the ends of conducting wires, so as to be in all respect like that of copper before described (3137.). In this case the iron not only rose up to the copper in effect, but even surpassed it; the results being for copper 35°·66, and for iron 38°·32. Thus, under these circumstances of mass, the difference between iron and copper disappears. The apparent inferiority of copper is probably due to the lateral discharge, which before reduced the effect of a cylinder below that of a thick wire (3137.). The iron being a worse conductor in itself, and having equally good conductors in the prolongation of the circuit as when it was employed as wire, would, I think, have proportionately less lateral discharge in it than the copper.

3151. For a comparison, both as regards the particular substance and the mass, I attached a similar cylinder of bismuth to conductors. Its effect, with the same magnet and force, was 23°; a very high proportion in relation to the copper, and no doubt due to its mass. If it could have been compared as a wire, only 0·04 in diameter (3145.), it would probably have appeared almost indifferent (3127.)*.

3152. So the current of electricity excited in different substances, moving across lines of magnetic force, appears to be directly as the conducting power of the substance. It appears to have no particular reference to the magnetic character of the body, for iron comes between tin and platinum, presenting no other distinction than

* When bismuth is soldered into the circuit, it requires to be left a long time before it is used for experiments, and should then be covered up, and the loop handled with great care; otherwise thermo-currents are produced. For an hour or two after soldering it generates electrical currents, which appear at the galvanometer very irregularly, being probably due to internal molecular changes, which occur from time to time until the whole has acquired a permanent state of equilibrium.

that due to conducting power, and differing far less from them, than they do from other metals not magnetic.

3153. The amount of *lines of force* (and of the force represented by them) appears, therefore, to be equal for equal spaces occupied by tin, iron, and platina under the circumstances; for the difference in result is in no proportion to the ordinary magnetic difference, and only as the conducting power. This agrees with the conclusion before arrived at, that, for air, water, bismuth, oxygen, nitrogen, or a vacuum, the lines of force are the same in amount, except as they are more or less concentrated in the substance across which they pass (2807.), according as it is more or less competent to conduct (2797.) or transmit the magnetic force.

3154. Such a conclusion as that just arrived at, brings on the question of what is *magnetic polarity*, and how is it to be defined? For my own part, I should understand the term to mean, the opposite and antithetical actions which are manifested at the opposite ends, or the opposite sides, of a limited (or unlimited) portion of a line of force (2835.). The line of dip of the earth, or a part of it, may again be referred to as the natural case; and a free needle above or below the part, or a wire moving across it (3076. 3079.), will give the direction of the polarity. If we refer to an entirely different and artificial standard as the electro-magnetic helix, the same meaning and description will apply.

3155. If the term *polarity* have any meaning, which has reference to experimental facts and not to hypotheses only, beyond that included in the above description, I am not aware that it has ever been distinctly and clearly expressed. It may be so, for I dare not venture to say that I recollect all I have read, or even all the conclusions I myself have at different times come to. But if it neither have, nor should have, any other meaning, then the question arises, is it correctly exhibited or indicated in every case by attractions and repulsions, *i. e.* by such like mutual action of particular bodies on each other under the magnetic influence? A weak solution of protosulphate of iron, if surrounded by water, will, in the magnetic field, point axially; if in a stronger solution than itself, it will point equatorially (2357. 2366. 2422.). The same is true with stronger cases. We cannot doubt it will be true even up to iron, nickel, and cobalt, if we could render these bodies fluid in turn without altering their paramagnetic power, or if we had the command of magnets and of paramagnetic and diamagnetic media, stronger or weaker at pleasure. But in the case of the solutions, we cannot suppose that the weaker has one polarity in the stronger solution and another in the water. The lines of force across the magnetic field have the same general polarity in all the cases, and would be shown experimentally to have it, by the moving wire (3076.), though not by the attractions and repulsions.

3156. Here, therefore, we have a *difference* in the two modes of experimental indication; not merely as to the method, but as to the nature of the results, and the very

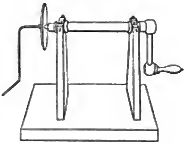
principles which are concerned in their production. Hence the value I think of the moving wire as an investigator; for it leads us into inquiries which touch upon the very nature of the magnetic force. There is no doubt that the needle gives true experimental indications; but it is not so sure that we always interpret them correctly. To assume that pointing is always the direct effect of attractive and repulsive forces acting in couples (as in the cases in question, or as in bismuth crystals), is to shut out ideas, in relation to magnetism, which are already applied in the theories of the nature of light and electricity; and the shutting out of such ideas *may be* an obstruction to the advancement of truth and a defence of wrong assumptions and error.

3157. What is the idea of polarity in a field of *equal force*? (whether it be occupied by air or by a mass of soft iron?) A magnetic needle, or an oblong piece of iron, would not show it in the air or elsewhere, except by disturbing the equal arrangement of the force and rendering it unequal; for on that the pointing of the needle or the iron, or the motions of either towards the walls of the magnetic field, if limited (2828.), would depend. A crystal of bismuth in showing this polarity by position (2464. 2839.), does it without much altering the distribution of the force, and the alteration which does take place is in the contrary direction to that effected by iron (2807.), for it expands the lines of force. It seems readily possible that a magne-crystal might exist, which, when in its stable position, should neither cause the convergence nor divergence of the lines of force within it. It need only be neutral in relation to space or any surrounding medium in that direction, and diamagnetic in its relation in the transverse direction, and the conditions would be fulfilled.

3158. But though an ordinary magnetic needle* cannot show polarity in a field of equal force, having no reference to it, and in fact ignoring such a condition of things, a moving wire makes it manifest instantly, and also shows the full amount of magnetic power to which such polarity belongs; and this it does without disturbing the distribution of the power, as far as we comprehend or understand distribution, when thinking of magnetic needles. At least such at present appears to me to be the case, from the consideration of the action of thin and thick wires (3141.) and wires of different substances (3153.).

3159. As an experimentalist, I feel bound to let experiment guide me into any train of thought which it may justify; being satisfied that experiment, like analysis, must lead to strict truth if rightly interpreted; and believing also, that it is in its nature far more suggestive of new trains of thought and new conditions of natural power. In order to extend its indications, and vary the form in which the principle of the moving wire may be applied, I had an apparatus constructed, fig. 19, consisting of a wooden axis, one extremity of which was terminated by a copper screw,

Fig. 19.



* One could easily imagine hypothetically a needle that should do so.

intended to receive and carry one or more discs of metal that might be screwed on to it. This end projected so far beyond the support, that such discs could be partly introduced between the poles of a horseshoe magnet, so as when revolving, to move across the lines of force at their most intense place of action; and, whilst the magnet and the apparatus continued fixed, to revolve continuously across the same lines of force. One of the galvanometer wires was pointed, and so held as to bear into and against the surface of a cup-shaped cavity at the end of the axial screw; and the other was applied by the hand, or so fixed as to bear by a rounded part against the rim of the disc, at that point which was furthest within the poles of the magnet.

3160. Discs of metal were prepared for this apparatus, each 2·5 inches in diameter, and of different thicknesses and material. When a disc of copper was fixed on the axis, and adjusted in association with the large horseshoe magnet (3159.), as described above, three, or even two revolutions of it, would deflect the needle of the thick wire galvanometer through a swing of 30°. In this apparatus, the most effectual part of the portion of the disc which is at any moment passing across the magnetic axis, is that which is near the circumference; for it has the greatest velocity, consequently moves through more space, and that in a part where the lines of force are most concentrated.

3161. The contact at the end of the axle should always be carefully watched and made good. The degree of pressure on the edge of the disc should not be too slight; otherwise the contact, under the circumstances of the motion, is not sufficient to carry forward the same constant proportion of current generated. Neither should it be made at the angles of the disc edge; if a grating or cutting friction occur, an electric current is generated by it. With a smooth hard friction of copper wire against the copper disc there is very little evolution of current. When the copper wire presses against the edge of an iron disc there is far more. In either case, however, the effect may be eliminated or compensated; for, in whichever direction the disc is revolved *without* the magnet, the deviation of the needle, if any be produced, remains the same; whereas, when the magnet is in place, the deviations produced by it are in the reversed direction for reversed revolutions. Hence, if an equal number of revolutions be made in the two directions, and the unequal deflections in opposite directions be noted, the half of their sum will give nearly the amount of deflection which would have occurred if no current had been exerted by friction at the edge, *i. e.* provided the deflections have not been through large arcs. These effects of friction are no doubt objections to the principle in this form; still the results are, as it appears to me, valuable in relation to copper and iron, and are as follows.

3162. A copper disc, 0·05 of an inch in thickness, gave a swing deflection for two revolutions, which, being the average of several experiments, $=20^\circ{\cdot}8$. A second copper disc, of 0·1 of an inch in thickness, gave an average deflection of $27^\circ{\cdot}8$. A third copper disc, of 0·2 in thickness, gave a deflection of $26^\circ{\cdot}5$. Here, therefore,

not only has the thickness (with these conditions of contact) been attained for the maximum effect, but even surpassed (3137.). Then an *iron* disc of 0·05 in thickness, was placed on the axle, and gave, as its mean result, a deflection of 15°·4. Another iron disc of four times the thickness (or 0·2) gave a deflection only of 14°. So here also, as before, the thickness of maximum effect had been surpassed.

3163. The two discs of copper and iron of 0·2 in thickness each, which had produced separately the respective deviations of 26°·5 and 14°, were then both fixed on the axle, being separated from mutual contact in respect of their mass, by a disc of paper, though both were of course in contact at the centre of motion with the copper axle, by means of which the electric communication was perfected. In arranging their place between the poles of the magnet, the iron was placed at mid-distance, and therefore the copper a little on one side. When the copper disc was brought into the circuit, it, by two revolutions, gave an average deviation of 23°·4; and when the iron disc was in the circuit, the deviation produced by it was 11°·91. Here, therefore, the proportions were nearly the same, when the two discs were subject at the same moment to the magnetic power, as when they were examined separately. Both have fallen a little, but not in any manner which seems to indicate that the iron has had any peculiar influence in altering or affecting the lines of force passing across the magnetic field. The effect which has taken place, appears to be one due to the action of the collateral mass of conducting matter.

3164. If the direction of the electric current induced by the magnetic force in the moving metal be taken as the true indication of polarity, and, I think, it cannot be denied that it represents that character of the force, which the term polarity is intended to express, and is unchangeably associated with that character; then these results show that the polarity of the lines of force within the iron is the same with that within the copper, when both are submitted in like manner to the magnetic force. In association with the former and new results with *bismuth* (2431. 3151. 3168.), and numerous other phenomena, the same conclusion may be drawn as to the lines of force within that substance, for the effects are the same with regard to the production of a current in it; and so further evidence is added to that which I have given, tending to show that bismuth is not polarized in the reverse direction as iron or a magnet (2429. 2640.). By reference to the phenomena presented by the relative actions of paramagnetic and diamagnetic substances, the same conclusions may be drawn with respect to all bodies and to space itself (2787. &c.).

3165. That the iron disc affects the disposition of the *lines of force*, is no doubt true, and the extent to which this is done is easily seen, by fixing a small magnetic needle, about 0·1 or 0·05 of an inch in length, across the middle of a piece of stretched thread as an axis, and then bringing it into the magnetic field and near the edges of the stationary disc. The lines of force will be seen (3071. 3076.) gathering in upon the iron at and near its edge, but only for a very little distance from it in any direction: the effect is that which I have considered proper to a para-

magnetic body (2807.). Elsewhere, the lines of force go with the same direction across the magnetic field where the iron is, as where it is not; and it is to me a proved fact, proved by the numerous results given, that a section of the lines of force taken across the magnetic field through the air, where there is no iron, is exactly equal in amount of force to a section taken across parallel to and through the iron disc (3163.). All iron under induction must have just as much force, *i. e.* lines of force in its internal parts, as is equivalent to the lines which fall on to, and are continued through and out of it; and the same is true, as it appears to me, of any other paramagnetic or diamagnetic substance whatever. The same is true *for the magnet itself*; for a section through the magnet has been shown to be exactly equal to a section anywhere through the outer lines of force (3121.), and these sections may be taken at the surface of the magnet, where they may be considered as either in the air or in the magnet indifferently; and therefore alike in size, shape, power, polarity, and every other point.

3166. I have used the phrase *conduction polarity* on a former occasion (2818. 2835.), but so limited, that it could lead to no mistake of my meaning, either then or now. It requires no words to show how it is included in the higher and general expression of the direction or polarity of the lines of force.

3167. Some other results with the disc apparatus (3159.) were obtained, which it may be useful to describe here. *Tin* was formed into a disc of 0·1 in thickness, and 2·5 inches in diameter. The effect of the friction of the copper conductor at its edge, was a feeble current, the reverse of that produced in the cases of copper and iron (3161.); but the current produced by the revolution, and dependent on the polarity of the lines of force, was the same as before. It produced a swing deflection of 14°·9 for two revolutions of the disc.

3168. A disc of *bismuth* produced far too strong a current by friction against the copper conductor, to allow of any useful result in its simple state. A ring of copper foil was therefore formed, and being placed tightly on the bismuth disc, was wedged up by plates of clean copper foil, so as to produce a clean hard contact; imperfect, no doubt, but as general as could be made under the circumstances. When this disc was rotated in the one direction, it gave a deflection in the same direction as if a copper or iron disc had been used; when rotated the other way, the deflection was little or nothing. This difference is due to the united influence of the rotation effect and the friction effect in the one case, and their opposition in the other; but the results show that the lines of force are in the same direction through bismuth, when between the magnetic poles, as they are through copper and iron. The induced current is small, both because of the bad conducting power of the bismuth and the imperfect contact at the edge. When the same copper rim was placed on the copper disc, it reduced the deflection of the needle from 26°·5 to 9°·34.

3169. In illustration of the effect produced by those parts of the disc, which, not being in the place of greatest action, are conducting back those currents formed by the radial parts in the place of maximum effect, I had a wooden disc constructed, 0·2 in thickness and 2·5 inches in diameter, the centre of which was copper, for the purpose of attachment and electrical connection, and the outer edge a ring of copper not more than $\frac{1}{30}$th of an inch in thickness. The two were connected by a single copper wire radius, in thickness 0·056 of an inch, which, as the disc revolved, was of course carried across and through the magnetic field. It gave a deflection of 14°. The copper disc of 0·05 thickness gave only an average of 28°. Now, though the matter of the copper ring round the wood will cause part of the current, yet the chief portion must be due to the copper radius, which, at the effectual part near the edge (3160.), is not more than the $\frac{1}{140}$th part of the full copper disc; and this indicates how much of the electricity put in motion there by the magnetic force must be returned back in short circuits in the other parts of the disc.

3170. The disc apparatus shows well the dependence of the induced current upon the *intersection* of the lines of force (3082. 3113.). If the disc be so arranged as to stand edgeways to the magnetic poles, and in the plane of the magnetic axis, so that it shall be *parallel* to the lines of force which pass by and through it, then no revolution of it, with the most powerful magnet, produces the slightest signs of a current at the galvanometer.

3171. The relation of the induced current to the electro-conducting power of the substance, amongst the metals (3152.), leads to the presumption that with other bodies, as water, wax, glass, &c., it is absent, only in consequence of the great deficiency of conducting power. I thought that processes analogous to those employed with the metals, might in such non-conductors as shell-lac, sulphur, &c., yield some results of static electricity (181. 192.); and have made many experiments with this view in the intense magnetic field, but without any distinct result.

3172. All the results described are those obtained with *moving metals*. But mere motion would not generate a relation, which had not a foundation in the existence of some previous state; and therefore the *quiescent* metals must be in some relation to the active centre of force, and that not necessarily dependent on their paramagnetic or diamagnetic condition, because a metal at zero, in that respect, would have an electric current generated in it as well as the others. The relation is not as the attractions or repulsions of the metals, and therefore not magnetic in the common sense of the word; but according to some other function of the power. Iron, copper, and bismuth are very different in the former sense, but when moving across the lines of force, give the same general result, modified only by electro-conducting power.

3173. If such a condition be hereafter verified by experiment, and the idea of an electrotonic state (60. 242. 1114. 1661. 1729.) be revived and established, then, such

bodies as water, oil, resin, &c., will probably be included in the same state; for the non-conducting condition, which prevents the formation of a current in them, does not militate against the existence of that condition which is prior to the effect of motion. A piece of copper, which cannot have the current, because it is not in a circuit (3087.), and a piece of lac, which cannot, because it is a non-conductor of electricity, may have peculiar but analogous states when moving across a field of magnetic power.

3174. On bringing this paper to a close, I cannot refrain from again expressing my conviction of the truthfulness of the representation, which the idea of lines of force affords in regard to magnetic action. All the points which are experimentally established with regard to that action, *i. e.* all that is not hypothetical, appear to be well and truly represented by it. Whatever idea we employ to represent the power, ought ultimately to include electric forces, for the two are so related that one expression ought to serve for both. In this respect, the idea of lines of force appears to me to have advantages over the method of representing magnetic forces by centres of action. In a straight wire, for instance, carrying an electric current, it is apparently impossible to represent the magnetic forces by centres of action, whereas the lines of force simply and truly represent them. The study of these lines has, at different times, been greatly influential in leading me to various results, which I think prove their utility as well as fertility. Thus, the law of magneto-electric induction (114.); the earth's inductive action (149. 161. 171.); the relation of magnetism and light (2146. and note); diamagnetic action and its law (2243.), and magnecrystallic action (2454.), are cases of this kind: and a similar influence of them, over my mind, will be seen in the further instances of the polarity of diamagnetic bodies (2640.); the relation of magnetic curves and the evolved electric currents (243.); the explication of Arago's phenomenon (81.), and the distinction between that and ordinary magnetism (243. 245.); the relation of electric and magnetic forces (1709.); the views regarding magnetic conduction (2797.), and atmospheric magnetism (2847.) I have been so accustomed, indeed, to employ them, and especially in my last Researches, that I may, unwittingly, have become prejudiced in their favour, and ceased to be a clear-sighted judge. Still, I have always endeavoured to make experiment the test and controler of theory and opinion; but neither by that nor by close cross examination in principle, have I been made aware of any error involved in their use.

3175. Whilst writing this paper, I perceive, that, in the late Series of these Researches, Nos. XXV. XXVI. XXVII., I have sometimes used the term *lines of force* so vaguely, as to leave the reader doubtful whether I intended it as a merely representative idea of the forces, or as the description of the path along which the power was continuously exerted. What I have said in the beginning of this paper (3075.) will render that matter clear. I have as yet found no reason to wish any part of those papers altered, except these doubtful expressions: but that will be rectified if

it be understood, that, wherever the expression *line of force* is taken simply to represent the disposition of the forces, it shall have the fullness of that meaning; but that wherever it may seem to represent the idea of the *physical mode* of transmission of the force, it expresses in that respect the opinion to which I incline at present. The opinion may be erroneous, and yet *all* that relates or refers to the disposition of the force will remain the same.

3176. The value of the moving wire or conductor, as an examiner of the magnetic forces, appears to me very great, because it touches the physics of the subject in a manner altogether different to the magnetic needle. It not only gives its indications upón a different principle and in a different manner, but in the mutual action of it and the source of power, it affects the power differently. The wire when quiescent does not sensibly disturb the arrangement of the force in the magnetic field; the needle when present does. When the wire is moving it does not sensibly disturb the forces external to it, unless perhaps in large masses, as in the discs (3163.), or when time is concerned (1730.), *i.e.* it does not disturb the disposition of the whole force, or the arrangement of the lines of force; a field of equal magnetic power is still equal to anything but the moving wire, whilst the wire moves across or through it. The moving wire also indicates quantity of force, independent of tension (2870.); it shows that the quantity within a magnet and that outside is the same, though the tension be very different. In addition to these advantageous points, the principle is available within magnets, and paramagnetic and diamagnetic bodies, so as to have an application beyond that of the needle, and thus give experimental evidence, of a nature not otherwise attainable.

Royal Institution,
October 9, 1851.

IV. *An Account of two cases, in which Ovules, or their Remains, were discovered in the Fallopian Tubes of Unimpregnated Women who had died during the period of Menstruation.* *By* H. LETHEBY, *M.B., Lond., Lecturer on Chemistry and Medical Jurisprudence in the Medical School of the London Hospital. Communicated by* T. B. CURLING, *Esq., F.R.S.*

Received February 20,—Read May 1, 1851.

THE observations that have been made at various times, during the last thirty years, by Messrs. POWER, LEE, BARRY, WHARTON JONES, GIRDWOOD, and others in this country, together with the experimental researches of MM. VALENTIN, GENDRIN, WAGNER, BISCHOFF, POUCHET and RACIBORSKI on the Continent, have, I think, clearly proved that the phenomena manifested during the period of the catamenia in women, are closely connected with those observed during the time of heat or rut in quadrupeds; and that both of these phenomena are dependent on one cause, namely, the maturation of ovules. But while this hypothesis has been very generally admitted, there is, I believe, a tendency in the minds of many physiologists of the present day, to doubt whether the ovules so matured are ever extruded from the ovary and carried into the Fallopian tubes, without the stimulus of impregnation, or, at any rate, without the congress of the male. In support of this view, or rather of these doubts, an appeal is often made to the fact, that an ovule has never yet been detected in either of the Fallopian tubes of a virgin, who has died during the period of the catamenia, notwithstanding that many subjects have been examined, that most careful search has been instituted, and that appearances have frequently been noticed indicating the recent rupture of a Graafian follicle. In point of fact, it is imagined by those who entertain such doubts, that the fecundation of the germ takes place while it is within the Graafian follicle, and consequently, that if the ovule fails to be the subject of impregnation it never quits the ovary, but perishes within its formative vesicle. On the other hand, the researches of BISCHOFF have led him to enunciate a law, the purport of which is the very reverse of the preceding; for he says, that "the ovules formed in the ovaries of females of the human species and of mammiferous animals, undergo a periodical maturation, quite independently of the male seminal fluid. At these periods, known as those of heat or the rut in animals, and menstruation in the human female, the ovules which have become mature, disengage themselves from the ovary and are extruded. If the union of the sexes takes place, the ovule is fecundated by the direct action of the semen upon it. If no union of the

sexes occurs, the ovule is nevertheless extruded from the ovary, and enters the Fallopian tube, but there perishes*."

The law, as thus expressed, is in conformity with the opinions entertained by Drs. Robert Lee, Paterson, Girdwood, Gendrin, Pouchet, Raciborski, Mr. Wharton Jones, and many other authorities of the present time. It is also in accordance with the more ancient doctrines of Malpighi, Sir Everard Home, and Dr. Power. Nevertheless, as the truth of this law, in its application to the human female, appears to be still open to the evidence of positive proof, I have thought it desirable to publish a report of the two following cases.

Case 1st.—November the 20th, 1850, a woman aged twenty-six, in a state of great mental excitement, attempted self-destruction by cutting her throat. The wound which she inflicted was not dangerous; and after having been attended to by a surgeon for a few days, the woman was removed to the London Hospital, where she became a patient under Mr. Curling. She lingered until the 14th day of December, when she died. On the following day the body was examined, and it was noticed that the pelvic viscera were highly congested, that the uterus was considerably enlarged, that the vagina contained a sero-sanguineous fluid, and that the hymen was unruptured. In consequence of these appearances the parts were removed for further examination.

On cutting into the uterus, I discovered that it contained a small quantity of sanious fluid; I noticed, moreover, that both of the ovaries presented a number of stellate fissures or cicatrices on their surfaces; and that at one part of the left organ, namely, at its inferior, inner and posterior border, there was a distinct purple spot of the size of a small pea. In the centre of this spot there was a ragged opening, that had, in all probability, recently given exit to an unimpregnated germ (*vide* Plate III. fig. 1). An incision was made into the gland, so as to cut through the discoloured portion of it, and it was then remarked that the aperture led into a small cavity, the existence of which is still evident in the wet preparation. The cavity was situated at the very summit of the spot, immediately within the opening on the peritoneal surface; and it was surrounded by a large quantity of what appeared to be extravasated blood (*vide* fig. 2). After the preparation had been immersed in spirit of wine for a few days, I was able to perceive that the spot was made up of four distinct parts;—1st, of an outer vascular layer which surrounded the mass, and extended into it to the depth of $\frac{1}{16}$th of an inch; this layer was gradually thinned as it approached the aperture on the surface of the follicle; and it seemed to consist of *the stroma of the ovary* in a highly congested state. 2nd. Of a thin layer of very dark matter, which appeared to be the remains of *the ovisac.* 3rd. Of a mass of coagulated blood strengthened by a network of intersecting fibres; this mass had the bulk of a hemp-seed, and it was found to be composed of blood-discs, fibrin, and large granular corpuscles; it was, doubtless, therefore, the remains of *the tunica granulosa* in-

* Beweis der von der Begattung unabhängigen periodischen Reifung und Loslösung der Eier, p. 4, quoted by Drs. Baly and Kirkes in their Supplement to the second volume of Müller's Physiology, p. 45.

filtrated with blood. 4thly, and lastly, there was *the cavity* to which I have before alluded.

When the ovaries had become firm and hard by the coagulating action of the spirit, sections were made into them at various parts; by which means a number of yellow bodies (false *corpora lutea*) in different stages of degeneration were brought into view. Some of these bodies were rather large, and one of them contained a well-defined clot, the summit of which communicated with a cicatrix on the surface of the ovary (see the preparation). This circumstance led me to conclude that it was the remains of an old Graafian follicle, from which, at perhaps the last catamenial period, an ovule had escaped.

The Fallopian tubes were highly congested, especially at their fimbriated extremities, where, from the abundance of turgid capillary vessels, the tubes assumed a bright scarlet appearance. The cavities of the oviducts were filled with, and much distended by, a thick bloody mucus, which readily escaped from their peritoneal apertures when the tubes were subjected to slight pressure between the finger and thumb. Both of the Fallopian tubes were carefully laid open by means of a pair of fine scissors—the operation being conducted on a clean white plate, containing a little water,—and their contents were minutely examined. The right tube did not present any object worthy of notice; but the left one contained, at about 1 inch from its distal extremity, a small white vesicular-looking body, which on being floated out into the water, was found to be rather ragged on its surface, and to have the size of the cavity noticed in the recently ruptured Graafian follicle. This body was submitted to microscopical examination. When viewed as an opake object, nothing could be made out beyond the fact that it was covered with white flocculi. It was then placed between two pieces of glass, and examined by the aid of transmitted light; but it was too opake for the eye to distinguish its structure, notwithstanding that the flocculi were very translucent and were seen to be made up of oval nucleated cells (see fig. 6). By the employment of slight pressure the body was readily crushed, and then I could perceive that it was composed of a mass of nucleated cells, among which, at one part, there was a number of highly refractive oil-globules (see fig. 5). The result of this investigation led me to think that the body in question was an ovule, the elements of which had been so far disarranged by the pressure, that the *membrana granulosa* and *yelk-globules* were the only recognisable constituents of it.

The fluid contained in the uterus and Fallopian tubes were likewise subjected to microscopical examination. That removed from the former was found to consist of numerous blood-discs, most of which were strongly beaded at their edges; of much cylindrical epithelium, some of which was distinctly ciliated; of a large quantity of granular corpuscles, like exudation cells; of a few white globules, similar to those found in blood, many of which had apparently passed into the form of spindle-shaped or fusiform bodies by the elongation of their opposite ends; and of a thick gelatinous fluid which united all the elements together (see fig. 3). That from the latter, namely,

the Fallopian tubes, was much the same as the preceding, excepting that the number of the blood-discs was considerably less; that there was a greater abundance of ciliated epithelium; and that the fluid in which the elements floated was not gelatinous, but serous (see fig. 4).

A consideration of the facts thus presented to notice, led me to conclude that the girl had died at the very onset of a catamenial period, for I could not discover any evidence of the occurrence of an external flow; in fact, the secretion found in the vagina was not very abundant, and it had acquired only a pale rose tint.

On instituting a further inquiry into the case, I ascertained that the periodical flux had taken place exactly one week before the woman made the attempt on her life; and with regard to the subsequent history of the case, it may be said that she was laid up with the wound in her throat for a period of twenty-four days before her death, nineteen of which were passed in a separate ward of the Hospital, where, in consequence of her very distressing condition, she was closely watched by a female attendant, so that it is hardly possible that sexual intercourse could have been effected during that period of time.

While I was engaged in the investigation of the preceding case, I received from my friend Dr. PARKER of Finsbury Square, who had assisted me in the foregoing inquiry, another uterus and its appendages, which he had removed from the body of a lunatic aged twenty-three. This girl had died, and was examined in St. Luke's Hospital, where she had been a patient for eleven months, under circumstances which deprived her of the opportunity of associating with a male for a long period before her death.

The information obtained by inquiries of the attendant at St. Luke's, as well as by an examination of the organs themselves, led me to conclude that the girl had quitted life during the catamenial period; for the pelvic viscera were much congested, the uterus was considerably enlarged, its vessels were turgid, and its cavity contained a red jelly-like matter; besides which, the Fallopian tubes were filled with a thick muco-sanguineous secretion, and the right ovary presented a dark livid spot on its outer and lower part: many cicatrices were also found on the surfaces of both the ovaries. As in the last case, the livid spot had a hole in its centre; and, on making a section of the ovary so as to divide it through the spot and an adjacent cicatrix, I perceived that the hole led into a cavity, which was surrounded by a deep red tissue, and that the cicatrix communicated with a very perfectly formed *corpus luteum*, having a central cavity containing a dark red clot (see fig. 7 and preparation).

The matter contained in the right Fallopian tube was submitted to careful examination, by which means I discovered a little globular body that had the size of a small pin's head. This body was transferred, as in the last case, to water; then placed between two pieces of glass, and examined under the microscope with a power of 100. The outer constituents of the mass were precisely like those of the preceding; that is, they consisted of nucleated cells, arranged so as to form a shaggy, but tolerably

compact tunic. At one end of the object, near to its surface, there was a transparent ring, enclosing a rather opake granular mass, in which there was an eccentric, highly pellucid spot (see fig. 9). I had no doubt that this was the ovule, consisting of the *zona pellucida*, the *yelk* and the *germinal vesicle*; but, to test the truth of my opinion, I subjected the mass to the action of a little strong acetic acid, by which means the *zona pellucida* was still more clearly brought into view; for it happened that the nucleated cells of the *membrana granulosa* were slightly corrugated by the acid, and, as it were, contracted on their contents. They also acquired a greater transparency; and by using a power of 300, the *zona* presented a distinctly striated appearance, and the granules within it were seen to be highly refractive. Lastly, the whole object was washed with ether, which dissolved away the fat granules, and thus left the *zona* and *vesicle* still more distinct. In the place of the yelk there was now left a somewhat opake structureless material, which seemed to have been the bond of union between the numerous fatty elements of the vitelline mass (see fig. 10).

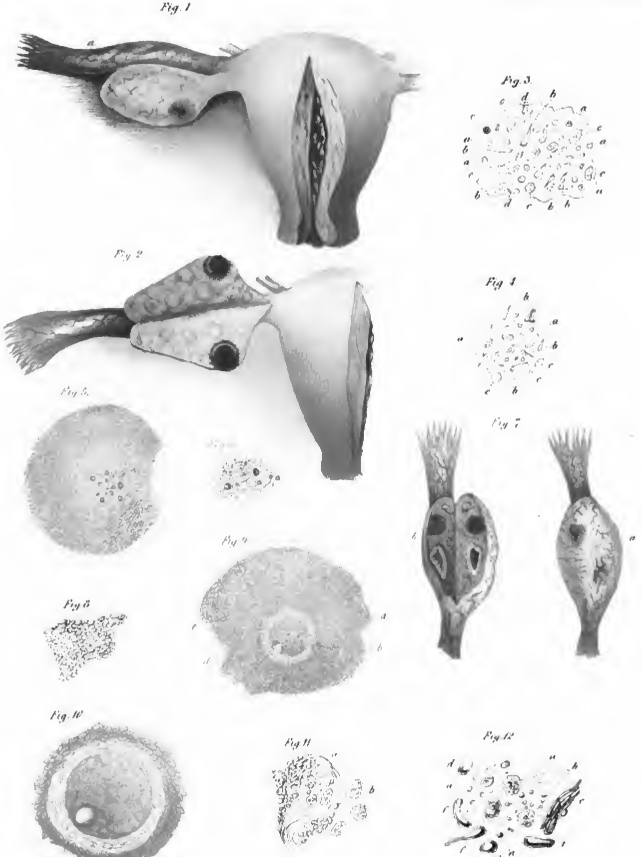
The fluid matters contained in the uterus and Fallopian tubes, were identical in their physical characters with those noticed in the last case; and the materials composing the recently ruptured Graafian follicle were likewise found to consist of blood-discs, fibrin and large nucleated cells (see fig. 11). The yellow tissue of the *corpus luteum* was made up of a fibro-cellular stroma, in which there was enclosed a number of large granular corpuscles, some of which contained oil-globules; besides these, there was a liquid fat diffused through the tissue, which appeared either as minute particles, or as large elongated highly refractive globules that had been produced by the union of the smaller elements (see fig. 10).

Remarks.—An examination of the preceding facts will, I think, justify me in coming to the following conclusions; namely, *that ovules escape from the ovaries of women during the catamenial flux; and that the escape of these bodies is spontaneous,* id est, *that their extrusion takes place independently of sexual intercourse.* In these respects, the phenomena witnessed are perfectly analogous with those observed by Bischoff, Raciborski, Pouchet and others, while they were instituting inquiries into the changes which occur in the ovaries of other mammals during the periods of heat and rut; and they are also in accordance with a fact long since ascertained by physiologists, namely, that there is a periodical maturation and spontaneous extrusion of ovules from the ovaries of animals still lower in the scale of creation. Hitherto, however, these several circumstances have been regarded only in the light of probable analogies; but I am of opinion that the facts detailed in this paper are quite sufficient to identify the phenomena in question, and also to establish the truth of a great part of the law enunciated by Bischoff, that is, that the ovules formed in the ovaries of females of the human species, and of mammiferous animals, undergo a periodical maturation, quite independently of the influence of the male seminal fluid; and that at the periods of menstruation in the one, and heat in the other, the ova which have become mature disengage themselves from the ovary, and are extruded

whether there be access of the male or no. But admitting that this law is in great part the expression of truth, we have yet to determine whether in human females, as in the females of other animals, the maturation and escape of ovules take place only at the fixed periods mentioned, or whether the discharge of these bodies is always occurring, thus rendering the human female susceptible of impregnation at all times. The final solution of these questions, together with that of the problem concerning the length of time that the ovule retains its faculty of being fecundated after it has quitted the ovary, is of great interest to the physiologist, the obstetric physician, and the medical jurist; for it would not only enable them to pronounce with certainty at what time conception usually takes place, but it would also furnish a starting-point for the determination of other equally important problems, viz. the ordinary, and the most extended periods of human gestation. It must, however, be evident that a great number of independent observations have yet to be made, before any hypothesis relating to this part of the subject can take its place amongst the well-recognised doctrines of science; and, believing that each observation must have some weight and value, I have been led to record the results of my own inquiries, in order that they may be placed in juxtaposition with the facts already elicited by more able investigators. Here, perhaps, I may be allowed to remark, that in addition to the experiments and observations so admirably reviewed by Drs. Baly and Kirkes in their Supplement to the second volume of Professor Müller's Physiology, two cases have been published, in which the phenomena witnessed are in many respects very similar to those noticed by myself. In one of these cases, the girl died shortly after menstruation; and the reporter, M. Janzer, states that, in making an examination of her body, he found on the surface of the left ovary a dark red spot, which had a fissure in its centre. Judging from all the circumstances of the case, Janzer was led to conclude that the spot in question was a recently ruptured Graafian follicle gorged with blood. He instituted a diligent search for the liberated ovule, but he failed to discover it*. The second case is recorded by M. Locatelli†; and he informs us that the woman who was the subject of his investigations died from the effects of an operation made for the relief of an imperforate hymen. He noticed that on one of the ovaries there was a livid spot, in the centre of which there was an aperture. The parts were carefully examined, and the author believes that the spot seen in the ovary was a recently ruptured follicle filled with blood. As in the last case, however, the liberated ovule escaped detection.

* Heidelberger Annalen, Bd. xiii. p. 601–604.

† Froriep's Neue Notizen, Bd. vii. p. 348–350.



J. Basire lith.

Explanation of the Plate.

PLATE III.

Fig. 1. *The uterus and its appendages* as seen from behind. The uterus is laid open by a longitudinal incision in order to show the contained menstrual fluid. The Fallopian tube is highly congested, and its cavity is much distended with sanguinolent mucus. The ovary presents several cicatrices on its surface; and at the inner and lower margin it exhibits a purple spot, in the centre of which there is a small opening through which the ovule had recently escaped.

(*a.*) The spot at which the ovule was found.

Fig. 2. *The left ovary* laid open in a line through the centre of the purple spot, showing the cavity from which the ovule had escaped, surrounded by a clot of dark blood, most of which was effused into the substance of the Graafian follicle.

Fig. 3. *A portion of the sanguinolent matter taken from the uterus*, and magnified about 200 times. It was found to consist of blood-discs, most of which were beaded at the edges (*a a*); of ciliated and cylindrical epithelium (*b b*); of granular corpuscles, some of which were oval and others round (*c c*); of spindle-shaped bodies (*d d*); of numerous minute granules; and of a thick gelatinous fluid which bound all the elements together.

Fig. 4. *Fluid from the Fallopian tube*, treated in like manner: it consisted of nearly the same elements. The fluid in which the corpuscles floated was, however, of a serous, not a gelatinous character.

N.B. To prevent obscurity, only a few of each of the elements have been figured.

Fig. 5. *A portion of the ovule mass* taken from the left Fallopian tube, and magnified 200 times. The entire mass consisted of nucleated cells (*membrana granulosa*) arranged in an irregular manner. Interspersed through the cells were many oil-globules, probably the broken-down yelk.

Fig. 6. *A part of the middle of the mass*, magnified still more, showing the arrangement of the cells and the position of the oil-globules.

Fig. 7. *The right ovary* removed from the uterus and placed vertically.

(*a.*) The external surface showing several stellate cicatrices; and at its

outer and lower part it also exhibits a purple spot, in which there is an aperture from which the ovule has escaped.

(*b*.) The same ovary divided longitudinally through the cicatrix and spot, by which means two false *corpora lutea*, and the ruptured Graafian vesicle were brought into view.

Fig. 8. *A part of the shaggy surface of the ovule*, magnified 200 times, showing the arrangement of the nucleated cells which compose it.

Fig. 9. *The ovule* crushed between glass and magnified 100 times: exhibiting the cellular envelope (*a*); the transparent zona pellucida (*b*); the yelk (*c*); and the germinal vesicle (*d*).

Fig. 10. *The same*, after having been treated with acetic acid and then with ether; magnified 300 times.

Fig. 11. *A portion of the Graafian follicle* from which the ovule has just escaped, magnified 200 times: (*a*) a clot containing blood-discs, fibrin and large nucleated cells; (*b*) some of the cells disengaged.

Fig. 12. *A portion of a corpus luteum*, magnified 200 times; consisting of small fat-globules (*a a*); large nucleated cells containing fat-globules (*b b*); (*c c*) a part of the tissue showing a structure composed of fusiform or fibrous cells; and (*d d*) large fat-globules formed by the aggregation of smaller ones; at *f f* they have assumed other forms, in consequence of their having been pressed between the glass.

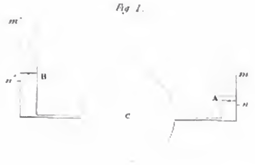


Fig. 1.
m'
n'
B
c
m
A
n

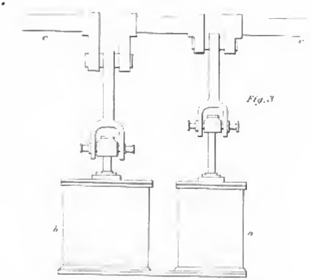
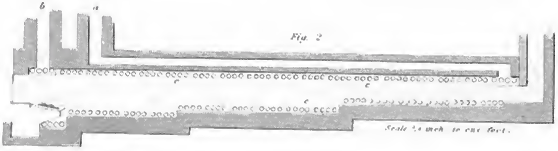


Fig. 3
c
c
b
a


b
a
Fig. 2
c
c
c
Scale ½ inch to one foot.

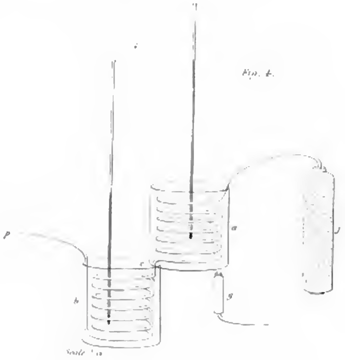


Fig. 4.
p
c
a
b
g
Scale ½

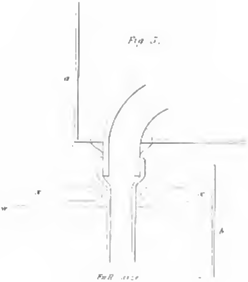


Fig. 5.
a
x
w
c
b
Full size

V. *On the Air-Engine.* By James Prescott Joule, *F.R.S., F.C.S., Corr. Mem. R.A. Turin, Sec. Lit. and Phil. Soc. Manchester, &c.*

Received May 13,—Read June 19, 1851.

It has long been suspected that important advantages might be derived from the substitution of air for steam as a prime mover of machinery. It has been alleged that the air-engine would be safer, lighter, and more economical in the expenditure of fuel than the steam-engine. Until comparatively recent times, however, experimental science was hardly in the state of advancement requisite to enable the physicist, in his investigation of this important subject, to arrive at conclusions sufficiently certain to give confidence to the practical machinist. Professor Thomson, Mr. Rankine, and M. Clausius have of late, however, published papers of great value on the mechanical action of gases, and particularly of steam, founded on tolerably correct experimental data. I hope that the following remarks founded on the same general principles, but applied to a particular kind of air-engine, may be interesting to the Royal Society.

The air-engine, the performance of which I propose to discuss, consists of two parts, in one of which the air is compressed into a receiver, where its elasticity is increased by the application of heat, and in the other it is allowed to escape again from the receiver into the atmosphere. By the former work is absorbed, by the latter it is evolved in a larger quantity, the excess constituting the work evolved by the engine on the whole. The simple question, therefore, is to determine the quantity of work so evolved, together with the heat applied to increase the elasticity of the air in the receiver.

In Plate VI. fig. 1 let A be the pump by which air is forced into the receiver C, where heat may be communicated to it from an external source, and B the cylinder, by which the same quantity is allowed to escape again into the atmosphere. Moreover, let the material of which the apparatus is made, with the exception of that part through which heat may be communicated to the air in C, be impervious to, and destitute of capacity for heat. Such a machine may be conceived to work in the following manner.

The cylinder of the pump A being filled with air of the atmospheric temperature and pressure, the piston compresses the air until, at a point n, its pressure is rendered equal to that of the air in the receiver C, which has been previously filled with air of an elevated temperature and pressure. The work absorbed by this action will be that communicated to the air in the cylinder, minus the work due to the atmospheric

pressure through *m n*. The moment the piston has passed the point *n* the valve will open, admitting the air into the receiver C; and as this receiver may be conceived to be of indefinite magnitude, the alteration of pressure in it, consequent upon the introduction of fresh air, may be neglected. Heat is then communicated to the air in the receiver, in order to restore its temperature to the intensity which existed before the admission of air at a lower temperature. The air is then allowed to escape from the receiver into the base of the cylinder B, evolving work until, on the arrival of the piston at *n'*, the same quantity has been removed from the receiver as was forced into it by the pump. The further supply of air from the receiver is then cut off, and that which has entered the cylinder expands, evolving work until, on the arrival of the piston at *m'*, its pressure is reduced to that of the atmosphere. By opening valves at the bases of A and B, the pistons are then brought to their first positions.

The problem which must be solved in order to estimate the power and consumption of fuel in an engine similar to that just described, is as follows:—To determine the pressure and temperature for any point of the stroke of a piston which compresses a given volume of air, and the quantity of work absorbed in forcing the piston to that point. For the temperature and pressure Poisson has furnished the following formulæ,—

$$\frac{T'}{T}=\left(\frac{V}{V'}\right)^{k-1},$$

and

$$\frac{P'}{P}=\left(\frac{V}{V'}\right)^{k},$$

where T, P, and V are the temperature from absolute zero (estimated at 491° Fahr. below the freezing-point of water), pressure, and volume of the air before compression; T', P', and V' the temperature from absolute zero, pressure, and volume of air after compression; and k is the ratio of the specific heat of air at constant pressure to that at constant volume. Professor W. Thomson has deduced, as a consequence of the above, the following formula for the work absorbed,

$$W=PV\frac{1}{k-1}\left\{\left(\frac{V}{V'}\right)^{k-1}-1\right\}^{*}.$$

From the foregoing formulæ I have calculated the work absorbed by compressing air in a cylinder 1 foot long, and of the capacity of 12 cubic inches, the absolute temperature of the air, and its pressure at each tenth of an inch of the piston's progress. The following data were employed in the computation:—Weight of 100 cubic inches of atmospheric air of 15 lbs. pressure on the square inch, and 491°

* The above formula was kindly communicated to the author by Professor Thomson, in a letter dated January 15, 1851, from which the following is an extract:—"It is required to find the work necessary to compress a given mass of air to a given fraction of its volume, when no heat is permitted to leave the air. Let P, V, T be the primitive pressure, volume, and temperature, respectively; let p, v, and t be the pressure, volume, and temperature at any instant during the compression; and let P', V', and T' be what they become

Fahr. from the absolute zero, 33·2237 grs.; specific heat of air at constant volume, 0·19742. Ratio of the specific heat of air at constant pressure to that at constant volume, as determined from the experiments of Delaroche and Berard, and the mechanical equivalent of heat, 1·3519325*. The results are shown in Table I.

I now proceed to give some estimates of the performance of an air-engine similar in principle to that already described, worked at various pressures and temperatures, those of the atmospheric air being 15 lbs. on the square inch, and 32° Fahr. or 491° Fahr. from the absolute zero. In order to render the results easily available in calculating the duty of engines of greater size, I shall assume that the condensing pump is 12 inches long, and has a sectional area equal to 1 square inch, and that the cylinder, also of 1 inch section, has a length which may be made to vary according to the pressure and temperature employed.

I take as the first example, a case in which the receiver C contains air of the atmospheric density, and of which the absolute temperature is 849°·464 Fahr. or 390°·464 of the scale of Fahrenheit's thermometer. The pressure in the receiver will then be 25·95104 lbs. on the square inch, as given in the third column of Table II. The air in the pump A will be brought to the same pressure, and to the absolute temperature 566°·3094 after the piston has traversed 4 inches. The work absorbed by the air will be 6·537154 foot-pounds, from which, by subtracting 5 foot-pounds, the work communicated by the pressure of the atmosphere following the piston, we obtain 1·537154 foot-pounds as the work of the engine absorbed by the first part of the stroke. This result is consigned to column 6. Immediately after the piston has passed the fourth inch of the pump, the valve will be opened admitting the compressed air into the receiver C. The work of the engine absorbed by the re-

when the compression is concluded. Then if k denote the ratio of the specific heat of air at constant pressure to the specific heat of air kept in a space of constant volume, and if, as appears to be nearly, if not rigorously true, k be constant for varying temperatures and pressures, we shall have by the investigation in Miller's 'Hydrostatics' (Edit. 1835, p. 22)—

$$\frac{1+Et}{1+ET}=\left(\frac{V}{v}\right)^{k-1}.$$

But

$$\frac{pv}{PV}=\frac{1+Et}{1+ET},$$

therefore

$$pv=PV\left(\frac{V}{v}\right)^{k-1}.$$

Now the work done in compressing the mass from volume v to volume $v-dv$ will be pdv, or by what precedes,

$$PV\,.\,V^{k-1}\frac{dv}{v^k}.$$

Hence by the integral calculus we readily find, for the work, W, necessary to compress from V to V′,

$$W=PV\,.\,\frac{1}{k-1}\left\{\left(\frac{V}{V'}\right)^{k-1}-1\right\}."$$

* The experiments of Desormes and Clement give 1·354; those of Gay-Lussac and Welter 1·375; and those described under the article 'Hygrometry' (Enc. Brit.), 1·333. See Art. 'Sound,' Enc. Brit., 7th Edit.

maining 8 inches of the piston's stroke will be $\frac{8}{12}(25 \cdot 95104-15)=7 \cdot 300693$ foot-pounds, as given in the seventh column. The air thus forced into the receiver at the absolute temperature $566^{\circ} \cdot 3094$ FAHR. must then be raised to $849^{\circ} \cdot 464$ FAHR., the constant absolute temperature of the receiver. The heat necessary for this purpose, being that due to the capacity for heat of air at constant pressure, will be that which is able to raise the temperature of 1 lb. of water $0^{\circ} \cdot 04304312$ FAHR., as given in column 15. On leaving the receiver, the air enters the cylinder of expansion B, and having propelled the piston through 12 inches, the same quantity of air will have passed out of the receiver as was pumped into it by A. The further supply of air is then cut off, and the air after expanding through the remaining 6 inches of the cylinder (which in this case must be 18 inches long), will be reduced to the pressure of 15 lbs. on the square inch, and the absolute temperature $\frac{3}{2}(491^{\circ})=736^{\circ} \cdot 5$. The work evolved by the piston will also be to that absorbed in the condensing pump, as the volume of the cylinder B is to that of the pump A; from which we find $\frac{3}{2}(7 \cdot 300693)=10 \cdot 95104$ foot-pounds, and $\frac{3}{2}(1 \cdot 537154)=2 \cdot 305731$ foot-pounds, the work evolved by the first and second parts of the piston's stroke, as given in columns 11 and 12. The work evolved by the engine on the whole, being the difference between the work evolved by B, and the work absorbed by A, will be equal to one-third of the former, or one-half of the latter, or 4·418924 foot-pounds, as given in column 14. Dividing this by $0^{\circ} \cdot 04304312$, we obtain 102·66276 foot-pounds as the work evolved by the engine out of each 1° FAHR. per lb. of water communicated to the receiver. This result, which is consigned to the sixteenth column, informs us of the economical value of the engine, which is of course great in proportion to its approach to 772 foot-pounds, the theoretical maximum. The seventeenth column contains the theoretical duty according to Professor THOMSON's law, viz. that the range of temperature divided by the maximum absolute temperature is equal to the fraction of heat converted into force by any perfect engine*.

It will be observed that the numbers in column 16, representing the work evolved out of each unit of heat, increase with the temperature and pressure of the air in the receiver. In every example given, with the exception of the first, the economical value of the air-engine in question is greater than that of the steam-engine calculated by Mr. RANKINE in his paper on the Mechanical Action of Heat†. In considering the relative merits of the engines, we must not, however, lose sight of a most important fact discovered by RANKINE and CLAUSIUS, viz. that a portion of the heat

* See Professor THOMSON's "Investigation of the Duty of a perfect Thermo-Dynamic Engine," at the end of this paper.

† Transactions of the Royal Society of Edinburgh, vol. xx. part 1. Professor THOMSON, in a paper "On the Dynamical Theory of Heat," recently read before the Royal Society, Edinburgh, gives 209 foot-pounds as the duty of an absolutely perfect steam-engine, with a range of temperature between 30° and 140° Centigrade.

Table I.

Distance traversed by piston, in inches.	Work absorbed, in foot-pounds.	Temperature from absolute zero, in degrees Fahr.	Pressure on the piston, in lbs.	Distance traversed by piston, in inches.	Work absorbed, in foot-pounds.	Temperature from absolute zero, in degrees Fahr.	Pressure on the piston, in lbs.
0	0·	491	15	6·0	11·77479	626·6480	38·28805
0·1	0·1257008	492·4481	15·17066	6·1	12·09828	630·3747	39·16857
0·2	0·2528426	493·9128	15·34473	6·2	12·42768	634·1694	40·08375
0·3	0·3814514	495·3944	15·52230	6·3	12·76566	638·0630	41·03738
0·4	0·5113882	496·8913	15·70343	6·4	13·11172	642·0498	42·03120
0·5	0·6432696	498·4106	15·88841	6·5	13·46626	646·1341	43·06763
0·6	0·7763749	499·9440	16·07709	6·6	13·82962	650·3201	44·14936
0·7	0·9111464	501·4966	16·26974	6·7	14·20221	654·6124	45·27926
0·8	1·047533	503·0678	16·46643	6·8	14·58441	659·0155	46·46043
0·9	1·185586	504·6582	16·66731	6·9	14·97668	663·5345	47·69625
1·0	1·325341	506·2682	16·87248	7·0	15·37948	668·1749	48·99042
1·1	1·466805	507·8979	17·08209	7·1	15·79334	672·9426	50·34692
1·2	1·610032	509·5479	17·29627	7·2	16·21879	677·8438	51·77015
1·3	1·755090	511·2190	17·51516	7·3	16·65633	682·8845	53·26481
1·4	1·901962	512·9110	17·73892	7·4	17·10672	688·0730	54·83624
1·5	2·050727	514·6248	17·96770	7·5	17·57047	693·4156	56·49008
1·6	2·201444	516·3611	18·20167	7·6	18·04842	698·9216	58·23268
1·7	2·354089	518·1196	18·44097	7·7	18·54128	704·5995	60·07098
1·8	2·508791	519·9018	18·68582	7·8	19·04988	710·4585	62·01267
1·9	2·665542	521·7076	18·93638	7·9	19·57510	716·5093	64·06620
2·0	2·824402	523·5377	19·19283	8·0	20·11797	722·7632	66·24102
2·1	2·985424	525·3927	19·45538	8·1	20·67947	729·2318	68·54755
2·2	3·148667	527·2733	19·72426	8·2	21·26078	735·9287	70·99750
2·3	3·314184	529·1801	19·99966	8·3	21·86317	742·8683	73·60395
2·4	3·481993	531·1133	20·28182	8·4	22·48795	750·0659	76·38145
2·5	3·652224	533·0744	20·57099	8·5	23·13667	757·5393	79·34657
2·6	3·824870	535·0633	20·86740	8·6	23·81096	765·3072	82·51786
2·7	4·000022	537·0811	21·17132	8·7	24·51263	773·3907	85·91642
2·8	4·177719	539·1282	21·48302	8·8	25·24353	781·8109	89·56590
2·9	4·358080	541·2060	21·80279	8·9	26·00607	790·5954	93·49398
3·0	4·541123	543·3147	22·13095	9·0	26·80260	799·7716	97·73172
3·1	4·726945	545·4554	22·46778	9·1	27·63584	809·3707	102·3153
3·2	4·915606	547·6288	22·81364	9·2	28·50889	819·4284	107·2862
3·3	5·107206	549·8361	23·16887	9·3	29·42512	829·9836	112·6929
3·4	5·301779	552·0776	23·53383	9·4	30·38842	841·0810	118·5920
3·5	5·499456	554·3549	23·90892	9·5	31·40316	852·7710	125·0499
3·6	5·700287	556·6685	24·29452	9·6	32·47432	865·1110	132·1452
3·7	5·904370	559·0196	24·69107	9·7	33·60755	878·1660	139·9716
3·8	6·111816	561·4094	25·09902	9·8	34·80946	892·0122	148·6412
3·9	6·322706	563·8389	25·51884	9·9	36·08755	906·7362	158·2897
4·0	6·537154	566·3094	25·95104	10·0	37·45073	922·4402	169·0827
4·1	6·755242	568·8218	26·39613	10·1	38·90927	939·2430	181·2239
4·2	6·977122	571·3779	26·85467	10·2	40·47547	957·2860	194·9666
4·3	7·202863	573·9785	27·32725	10·3	42·16396	976·7377	210·6300
4·4	7·432590	576·6250	27·81448	10·4	43·99234	997·8010	228·6204
4·5	7·666465	579·3193	28·31703	10·5	45·98215	1020·724	249·4640
4·6	7·904577	582·0624	28·83559	10·6	48·15980	1045·811	273·8522
4·7	8·147090	584·8562	29·37090	10·7	50·55854	1073·445	302·7106
4·8	8·394129	587·7021	29·92373	10·8	53·22073	1104·114	337·3058
4·9	8·645876	590·6023	30·49494	10·9	56·20106	1138·448	379·4122
5·0	8·902414	593·5577	31·08536	11·0	59·57200	1177·282	431·5900
5·1	9·164006	596·5713	31·69598	11·1	63·43234	1221·754	497·6597
5·2	9·430732	599·6440	32·32776	11·2	67·92089	1273·463	583·5623
5·3	9·702832	602·7787	32·98178	11·3	73·23965	1334·736	699·0180
5·4	9·980470	605·9771	33·65917	11·4	79·69880	1409·147	860·9854
5·5	10·26387	609·2420	34·36112	11·5	87·80466	1502·528	1101·650
5·6	10·55323	612·5754	35·08897	11·6	98·46010	1625·281	1489·565
5·7	10·84876	615·9800	35·84405	11·7	113·4919	1798·450	2197·699
5·8	11·15067	619·4581	36·62784	11·8	137·4363	2074·295	3802·170
5·9	11·45928	623·0133	37·44196	11·9	187·1806	2647·359	9705·187

TABLE II.

No. of Example.	Receiver C. Density of the air, that of the atmosphere being called unity.	Receiver C. Pressure of the air in lbs. on the square inch.	Receiver C. Absolute temperature of the air, in degrees FAHR. from the absolute zero.	Pump of Compression A. Length 12 inches. Sectional Area = 1 square inch. Length of the first part of the stroke.	Pump of Compression A. Work of the engine absorbed by the first part of the stroke of the piston, in foot-pounds.	Pump of Compression A. Work of the engine absorbed by the second part of the stroke of the piston, in foot-pounds.	Pump of Compression A. Absolute temperature of the air forced into receiver C, in degrees FAHR. from the absolute zero.	Cylinder of Expansion B. Sectional Area = 1 square inch. Length of cylinder B, in inches.	Cylinder of Expansion B. Length of the first part of the piston's stroke, in inches.	Cylinder of Expansion B. Work communicated to the engine by the first part of the stroke of the piston, in foot-pounds.	Cylinder of Expansion B. Work communicated to the engine by the second part of the stroke, in foot-pounds.	Cylinder of Expansion B. Absolute temperature of the air escaping into the atmosphere, in degrees FAHR. from absolute zero.	Work evolved by the engine by each stroke of the piston, in foot-pounds.	Heat communicated to the air in receiver C, in degrees FAHR. per capacity of a lb. of water.	Work evolved out of each degree FAHR. in the capacity of a lb. of water, in foot-pounds.	Difference between the numbers in columns 4 and 13, divided by the numbers in column 4, and multiplied by the mechanical equivalent of heat.
1	1	25·95104	849·464	4	1·537154	7·300693	566·3094	18	12	10·95104	2·305731	736·5	4·418924	0·04304312	102·6628	102·6626
2	1	66·24102	2168·289	8	10·11797	17·08034	722·7632	36	12	51·24102	30·35391	1473·0	54·39662	0·2197384	247·5517	247·5515
3	2	66·24102	1084·145	8	10·11797	17·08034	722·7632	18	6	25·62051	15·17695	736·5	13·59915	0·0549346	247·5517	247·5515
4	2	169·0827	2767·321	10	24·95073	25·68045	922·4402	36	6	77·04135	74·85219	1473·0	101·2624	0·2804454	361·0769	361·0770
5	4	169·0827	1383·660	10	24·95073	25·68045	922·4402	18	3	38·52067	37·42610	736·5	25·31569	0·07011137	361·0769	361·0770
6	8	431·59	1765·923	11	45·82200	34·71583	1177·282	18	1·5	52·07375	68·73300	736·5	40·26892	0·08948094	450·0278	450·0279
7	20	1101·65	1803·034	11·5	73·42966	45·27708	1502·528	14·4	0·6	54·33250	88·11559	589·2	23·74135	0·04568073	519·7236	519·7238
8	100	9705·187	3176·831	11·9	172·3056	80·75156	2647·359	14·4	0·12	96·90187	206·7667	589·2	50·61143	0·0804865	628·8189	628·8189
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

employed to evaporate water in the boiler is afterwards evolved in the form of work, in consequence of the liquefaction, in the cylinder, of a portion of the expanding vapour. This fact would induce the hope that a great portion of the latent heat of evaporation, which is at present almost entirely lost, might by an increase of temperature, and by extending the principle of expansion, be converted into mechanical effect.

If, as would appear from the experiments of De la Rive and Marcet, Haycraft and Dulong, the capacity for heat of a given volume is the same in all gases taken at the same pressure and temperature, the results of the above Tables will be equally true whatever elastic fluid be employed.

It now only remains to offer a few observations, with a view to facilitate the labours of those who may be desirous of constructing a good practical air-engine.

It may be remarked, in the first place, that the receiver C need not be of much greater capacity than the cylinder B. For in the reciprocating engine, the air could be introduced from the pump A, at the same time that an equal amount would be expelled into the cylinder B. It would therefore be only requisite to pass the air through tubes heated by a proper furnace, as in Neilson's *hot-blast*, the tubes themselves constituting the receiver C. For a temperature under the red heat, these tubes might be constructed of wrought or cast iron. They might be either straight, like the tubes of a locomotive boiler, or arranged in the form of a coil, as represented by fig. 2, in which *a* is the pipe which conveys the air from the pump, *c*, *c*, *c*, &c. is the coil of wrought or cast-iron tubing, and *b* is the pipe which conveys the heated air to the cylinder. The coil is surrounded by a massive arch of brickwork, which serves at once to support the pipes, and to prevent waste of heat. To prevent the temperature exceeding the proper limits, the pipe *b* might, as it expands by the heat of the inclosed air, move a piece of mechanism in connection with the damper of the flue. I may remark that, on the scales adopted, fig. 2 represents the size of receiver which would be required for an engine the cylinder of which is 3 feet in diameter.

I would here venture to suggest whether the combustion of the fuel could not, by suitable mechanical arrangements, be carried on within the receiver C; if this could be accomplished, the heat, which in the form of receiver already described is lost up the chimney, would be economized, and a great saving of weight and space would be effected. An engine furnished with a receiver of this kind would be strikingly analogous to the electro-magnetic engine, and present a beautiful illustration of the evolution of mechanical effect from chemical forces.

In both of the above forms of receiver, it would be desirable, as already hinted, that the introduction of the air into the receiver should be simultaneous with the expulsion of the same quantity into the cylinder. This is necessary in order both to keep the pressure in the receiver uniform and to promote the smooth action of the engine. For this purpose the piston-rods of the pump and cylinder, *a* and *b* (fig. 3), must be attached to cranks on different parts of the circumference of the revolving shaft *c c*, so contrived that the piston shall arrive at the top or bottom of the cylinder

the moment that the pump-valve opens admitting a fresh supply of air into the receiver. The cylinder should of course be provided with proper expansion gear to cut off the air at the required part of the stroke, which must be a constant quantity for each engine. The valves of the pump would of course be self-acting.

In an engine similar to that described, it will be obvious that if the temperature of the receiver be kept constant, the pressure of air in it will also remain constant. For whilst the same quantity of air is always introduced into the receiver by each stroke of the pump, the quantity expelled out of it would increase with an augmentation and decrease with a diminution of pressure.

In conclusion, I would recommend the examples No. 3 and No. 5 of Table II. to the attention of those who may be willing to construct an air-engine. In both of these cases the capacity of the pump is two-thirds of that of the cylinder. In the cylinder of No. 3 the air is to be cut off at one-third of the stroke; and in that of No. 5 at one-sixth of the stroke. The temperature of the air in the receiver (supposing that of the atmosphere to be 32° Fahr.) is 625°·145 Fahr. in No. 3, and 924°·66 Fahr. in No. 5. The consumption of fuel in No. 3 need not exceed one-half, nor that in No. 5 one-third of that in the most perfect steam-engines at present constructed.

Acton Square, Salford, Manchester,
May 6, 1851.

Note to the foregoing Paper, with a New Experimental Determination of the Specific Heat of Atmospheric Air.

Received March 23, 1852.

Since the above was written, Professor W. H. Miller has directed my attention to the probable incorrectness of the value of k, as deduced from the experiments of Delaroche and Berard on the specific heat of air, and my own determination of the mechanical equivalent of heat; in comparison with the value deduced from the numerous and excellent experiments on the velocity of sound. Mr. Rankine considers that the discrepancy between the two values arises from the incorrectness of Delaroche and Berard's result, an opinion which would seem to be justified by the entire want of accordance between the determination of these philosophers, and those of Suermann, and Clement and Desormes. I have therefore been induced to make the following careful experiments in order to obtain a fresh and, if possible, more correct value of the specific heat of air at constant pressure.

The apparatus I employed is represented by fig. 4, in which a and b are two vessels, each of which contains a coil of leaden piping, eight yards long and one quarter of an inch in internal diameter. The coil of the upper vessel passes three-eighths of an inch through the bottom, to which it is soldered at c, and is thence connected with the coil of the lower vessel by a piece of vulcanized india rubber tubing. This part

of the apparatus will be better understood by a reference to fig. 5, in which a section of it is represented, *a* being the upper, *b* the lower vessel, and *w* the surface of the water in the latter. *x x* are a pair of wooden pincers by means of which the india rubber tube could be compressed so as to prevent, when desired, any communication between the air in the two coils of piping. Referring again to fig. 4, *g* is a gas-lamp to maintain the water in the upper vessel at a constant high temperature, and *j* is a tall jar filled with coarsely pounded chloride of calcium, in passing through which the air was entirely deprived of aqueous vapour; a length of vulcanized india rubber tubing, *p*, connects the coil of the lower vessel with a good air-pump, each barrel of which was found to have a capacity of 12·77 cubic inches. The temperature of the pump could be ascertained by means of a small thermometer, the bulb of which was kept in contact with one of the barrels.

The method of experimenting was as follows:—The lower vessel being filled with cold water, and the upper with water raised to about 190°, their exact temperatures were read off, with the usual precautions, from the scales of delicate and accurate thermometers. The pump was then worked at a uniform velocity for twenty-six minutes, the water in the lower vessel being agitated from time to time by a stirrer. The examination of the barometer and thermometers a second time occupied four minutes more; so that the whole time occupied by each experiment was exactly half an hour. The pincers were now applied so as to cut off all communication between the air in the two coils, and the effect of the various causes of a change of temperature in the lower vessel, unconnected with the current of heated air, was observed during another half-hour. Experiments of both the above kinds were repeated several times with the results tabulated below.

I may remark in this place that I had ascertained, by preliminary experiments, that the air passed from the coils of the vessels sensibly at the temperatures registered by the thermometers plunged into the surrounding water.

Series I.—Pump worked 26′, at the rate of twenty-four strokes per minute.

No. of Experiment.	Source of calorific effect.	Height of Barometer.	Temperature of Barometer.	Temperature of Air-pump.	Temperature of upper vessel.	Temperature of the room.	Temperature of the lower vessel.		Increase of temperature.
							Commencement of Experiment.	Termination of Experiment.	
1	Radiation		°	°	°	46°·081	41°·270	41°·814	0°·544
1	Heated air and radiation	30·195	46	49·3	189·28	46·188	41·814	42·802	0·988
2	Radiation					46·497	42·802	43·304	0·502
2	Heated air and radiation	30·205	46·75	50·3	189·43	46·785	43·304	44·246	0·942
3	Radiation					46·948	44·246	44·694	0·448
3	Heated air and radiation	30·22	47·5	51·1	189·89	47·068	44·694	45·590	0·896
4	Radiation					47·197	45·590	45·983	0·393
4	Heated air and radiation	30·235	48·	51·7	194·85	47·283	45·983	46·856	0·873
5	Radiation					47·455	46·856	47·211	0·355
Mean.	Heated air and radiation	30·214	47·06	50·6	190·862	46·831	43·949	44·874	0·925
Mean.	Radiation					46·836	44·153	44·601	0·448

It will be observed that the excess of the temperature of the room above the mean temperature of the water in the lower vessel, was, in the experiments with heated air, $2^{\circ}\cdot42$, but in the experiments on the effect of radiation $2^{\circ}\cdot459$. A comparison of the several experiments with one another, furnished the means of determining the amount of the small correction due to this circumstance. Hence $0^{\circ}\cdot925+0^{\circ}\cdot002-0^{\circ}\cdot448=0^{\circ}\cdot479$ will be the corrected mean increase of temperature due to the current of heated air. The material in which this increase took place consisted of 175500 grs. of water, 15635 grs. of copper, and 53370 grs. of lead, the whole having a capacity for heat equivalent to that of 178535 grs. of water. The volume of air passed through the pump was $12\cdot77\times26\times24=7968\cdot48$ cubic inches, which, at the observed barometric pressure and the temperature $50^{\circ}\cdot6$, would weigh $2537\cdot94$ grs. We have therefore for the specific heat of atmospheric air at constant pressure—

$$\frac{178535\times0\cdot479}{2537\cdot94\times146\cdot45}=0\cdot23008.$$

SERIES II.—Pump worked 26′, at the rate of forty strokes per minute.

No. of Experiment.	Source of calorific effect.	Height of Barometer.	Temperature of Barometer.	Temperature of Air-pump.	Temperature of upper vessel.	Temperature of the room.	Temperature of the lower vessel. Commencement of Experiment.	Termination of Experiment.	Increase of temperature.
			°	°	°	°	°	°	°
1	Radiation	....	....	....	....	47·223	44·200	44·648	0·448
1	Heated air and radiation ...	30·6	47·75	52	197·71	47·558	44·648	45·902	1·254
2	Radiation	....	....	....	....	47·841	45·902	46·319	0·417
2	Heated air and radiation ...	30·602	48·25	53·5	198·63	48·099	46·319	47·516	1·197
3	Radiation	....	....	....	....	48·339	47·516	47·860	0·344
3	Heated air and radiation ...	30·61	49·5	55·4	202·42	49·107	49·327	50·443	1·116
4	Radiation	....	....	....	....	49·524	50·443	50·728	0·285
4	Heated air and radiation ...	30·607	50·25	56·4	203·13	49·850	50·728	51·809	1·081
5	Radiation	....	....	....	....	50·030	51·809	52·037	0·228
Mean.	Heated air and radiation ...	30·605	48·94	54·32	200·472	48·653	47·755	48·917	1·162
Mean.	Radiation	....	....	....	....	48·591	47·974	48·318	0·344

In the above series $1^{\circ}\cdot162+0^{\circ}\cdot006-0^{\circ}\cdot344=0^{\circ}\cdot824$ will be the corrected mean increase of temperature due to the current of heated air. The material in which this increase took place consisted of 175000 grs. of water, 15635 grs. of copper, and 53370 grs. of lead, the whole having a capacity for heat equivalent to that of 178035 grs. of water. The volume of air passed through the pump was $12\cdot77\times26\times40=13280\cdot8$ cubic inches, which, at the observed barometric pressure and the temperature $54^{\circ}\cdot32$, would weigh $4252\cdot7$ grs. Hence we have for the specific heat—

$$\frac{178035\times0\cdot824}{4252\cdot7\times152\cdot136}=0\cdot22674.$$

By another series of experiments, in which the air-pump was worked at the velocity of twenty strokes per minute for twenty minutes, I obtained the value 0·2325. The mean of the three results is 0·22977, or nearly 0·23, which we may take as the specific heat of air at constant pressure determined by the above experiments.

Professor W. H. Miller has remarked that Moll's experiments, when correctly reduced, give a velocity of sound equal to 332·475 metres per second in dry air at 32°. Hence he deduces 1·41029 as the value of k. Calling it in round numbers 1·41, and the mechanical equivalent of heat 772, we obtain 0·238944 as the value of the specific heat of air at constant pressure, a result sufficiently near the experimental determination to show that the value of k, as deduced by Professor Miller, is much nearer the truth than that upon which the tables of the foregoing paper are founded.

The values of k, as determined by the experiments of Desormes and Clement, Gay-Lussac and Welter, and Mr. Meikle, referred to in the note to page 67, are respectively only 1·354, 1·375, and 1·333. In these experiments a small portion of air having been withdrawn from a large receiver, the equilibrium was re-established by opening for an instant a large aperture communicating with the external air, and then, after the receiver and its contents had regained their original temperature, the alteration of pressure, indicating the sudden rise of temperature which had taken place on the admission of the air, was noted. But it is obvious that the sudden admission of the air would cause the development of *sound*, and that, a portion of the *vis viva* escaping in this form, the increase of temperature and the deduced ratio of the specific heats would be diminished accordingly.

I subjoin Tables, similar to Tables I. and II., calculated from the data $k=1·41$, and the specific heat of air at constant volume $=0·169464$, or at constant pressure $=0·238944$.

In Table IV., the examples 9, 10 and 11 may be suggested to the notice of the practical engineer, the temperature of the receiver being in all those cases below that of redness. I may remind the reader that the Table is founded on the supposition that the air which enters the pump has 491° of temperature from the absolute zero, and that its pressure is 15 lbs. on the square inch. If this initial temperature be altered, the whole of the other temperatures in the Table must be altered in the same proportion, but the pressure, work and economical duty will remain unchanged. If the initial pressure be altered, all the other pressures and work will suffer a proportionate change, but the temperatures and economical duty will remain the same. The above are obvious deductions from the formulæ on which the Tables are founded.

Acton Square, Salford,
March 20, 1852.

Table III.

Distance traversed by piston, in inches.	Work absorbed, in foot-pounds.	Temperature from absolute zero, in degrees Fahr.	Pressure on the piston, in pounds avoirdupois.	Distance traversed by piston, in inches.	Work absorbed, in foot-pounds.	Temperature from absolute zero, in degrees Fahr.	Pressure on the piston, in pounds avoirdupois.
0	0	491	15	6·0	12·025096	652·3847	39·86055
0·1	0·1257463	492·6876	15·17803	6·1	12·36122	656·8958	40·81647
0·2	0·2529680	494·3950	15·35970	6·2	12·70547	661·5159	41·81223
0·3	0·3817395	496·1232	15·54513	6·3	13·05820	666·2498	42·85023
0·4	0·5120683	497·8723	15·73438	6·4	13·41977	671·1023	43·93309
0·5	0·6439993	499·6429	15·92769	6·5	13·79055	676·0785	45·06354
0·6	0·7775396	501·4351	16·12503	6·6	14·17096	681·1838	46·24464
0·7	0·9127567	503·2498	16·32661	6·7	14·56146	686·4245	47·47966
0·8	1·049665	505·0872	16·53253	6·8	14·96245	691·8060	48·77217
0·9	1·188301	506·9478	16·74292	6·9	15·37448	697·3358	50·12596
1·0	1·328719	508·8323	16·95793	7·0	15·79807	703·0207	51·54529
1·1	1·471827	510·7529	17·17773	7·1	16·23377	708·8680	53·03472
1·2	1·615031	512·6748	17·40240	7·2	16·68220	714·8862	54·59923
1·3	1·761007	514·6339	17·63216	7·3	17·14397	721·0835	56·24433
1·4	1·908922	516·6190	17·86716	7·4	17·61984	727·4700	57·97600
1·5	2·058809	518·6306	18·10755	7·5	18·11050	734·0550	59·80080
1·6	2·210724	520·6694	18·35353	7·6	18·61680	740·8498	61·72605
1·7	2·364719	522·7361	18·60529	7·7	19·13958	747·8659	63·75969
1·8	2·520828	524·8312	18·86299	7·8	19·67980	755·1160	65·91058
1·9	2·679114	526·9555	19·12686	7·9	20·23844	762·6133	68·18854
2·0	2·839636	529·1098	19·39710	8·0	20·81664	770·3732	70·60445
2·1	3·002421	531·2945	19·67392	8·1	21·41559	778·4115	73·17045
2·2	3·167547	533·5106	19·95757	8·2	22·03660	786·7459	75·90002
2·3	3·335073	535·7589	20·24830	8·3	22·68110	795·3954	78·80838
2·4	3·505041	538·0400	20·54632	8·4	23·35061	804·3808	81·91249
2·5	3·677529	540·3549	20·85193	8·5	24·04690	813·7254	85·23164
2·6	3·852596	542·7044	21·16540	8·6	24·77180	823·4542	88·78743
2·7	4·030240	545·0885	21·48700	8·7	25·52742	833·5950	92·60451
2·8	4·210744	547·5110	21·81705	8·8	26·31602	844·1786	96·71086
2·9	4·393947	549·9697	22·15585	8·9	27·14016	855·2390	101·1385
3·0	4·580211	552·4695	22·50375	9·0	28·00265	866·8144	105·9244
3·1	4·769047	555·0038	22·86105	9·1	28·90667	878·9468	111·1106
3·2	4·961064	557·5808	23·22823	9·2	29·85575	891·6840	116·7460
3·3	5·156197	560·1996	23·60557	9·3	30·85385	905·0792	122·8892
3·4	5·354524	562·8613	23·99352	9·4	31·90550	919·1930	129·6058
3·5	5·556132	565·5670	24·39246	9·5	33·01577	934·0936	136·9750
3·6	5·761092	568·3177	24·80292	9·6	34·19049	949·8591	145·0905
3·7	5·969523	571·1150	25·22533	9·7	35·43632	966·5791	154·0638
3·8	6·181561	573·9607	25·66015	9·8	36·76094	984·3564	164·0289
3·9	6·397243	576·8553	26·10795	9·9	38·17335	1003·312	175·1490
4·0	6·616735	579·8010	26·56929	10·0	39·68387	1023·584	187·6224
4·1	6·840106	582·7988	27·04473	10·1	41·30480	1045·338	201·6945
4·2	7·067466	585·8501	27·53490	10·2	43·05077	1068·770	217·6720
4·3	7·299071	588·9584	28·04045	10·3	44·93904	1094·112	235·9413
4·4	7·534902	592·1234	28·56207	10·4	46·99081	1121·648	256·9966
4·5	7·775119	595·3475	29·10049	10·5	49·23177	1151·723	281·4802
4·6	8·019950	598·6331	29·65651	10·6	51·69402	1184·768	310·2390
4·7	8·269468	601·9818	30·23092	10·7	54·41741	1221·318	344·4105
4·8	8·523854	605·3958	30·82462	10·8	57·45355	1262·065	385·5594
4·9	8·783274	608·8774	31·43856	10·9	60·86887	1307·901	435·8856
5·0	9·047890	612·4287	32·07366	11·0	64·75245	1360·021	498·5821
5·1	9·317890	616·0523	32·73102	11·1	69·22596	1420·059	578·4345
5·2	9·593477	619·7509	33·41175	11·2	74·46110	1490·318	682·9350
5·3	9·874827	623·5267	34·11703	11·3	80·71017	1574·184	824·4195
5·4	10·16216	627·3830	34·84815	11·4	88·36276	1676·887	1024·575
5·5	10·45581	631·3240	35·60646	11·5	98·06077	1807·041	1324·918
5·6	10·75570	635·3486	36·39342	11·6	110·9605	1980·164	1814·815
5·7	11·06235	639·4641	37·21060	11·7	129·4314	2228·056	2722·671
5·8	11·37594	643·6727	38·05962	11·8	159·4567	2631·016	4822·635
5·9	11·69678	647·9786	38·94231	11·9	223·8930	3495·794	12815·505

TABLE IV.

No. of Example.	Receiver C.			Pump of Compression A. Length 12 inches. Sectional Area = 1 square inch.				Cylinder of Expansion B. Sectional Area = 1 square inch.					Work evolved by the engine by each stroke of the piston, in foot-pounds.	Heat communicated to the air in receiver C, in degrees Fahr. per capacity of a lb. of water.	Work evolved out of each degree Fahr. in the capacity of a lb. of water, in foot-pounds.	Difference between the numbers in columns 4 and 13, divided by the numbers in column 4, and multiplied by the mechanical equivalent of heat.
	Density of the air, that of the atmosphere being called unity.	Pressure of the air in pounds on the square inch.	Absolute temperature of the air, in degrees Fahr. from the absolute zero.	Length of the first part of the stroke.	Work of the engine absorbed by the first part of the stroke of the piston, in foot-pounds.	Work of the engine absorbed by the second part of the stroke of the piston, in foot-pounds.	Absolute temperature of the air forced into the receiver C, in degrees Fahr. from the absolute zero.	Length of the cylinder B, in inches.	Length of the first part of the piston's stroke, in inches.	Work communicated to the engine by the first part of the stroke, in foot-pounds.	Work communicated to the engine by the second part of the stroke, in foot-pounds.	Absolute temperature of the air escaping into the atmosphere, in degrees Fahr. from the absolute zero.				
1	1	26·56929	869·7014	4	1·616735	7·71286	579·801	18	12	11·56929	2·425102	736·5	4·664797	0·03945268	118·2378	118·2377
2	1	70·60445	2311·119	8	10·81664	18·53482	770·3732	36	12	55·60446	32·44992	1473·0	58·70292	0·2096809	279·9632	279·9630
3	2	70·60445	1155·559	8	10·81664	18·53482	770·3732	18	6	27·80223	16·22496	736·5	14·67573	0·05242022	279·9632	279·9628
4	2	187·6224	3070·753	10	27·18387	28·7704	1023·584	36	6	86·3112	81·55161	1473·0	111·90854	0·2786002	401·6815	401·6817
5	4	187·6224	1535·3765	10	27·18387	28·7704	1023·584	18	3	43·1556	40·7758	736·5	27·97713	0·06965	401·6815	401·6815
6	8	498·5821	2040·032	11	51·00245	40·29851	1360·021	18	1·5	60·44776	76·50367	736·5	45·65047	0·092543	493·2893	493·2897
7	20	1324·918	2168·449	11·5	83·68577	54·57992	1807·041	14·4	0·6	65·4959	100·4229	589·2	27·65313	0·04918419	562·2364	562·2361
8	100	12815·505	4194·953	11·9	209·018	106·6709	3495·794	14·4	0·12	128·0051	250·8216	589·2	63·1378	0·0951489	663·5685	663·5692
9	3	105·92437	1155·7525	9	16·75265	22·73109	866·8144	16	4	30·30812	22·33687	654·666	13·16125	0·03932171	334·7069	334·7069
10	2·5	105·92437	1386·903	9	16·75265	22·73109	866·8144	19·2	4·8	36·36974	26·80424	785·6	23·69024	0·0707791	334·7068	334·7068
11	4·5	187·6224	1364·779	10	27·18387	28·7704	1023·584	16	2·$\frac{1}{3}$	38·36053	36·24516	654·666	18·65142	0·0464334	401·6814	401·6816
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Additional Note on the preceding Paper. By WILLIAM THOMSON, *M.A., F.R.S., F.R.S.E., Fellow of St. Peter's College, Cambridge, and Professor of Natural Philosophy in the University of Glasgow.*

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1. *Synthetical Investigation of the Duty of a Perfect Thermo-Dynamic Engine founded on the Expansions and Condensations of a Fluid, for which the gaseous laws hold and the ratio* (k) *of the specific heat under constant pressure to the specific heat in constant volume is constant; and modification of the result by the assumption of* MAYER'S *hypothesis.*

Let the source from which the heat is supplied be at the temperature S, and let T denote the temperature of the coldest body that can be obtained as a refrigerator. A cycle of the following four operations, *being reversible in every respect*, gives, according to CARNOT'S principle, first demonstrated for the Dynamical Theory by CLAUSIUS, the greatest possible statical mechanical effect that can be obtained in these circumstances from a quantity of heat supplied from the source.

(1.) Let a quantity of air contained in a cylinder and piston, at the temperature S, be allowed to expand to any extent, and let heat be supplied to it to keep its temperature constantly S.

(2.) Let the air expand farther, without being allowed to take heat from or to part with heat to surrounding matter, until its temperature sinks to T.

(3.) Let the air be allowed to part with heat so as to keep its temperature constantly T, while it is compressed to such an extent that at the end of the fourth operation the temperature may be S.

(4.) Let the air be farther compressed, and prevented from either gaining or parting with heat, till the piston reaches its primitive position.

The amount of mechanical effect gained on the whole of this cycle of operations will be the excess of the mechanical effect obtained by the first and second above the work spent in the third and fourth. Now if P and V denote the primitive pressure and volume of the air, and if P_1 and V_1, P_2 and V_2, P_3 and V_3, P_4 and V_4 denote the pressure and volume respectively, at the ends of the four successive operations, we have by the gaseous laws, and by POISSON'S formula and a conclusion from it quoted above, the following expressions:—

Mechanical effect obtained by the first operation $= PV \log \frac{V_1}{V}$.

Mechanical effect obtained by the second operation $= P_2V_2 \cdot \frac{1}{k-1} \cdot \left\{ \left(\frac{V_2}{V_1}\right)^{k-1} - 1 \right\}$.

Work spent in the third operation . . . . $= P_3V_3 \log \frac{V_2}{V_3}$.

Work spent in the fourth operation . . . . . . $=P_3V_3 \cdot \frac{1}{k-1}\left\{\left(\frac{V_3}{V_4}\right)^{k-1}-1\right\}$.

Now, according to the gaseous laws, we have

$$P_1V_1=PV; \quad P_2V_2=P_1V_1\frac{1+ET}{1+ES};$$

$$P_3V_3=P_2V_2; \quad \text{and (since } V_4=V\text{)}, \; P_4=P.$$

Also by Poisson's formula,

$$\left(\frac{V_2}{V_1}\right)^{k-1}=\left(\frac{V_3}{V}\right)^{k-1}=\frac{1+ES}{1+ET}.$$

By means of these we perceive that the work spent in the fourth operation is equal to the mechanical effect gained in the second; and we find, for the whole gain of mechanical effect (denoted by M), the expressions

$$M=(PV-P_2V_2)\log\frac{V_1}{V}=PV\log\frac{V_1}{V}\cdot\frac{E(S-T)}{1+ES}.$$

All the preceding formulæ are founded on the assumption of the gaseous laws and the constancy of the ratio (k) of the specific heat under constant pressure to the specific heat in constant volume, for the air contained in the cylinder and piston, and involve no other hypothesis*. If now we add the assumption of Mayer's hypothesis, which for the actual circumstance is $PV\log\frac{V_1}{V}=JH$, where H denotes the heat abstracted by the air from the surrounding matter in the first operation, and J the mechanical equivalent of a thermal unit, we have

$$M=JH\cdot\frac{E(S-T)}{1+ES}.$$

The investigation of this formula given in my paper on the Dynamical Theory of Heat, shows that it would be true for every perfect thermo-dynamic engine, if Mayer's hypothesis were true for a fluid subject to the gaseous laws of pressure and density, whether, for such a fluid (did it exist), k were constant or not.

It was first obtained by using, in the formula

$$M=JH\varepsilon^{-\frac{1}{J}\int_T^S \mu dt},$$

* From the sole hypothesis that k is constant for a single fluid fulfilling the gaseous laws, and having E for its coefficient of expansion, I find it follows, as a necessary consequence, that Carnot's function would have the form $\frac{JE}{1+Et+C}$; where C denotes an unknown absolute constant, and t the temperature measured by a thermometer founded on the equable expansions of that gas. From this it follows, that for such a gas subjected to the four operations described in the text, we must have $PV\log\frac{V_1}{V}=JH\frac{1+ES}{1+ES+C}$, and consequently, $M=JH\frac{E(S-T)}{1+ES+C}$, which is Mr. Rankine's general formula.

which involves no hypothesis, the expression

$$\mu=\frac{J}{\frac{1}{E}+t}$$

for CARNOT's function, which Mr. JOULE had suggested to me in a letter dated December 9, 1848, as the expression of MAYER's hypothesis, in terms of the notation of my "Account of CARNOT's Theory.*" Mr. RANKINE† has arrived at a formula agreeing with it (with the exception of a constant term in the denominator, which, as its value is unknown, but probably small, he neglects in the actual use of the formula), as a consequence of the fundamental principles assumed in his Theory of Molecular Vortices, when applied to any fluid whatever, experiencing a cycle of four operations satisfying CARNOT's criterion of reversibility (being in fact precisely analogous to those described above, and originally invented by CARNOT); and he thus establishes CARNOT's law as a consequence of the equations of the mutual conversion of heat and expansive power, which had been given in the first section of his paper on the Mechanical Action of Heat‡.

2. *Note on the Specific Heats of Air.*

Let N be the specific heat of unity of weight of a fluid at the temperature t, kept within constant volume, v; and let kN be the specific heat of the same fluid mass, under constant pressure, p. Without any other assumption than that of CARNOT's principle, the following equation is demonstrated in my paper§ on the Dynamical Theory of Heat, § 48,

$$kN-N=\frac{\left(\frac{dp}{dt}\right)^2}{\mu\times-\frac{dp}{dv}},$$

where μ denotes the value of CARNOT's function, for the temperature t, and the differentiations indicated are with reference to v and t considered as independent variables, of which p is a function. If the fluid be subject to BOYLE's and MARIOTTE's law of compression, we have

$$\frac{dp}{dv}=-\frac{p}{v};$$

and if it be subject also to GAY-LUSSAC's law of expansion,

$$\frac{dp}{dt}=\frac{Ep}{1+Et}.$$

* Royal Society of Edinburgh, January 2, 1849, Transactions, vol. xvi. part 5.

† On the Economy of Heat in Expansive Engines. Royal Society of Edinburgh, April 21, 1851, Transactions, vol. xx. part 2.

‡ Royal Society of Edinburgh, February 4, 1850, Transactions, vol. xx. part 1.

§ Royal Society of Edinburgh, March 17, 1851, Transactions, vol. xx. part 2.

Hence, for such a fluid,

$$kN-N=\frac{E^2pv}{\mu(1+Et)^2}\ *.$$

In the case of dry air these laws are fulfilled to a very high degree of approximation, and, for it, according to REGNAULT's observations,

$$\frac{pv}{1+Et}=26215, \quad E=\cdot 00366$$

(a British foot being the unit of length, and the weight of a British pound at Paris, the unit of force).

We have consequently, for dry air,

$$kN-N=\frac{26215E^2}{\mu(1+Et)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now it is demonstrated, without any other assumption than that of CARNOT's principle, in my "Account of CARNOT's Theory" (Appendix III.), that

$$\frac{E}{\mu(1+Et)}=\frac{H}{W},$$

if W denote the quantity of work that must be spent in compressing a fluid subject to the gaseous laws, to produce H units of heat when its temperature is kept at t. Hence

$$kN-N=26215E\times\frac{H}{W}=95\cdot 947\times\frac{H}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we adopt the values of μ shown in Table I. of the "Account of CARNOT's Theory," depending on no uncertain data except the densities of saturated steam at different temperatures, which, for want of accurate experimental data, were derived from the value $\frac{1}{1693\cdot 5}$ for the density of saturated vapour at 100°, by the assumption of the "gaseous laws" of variation with temperature and pressure; we find 1357 and 1369 for the values of $\frac{E}{\mu(1+Et)}$ at the temperatures 0 and 10° respectively; and hence, for these temperatures,

$$\left.\begin{array}{ll}(t=0) & kN-N=\dfrac{95\cdot 947}{1357}=\cdot 07071 \\ (t=10^\circ) & kN-N=\dfrac{95\cdot 947}{1369}=\cdot 07008\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a).$$

Or, if we adopt MAYER's hypothesis, according to which $\frac{W}{H}$ is equal to the mechanical equivalent of the thermal unit†, we have $\frac{W}{H}=1390$; and hence, for all temperatures,

$$kN-N=\frac{95\cdot 947}{1390}=\cdot 06903 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a').$$

* This equation expresses a proposition first demonstrated by CARNOT. See "Account of CARNOT's Theory," Appendix III. (Transactions Royal Society of Edinburgh, vol. xvi. part 5.)

† The number 1390, derived from Mr. JOULE's experiments on the friction of fluids, cannot differ by $\frac{1}{100}$, and probably does not differ by $\frac{1}{300}$, of its own value, from the true value of the mechanical equivalent of the thermal unit.

The very accurate observations which have been made on the velocity of sound in air, taken in connection with the results of REGNAULT's observations on its density, &c., lead to the value 1·410 for k, which is probably true in three if not in four of its figures. Now, k being known, the preceding equations enable us to determine the absolute values of the two specific heats (kN, and N) according to the hypotheses used in (a) and in (a') respectively; and we thus find,

	Specific heat of air under constant pressure (kN).	Specific heat of air in constant volume (N).	
for $t=0$,	·2431	·1724,	according to the tabulated values of CARNOT's function.
for $t=10$,	·2410	·1709,	
Or, for all temperatures,	·2374	·1684,	according to MAYER's hypothesis.

By the adoption of hypotheses involving that of MAYER, and taking 1389·6 and 1·4 as the values of J and k, respectively, Mr. RANKINE finds ·2404 and ·1717 as the values of the two specific heats.

Hence it is probable that the values of the specific heat of air under constant pressure, found by SUERMANN (·3046), and by DE LA ROCHE and BERARD (·2669), are both considerably too great; and the true value, to two significant figures, is probably ·24.

Glasgow College,
February 19, 1852.

VI. *On a General Law of Density in Saturated Vapours.* *By* J. J. Waterston, *Esq. Communicated by Lieut.-Col.* Sabine, *V.P. and Treas.*

Received June 19, 1851,—Read June 19, 1851.

THE relation between the pressure and temperature of vapours in contact with their generating liquids has been expressed by a variety of empirical formulæ, which, although convenient for practical purposes, do not claim to represent any general law. Some years ago, while examining a mathematical theory of gases, I endeavoured to find out from the French Academy's experiments, if the density of steam in contact with water followed any distinct law with reference to the temperature measured from the zero of gaseous tension. [By Rudberg's experiments, confirmed by Magnus and Regnault, this zero is $-461°$ in Fahr. scale, or $-273°{\cdot}89$ in the Centigrade scale. *Temperatures reckoned from this zero I shall call* G *temperatures to save circumlocution.*] If t represents the G temperature, Δ the density of a gas or a vapour, and p its elastic force, the equation

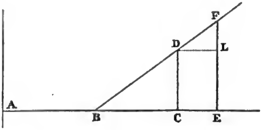
$$t\Delta=p \quad \ldots\ldots\ldots \quad (1.)$$

represents the well-known laws of Marriotte and of Dalton and Gay-Lussac. The function that expresses a general relation between p and t in vapours must include a more simple function, expressing a general relation between Δ and t. The proper course, therefore, seemed to be to tabulate the quotients $\frac{p}{t}$ from the experiments of the Academy and to project them into a curve. Now, for reasons connected with the *vis viva* theory of gases, which represents the G temperature as a square quantity, I projected these densities as ordinates to *the square root of the* G *temperatures as abscissæ*, and the curve traced out was of the parabolic kind, but of high power. To reduce this, because density is a cubic quantity, I tabulated their cube roots and set them off as ordinates to the same abscissæ. The result was gratifying, for the familiar conic parabola made its appearance. To ascertain whether this curve was exactly the conic parabola, I tabulated the square root of these ordinates, corresponding with the sixth root of the densities, and laid them off as new ordinates to the same abscissæ. The result is shown in the accompanying Chart, Plate VII., under the title *French Academy's Steam*. The observations are denoted by dots thus •, and it will be remarked that they range with great precision in a straight line, any slight divergence being sometimes to the right and sometimes to the left; precisely as might be expected from small errors of observation. Other series of experiments on steam were projected in a similar manner; and it was found that although no two exactly agreed with each other, yet that each set ranged in a straight line nearly.

The vapours of ether, alcohol, and sulphuret of carbon were tried in the same way, and found to conform to the same law. I have since added to the Chart M. AVOGADRO's observations on the vapour of mercury, which will be found remarkably in accordance; also Dr. FARADAY's experiments on liquefied gases, given in the Philosophical Transactions for 1845. Of these, olefiant gas (No. 1, p. 160) is remarkably in accordance; also the nitrous oxide (No. 2, p. 168), ammonia, cyanogen, sulphurous acid, and carbonic acid at the upper part of its range. Muriatic acid, sulphuretted and arseniuretted hydrogen do not show the same regularity.

The coordinates of the points being the square root of the G temperature and the sixth root of the densities, the equation to the straight line that passes through the points expresses the sixth root of the density in terms of the square root of the G temperature.

Thus let $t_1=$ the G temperature corresponding to Δ_1 density of vapour: set off $AC=\sqrt{t_1}$ and $CD=\sqrt[6]{\Delta_1}$. A second observation treated in the same way gives AE, EF, which determines the position of the line of vapour BF. Thus let $AC=g$, and $\cot FBE=h=\frac{DL}{FL}=\left\{\frac{\sqrt{t_2}-\sqrt{t_1}}{\sqrt[6]{\Delta_2}-\sqrt[6]{\Delta_1}}\right\}$ and $g=AB=\sqrt{t_1}-h\sqrt[6]{\Delta_1}$.



The constants g and h are thus determined from two observations, and the equation for the density Δ at any other G temperature is

$$\left\{\frac{\sqrt{t}-g}{h}\right\}^6=\Delta, \quad \ldots \quad (2.)$$

and for the pressure,

$$\left\{\frac{\sqrt{t}-g}{h}\right\}^6 t=p. \quad \ldots \quad (3.)$$

The following are the equations for the series of observations given on the Chart; the G temperatures are in degrees of FAHR. scale, and the values given to h are made to give the pressure in inches of mercury.

Mercury (AVOGADRO) . . . . $p=t\left\{\frac{\sqrt{t}-22\cdot606}{20\cdot00}\right\}^6$.

Oil of turpentine (URE).

Salt water saturated (WATT).

Water (French Academy and SOUTHERN) . $p=t\left\{\frac{\sqrt{t}-19\cdot492}{10\cdot830}\right\}^6$.

Alcohol, sp. gr. ·813 (URE) . . . . . . $p=t\left\{\frac{\sqrt{t}-19\cdot287}{9\cdot800}\right\}^6$.

Sulphuret of carbon (MARX) . . . . . . $p=t\left\{\frac{\sqrt{t}-16\cdot254}{12\cdot76}\right\}^6$.

Sulphuric ether (DALTON) . . . . $p=t\left\{\frac{\sqrt{t}-16\cdot860}{10\cdot990}\right\}^6$.

Sulphurous acid (FARADAY) . . . . . . $p=t\left\{\frac{\sqrt{t}-14\cdot667}{11\cdot194}\right\}^6$.

Cyanogen (FARADAY) . . . . $p=t\left\{\frac{\sqrt{t}-13\cdot846}{11\cdot542}\right\}^6$.

Ammonia (FARADAY) . . . . $p=t\left\{\frac{\sqrt{t}-13\cdot317}{11\cdot050}\right\}^6$.

Arseniuretted hydrogen (FARADAY) . . . $p=t\left\{\frac{\sqrt{t}-12\cdot929}{10\cdot264}\right\}^6$ (?).

Sulphuretted hydrogen (FARADAY) . . . $p=t\left\{\frac{\sqrt{t}-12\cdot957}{9\cdot878}\right\}^6$ (?).

Muriatic acid (FARADAY) . . . . $p=t\left\{\frac{\sqrt{t}-12\cdot060}{9\cdot413}\right\}^6$ (?).

Carbonic acid (FARADAY) . . . . $p=t\left\{\frac{\sqrt{t}-11\cdot997}{8\cdot857}\right\}^6$.

Nitrous oxide (FARADAY), No. 2, p. 168 . $p=t\left\{\frac{\sqrt{t}-8\cdot936}{11\cdot604}\right\}^6$.

Olefiant gas (FARADAY), No. 1, p. 160 . . $p=t\left\{\frac{\sqrt{t}-10\cdot352}{10\cdot152}\right\}^6$.

While projecting different sets of observations of the same vapour, the attention is forcibly drawn to the tendency which the line shows to alter its inclination on the most trifling change of circumstances. This inclination depends on the value of the denominator h. It will be remarked that this constant is nearly the same for the vapours of ether and of water, also for a considerable number of the liquified gases. In fact the difference is not greater than might be caused by a slight want of purity. Even the specific nature of the vessel in which the observations are made seems materially to affect the results. Dr. FARADAY employed tubes of from one-fifth to one-seventh inch bore. I believe it will be found that the density is considerably less in these tubes than in larger vessels; I have had proof of this in the case of ether heated in such tubes to 280°. The capillary attraction of the sides even at high pressures seems to affect the whole surface of the liquid, and to check the formation of vapour. We must not forget also, that Dr. FARADAY's temperatures are confessedly not so low as a correct thermometer would have indicated. The liquid he employed to measure the temperature was alcohol, and the degrees were graduated with the same capacity as between 32° and 212°. Now from the experiments that have been made on the expansion of alcohol by M. GAY-LUSSAC, and more recently by M. MUNCKE, it appears that the mean contraction between 32° and 212° is 6·2 thousandths of the volume at 173°, its boiling-point, and that this amount diminishes very sensibly with the temperature; and if it continued to do so at the same rate from 32° to −100°, we should have Dr. FARADAY's −100° equal to less than −200° by the air-thermometer. It is clear that until the value of these degrees are known with reference to the air-thermometer, the importance of Dr. FARADAY's observations cannot be fully estimated.

It will be observed from the Chart, that the observations on the vapour of water below 80° show a small excess of density above what is required by the line drawn through those at higher temperatures, and the contour of their projection is a curve convex to the axis, and very nearly the same as would be caused by the pressure of a very minute quantity of air; but the character of the late experiments of M. REGNAULT and MAGNUS forbid this mode of explaining the slight divergence. It is probable that it is connected with the change of condition that takes place at 32°, and the effects of which may be sensible up to 80° or 90°.

It is a curious circumstance that the law of expansibility of water also becomes disturbed at about the same temperature. As this has not been remarked, I have given the proof of it by projecting M. DESPRETZ's observations* on the Chart. If the volume is made ordinate to the square root of the G temperature as abscissa, these observations above 25° Centigrade or 77° FAHR., trace out *in the most exact manner a conic parabola.* The equation is

$$a(v-\theta)=(\sqrt{t}-\varphi)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

in which v is the volume of the G temperature t in terms of its volume unity at 39°·2 FAHR., or 4° Centigrade (its point of maximum density), $a=352{\cdot}38$, $\theta={\cdot}99872$, and $\varphi=21{\cdot}977$ or $\varphi^2=483°$.

In the Chart I have made $\sqrt{v-\theta}$ the ordinate, and $\sqrt{t}$ the abscissa. The proof of the position is, that in this mode of projection the observations range themselves exactly in a straight line from about 77° upwards†, and that below this temperature there is both less decrement of volume and less decrement of density than is required by the law that is followed above that temperature. Other liquids appear to follow the same rule of expansion, but the range of the observations is too small as yet to found upon them any general conclusion.

If these equations for the expansion of water and the density of its vapour hold good at high temperatures, they would have a common density at 1108° FAHR., its amount being $\frac{1}{1{\cdot}878}$.

The equation for the density of steam, at any G temperature under this, in terms of the density of water unity at 39·2, is

$$\Delta=\left\{\frac{\sqrt{t}-19{\cdot}492}{22{\cdot}2745}\right\}^6. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

M. CAGNIARD DE LA TOUR observed the sudden conversion of water into steam at a much lower temperature, but his observations are at variance with the laws of MARRIOTTE and DALTON, and GAY-LUSSAC; and although a slight divergence from these laws has been discovered by M. REGNAULT, it is quite in the opposite direction to that which M. CAGNIARD DE LA TOUR's observations require.

* Ann. de Chem. vol. lxx.

† If these ordinates are projected to ordinates $=t$ instead of $\sqrt{t}$, the line is a distinct and regular, though flat, curve convex to axis. This, if confirmed at high temperatures, will prove that the density of liquids as well as vapours has reference to the zero of gaseous tension.

VII. *On the Electro-Chemical Polarity of Gases.*

By W. R. GROVE, *Esq., M.A., F.R.S.*

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THE different effect of electricity upon gases and liquids has long been a subject of interest to physical inquirers. There are, as far as I am aware, no experiments which show any analogy in the electrization of gases to those effects now commonly comprehended under the term electrolysis. Whether gases at all conduct electricity, properly speaking, or whether its transmission is not always by the disruptive discharge, the discharge by convection, or something closely analogous, is perhaps a doubtful question; but I feel strongly convinced that gases do not conduct in any similar manner to metals or electrolytes.

In a paper published in the year 1849*, I have shown that hydrogen or atmospheric air intensely heated, showed no sign of conduction for voltaic electricity even when a battery of very high intensity was employed.

In the Eleventh, Twelfth and Thirteenth Series of FARADAY'S Experimental Researches, the line of demarcation between induction across a dielectric and electrolytic discharge is repeatedly adverted to; induction is regarded as an action of contiguous particles, and as a state of polarization anterior to discharge, whether disruptive, as in the case of dielectrics, or electrolytic, as in electrolytes. See §§ 1164—1298—1345—1368, &c.

Mr. GASSIOT, in a paper published in the year 1844†, has shown that the static effects, or effects of tension, produced by a voltaic battery, are in some direct ratio with the chemical energies of the substances of which the battery is composed; in other words, that in a voltaic series, whatever increases the decomposing power of the battery when the terminals are united by an electrolyte, also increases the effects of tension produced by it, when its terminals are separated by a dielectric.

In none of the above papers, and in no researches on electricity of which I am aware, is there any experimental evidence that the polarization of the dielectric is or may be chemical in its nature, that, assuming a dielectric to consist of two substances having antagonist chemical relations, as for instance, oxygen and hydrogen, the particles of the oxygen would be determined in one direction, and those of the hydrogen in the other; the only experimental result bearing on this point with which I am acquainted, is the curious fact which was observed by Mr. GASSIOT and

* Philosophical Transactions, 1849, p. 55.

† Philosophical Transactions, 1844, p. 39.

some other electricians who experimented with him in the year 1838, viz. that when two wires forming the terminals of a powerful battery were placed across each other, and the voltaic arc taken between them, the extremity of the wire proceeding from the positive end of the battery was rendered incandescent, while the negative wire remained comparatively cool; it was at that time believed that there was some effect exhibited here *extra* the voltaic circuit. Shortly afterwards I showed that with all, or at all events a great number of metals, the positive terminal was more heated than the negative, and that the portion of the crossed wire which was positive became more incandescent than that of the negative, from the greater heating effect developed at the point when the disruptive discharge took place. I suggested as an explanation of this phenomenon, the possibility that in air, as in water, or other electrolyte, the oxygen or electro-negative element was determined to the positive terminal, and that from the union of the metal with that oxygen a greater heating effect was developed. This, with some other impressions, I mentioned in a letter to my friend Dr. Schönbein, not intended for publication, but which shortly afterwards found its way into print*.

Though by no means thinking that this explanation was in every respect satisfactory, there were many arguments in its favour, and the fact strongly impressed my mind as evincing a very striking difference in character between the effect of the discharge at the positive and negative terminals, and as presenting, as far as it went, a distant analogy to the effect of electrolysis.

In the year 1848, while experimenting with Mr. Gassiot with a nitric acid battery consisting of 500 well insulated cells, I made the following experiment:—Two wires of platinum $\frac{1}{40}$th of an inch in diameter, forming the terminals of the battery, were immersed in distilled water; the negative wire was then gradually withdrawn until it reached a point a quarter of an inch distant from the surface of the water. A cone of blue flame was now perceptible, the water forming its base, and the point of the wire its apex; the wire rapidly fused, and became so brilliant that the cone of flame could be no longer perceived, and the globule of fused platinum was apparently suspended in air and hanging from the wire; it appeared sustained by a repulsive action like a cork ball on a *jet d'eau*, and threw out scintillations in a direction away from the water. The surface of the water at the base of the cone was depressed, and divided into little concave cups, which were in a continual agitation. When the conditions were reversed and the negative wire immersed, the positive wire being at the surface, similar phenomena ensued, but not nearly in so marked a manner; the cone was smaller, and its base much more narrow in proportion to its height.

This experiment, the beautiful effect of which requires to be seen to be appreciated, indicates a new mode of transmission of electricity partaking of the electrolytic and disruptive discharges. Not possessing a battery of this enormous intensity, I have not been able to examine this phenomenon more in detail; but I have from time to time

* Philosophical Magazine, 1840, p. 478.

made many other experiments on the voltaic arc taken in various gaseous media, with the view of ascertaining the state of the intervening media anterior to, during, and after the discharge; these experiments have hitherto given me no results of any value. In the voltaic arc, the intense heat developed so affects the terminals and so masks the proper electrical effect, that the difficulty of isolating the latter is extreme; and I have latterly sought for some modified form of electric discharge which should be intermediate between the voltaic arc and the ordinary Franklinic discharge, or that from the prime conductor of a frictional machine; for something, in short, which should yield greater quantitative effects than the electrical machine, but not dissipate the terminals, as is done by the voltaic arc.

An apparatus, to which M. DESPRETZ was kind enough to call my attention recently at Paris, seemed to promise me some aid in this respect. It was constructed by M. RUHMKORFF, on the ordinary plan for producing an induced current, viz. a coil of stout wire round a soft iron core, with a secondary coil of fine wire exterior to it, having an ingenious self-working contact breaker attached; from the attention paid to insulation in the construction of this apparatus, very exalted effects of induction could be procured. Thus in air rarefied by the air-pump, an aurora or discharge of 5 or 6 inches long could be obtained from the secondary coil, and in air of ordinary density a spark of one-eighth of an inch long.

I procured one of these apparatus from M. RUHMKORFF; the size of the coil portion of the apparatus is 6·5 inches long, 4 inches diameter; the length of the wires forming the coils are (I give M. RUHMKORFF's measurements) stout wire, 30 metres long, 2 millimetres diameter, 200 convolutions; fine wire, 2500 metres long, $\frac{1}{4}$ metre diameter, 10,000 convolutions. These measurements will only be taken as approximative, and indeed the exact size is immaterial to the consideration of the experiments which I am about to detail. I will not give my experiments in the order in which I made them, as I should have to describe many fruitless ones, but I will place first that which I consider the most important and fundamental.

1st. On the plate of a good air-pump was placed a silvered copper plate, such as is ordinarily used for Daguerreotypes, the polished silver surface being uppermost. A receiver, with a rod passing through a collar of leathers, was used, and to the lower extremity of this rod was affixed a steel needle, which could thus be brought to any required distance from the silver surface; a vessel containing potassa fusa was suspended in the receiver, and a bladder of hydrogen gas was attached to a stopcock, another orifice enabling me to pass atmospheric air into the receiver in such quantities as might be required*. A vacuum being made, hydrogen gas and air were allowed to enter the receiver in very small quantities, so as to form an attenuated atmosphere of the mixed gas: there was no barometer attached to my air-pump, but from separate experiments I found the most efficient extent of rarefaction for my purpose was that indicated by a barometric height of from half to three-quarters of

* See a figure and description of the apparatus at the end of this paper.

an inch of mercury; and except where otherwise stated, a similarly attenuated medium was employed for all the following experiments.

Two small cells of the nitric acid battery, each plate exposing 4 square inches of surface, were used to excite the coil machine, and the discharge from the secondary coil was taken between the steel point and the silver plate. The distance between these was generally $=0{\cdot}1$ of an inch, but this may be considerably varied. When the plate formed the positive terminal, a dark circular stain of oxide rapidly formed on the silver, presenting in succession yellow, orange and blue tints, very similar to the successive tints given by iodizing in the ordinary manner a Daguerreotype plate. Upon the poles being reversed and the plate made negative, this spot was entirely removed, and the plate became perfectly clean, leaving, however, a dark, polished spot occasioned by molecular disintegration, and therefore distinguishable from the remainder of the plate.

The experiment was repeated a great many times, and with varying proportions of gas, and I found that with proportions varying from equal volumes of hydrogen and air to those of one volume of the former to two and a half of the latter, the experiments succeeded; better, I should say, when there was rather an excess of hydrogen as compared with the equivalent of oxygen in the atmospheric air; about one volume of hydrogen to one and a half of air succeeded well; when excess of air was present, oxidation took place whether the plate was positive or negative, and when excess of hydrogen was present no oxidation took place.

2nd. I experimented with an air vacuum (to borrow an expression of Dr. FARADAY), and found that oxidation took place whether the plates were positive or negative, but in different degrees; when the plate was positive, a small circular spot was rapidly formed, quickly deepening in colour, and apparently eating into the plate; when the plate was negative, a large diffuse spot was formed, the oxidation was more slow, and the plate not so rapidly corroded.

3rd. I now operated with a hydrogen vacuum; when the plate was clean no discoloration took place, the plate retained its polish, though after a long continuance of the discharge a molecular change was perceptible, producing a frosted appearance similar to the mercurialized portions of a Daguerreotype.

When the plate had been previously oxidated by the discharge in an air vacuum, the oxidation was rapidly and beautifully cleared off by the discharge in the hydrogen vacuum, and this whether the plate was positive or negative, the effect being, however, better and more rapidly produced in the latter case.

4th. I substituted respectively for the steel needle, wires of copper, silver and platinum, and found the effect produced by all and with nearly equal facility; if there were any difference, the platinum point was the least efficient; this may be due to the peculiar effect of platinum in itself combining the gases, or to its inoxidable character, the oxygen being thrown off from its surface, and not uniting with it as with the more oxidable metals; the flame or luminous appearance which

surrounded the wire when the platinum was negative, was larger and more diffuse than with the other metals.

5th. As air, notwithstanding its containing a great excess of nitrogen, gave an effect of oxidation at both electrodes, though different in degree, I increased the proportion of nitrogen by passing into the receiver nitrogen which had been formed by the slow combustion of phosphorus, the phosphorous acid having been well washed away, and potash being always in the receiver; no more air was allowed to be present than the very small quantity contained in the apertures of the stopcock; with this mixture, viz. a maximum of nitrogen and a minimum of oxygen, and rarefied as before, a similar effect was produced to that shown in the mixture of air and hydrogen, the positive plate being oxidated by the discharge, and the spot when made negative being reduced. The effect of reduction was not so rapid or so readily produced as when hydrogen was used, but was very decided.

6th. With nitrogen, as much deprived of oxygen as I could procure, the colours of oxidation were not exhibited, but a dark spot apparently due to disintegration was produced, which was not removed by the plate being made negative; if, however, the coloured spot was produced by the plate being made positive in an air vacuum, they were removed by the plate being made negative in a nitrogen vacuum, leaving, however, a darker spot than that which was exhibited when they were reduced in hydrogen. Even when produced in an air vacuum, and then a very perfect exhaustion effected, such as would reduce the mercury in the barometer to the height of $\frac{1}{20}$th of an inch, the spot was partially reduced when the plate was made negative.

7th. An oxyhydrogen vacuum was formed, the gases being in the proportion in which they form water; and thanks to the attenuated atmosphere, it was easy to take the discharge in this mixture without producing detonation or any sudden combination of the gases, a possibility pointed out by Grotthus*. With this mixture the effect took place as with the mixture of atmospheric air and hydrogen. I expected it to have been more efficient, but it was rather less so than the mixture of air and hydrogen; whether it be that the presence of nitrogen lessens the tendency to combine of the gases oxygen and hydrogen, and thus enables the electrical polarization and discharge to operate more efficiently, whether the nitrogen has a specific effect in aiding the electro-chemical effect, as I have shown it has in one peculiar case†, or whether any unknown effect of nitrogen is concerned, I do not undertake to pronounce; I can only say that in several repetitions of the experiment, it appeared to me that the mixture of atmospheric air and hydrogen was more efficient in exhibiting this phenomenon than that of oxygen and hydrogen.

8th. Different proportions of oxygen and hydrogen were employed, and here also I found that within a tolerably wide margin I could vary the proportion of the gases; three volumes of hydrogen to one volume of oxygen I found to be a very efficient mixture.

* Annales de Chimie, vol. lxxxii.

† Philosophical Transactions, 1843, pp. 110, 111.

9th. I now substituted for the silver plate, plates of the following metals:—bismuth, lead, tin, zinc, copper, iron and platinum, the former three metals being burnished, the latter polished.

Bismuth showed the effect nearly, if not quite as well as silver; it was oxidated in an air vacuum, reduced in a hydrogen vacuum, and oxidated or reduced in the mixed gas according as it formed the positive or negative terminal.

Lead oxidated easily, but the spot of oxide could with difficulty be reduced. Tin, zinc and copper required the admission of a great quantity of air to produce oxidation; and I could not succeed in reducing the oxide by the electrical discharge, at least so as to restore the polish of the plate; a blackening effect was in some degree produced. Iron was not oxidated until the receiver was nearly filled with air, and then a small spot of rust was formed which I could not reduce. With all the metals a slight whitish film like the mercurialized portion of a Daguerreotype was visible beyond the circle marked by the discharge when the plate was rendered positive, which film was removed by negative electrolization in a hydrogen vacuum; it seemed to me that this film, as well as others among those I have described, was affected by light, but I did not turn aside to examine this effect. Platinum showed no effect either of oxidation or reduction.

10th. As it was impossible to operate with an atmosphere of chlorine with the apparatus which I possessed, and wishing to vary the electro-negative element, I iodized a silver plate by the vapour of iodine to a deep blue colour, and then made it negative in an atmosphere of hydrogen; the iodine was beautifully removed in a circle or disc opposite the point which formed the positive terminal.

11th. I now substituted for the coil apparatus a very good electrical machine, the cylinder of which was 16 inches diameter, and the prime conductor of which, when the machine was properly excited, gave a spark of 8 inches long. With this machine, and in an attenuated atmosphere of one volume hydrogen plus two of atmospheric air, I produced the effects of oxidation and reduction very distinctly, the plate being in turn connected with the conductor and with the ground; but the comparative minuteness of the spot after many turns of the machine, showed the great superiority of the coil machine for producing quantitative effects over the ordinary electrical machine; and I question whether I should have detected the phenomenon with the latter, had I not become previously well acquainted with it by the former apparatus. Probably an extensive series of the water battery or a steam hydro-electric machine would succeed equally well, or better than the coil machine.

12th. A solution of hyposulphite of soda removed the spots formed by electrization from the silver plate just as it removes the iodine from an iodized plate.

13th. In some of the above experiments I remarked a tendency in the spots produced by the discharge, to show circles or zones of oxidation in different degrees, and in a more marked manner than would be accounted for by the different colours of the thin films of oxide formed. I determined to examine this effect, and selected,

after some experiments, an atmosphere of one volume oxygen mixed with four volumes of hydrogen, and attenuated by the air-pump as in the previous experiments. The plate was made positive, and the point was placed successively opposite different portions of the silver plate, at distances of $\frac{1}{50}$th, $\frac{2}{50}$ths, $\frac{3}{50}$ths, $\frac{4}{50}$ths and $\frac{5}{50}$ths of an inch. The results are given, as nearly as I can copy them, in the accompanying Plate, figs. 1 to 5.

The colour of the central spot was a yellow-green in the centre, surrounded by a blue-green, then a clear ring of polished silver, then an outer ring crimson, with a slightly orange tint on the inner side, and deep purple on the outer; the exterior portion of the spot was, as far as my eye could judge, of a colour complementary to the interior of the external ring, and the central portion of the spot of a colour complementary to the exterior portion of the ring. The colours varied with the time, density of gas and other conditions, but generally showed this complementary tendency. Symptoms of a faint polished ring were visible beyond the outer ring, and could be rendered more distinct by breathing on the plate. As the distance between the point and the plate was increased the colours became fainter, and the rings more diffuse, and beyond the distance I have given nearly lost their defined character; but the first three distances, or those of $\frac{1}{50}$th, $\frac{2}{50}$ths and $\frac{3}{50}$ths of an inch, gave very beautifully defined rings. The luminous appearance on the needle in these experiments extended from three-fourths of an inch to an inch from the point. Frequently a small polished speck was visible, exactly opposite the point of the needle. See fig. 6. When the plate was made negative, the other conditions being the same, a polished space appeared opposite the point of the needle, surrounded by a dusky and ill-defined areola; its colour, when regarded from a point opposite the incident light, was brown tinged with purple; and when in the same direction as the light, a greenish white, similar to the tint seen on mildew or on some of the lichens: these spots were very different from the positive spots, and in some degree the converse of them; but they were not nearly so well defined or capable of being produced with the same uniformity. I have endeavoured to represent one of them at fig. 7.

14th. In order to ascertain whether the polished ring intervening between the oxidated central spot and oxidated external ring were a mere negation of effect or an antithetic polar effect, such as would occasion reduction, I formed in an air vacuum two large spots on a silver plate, with one the plate being made negative, and with the other positive, oxidating them until they began to pass from deep orange to purple. I then perfectly exhausted the receiver, swept it with the gas employed in the last experiment, and then took the discharge in a vacuum of that gas, viz. one volume oxygen+four hydrogen; the plate being positive and the needle $\frac{2}{50}$ths of an inch over the centre of each spot in turn, a ring of clear polish was formed rapidly in both the dark discs, just at the distance where the ring of polish appeared in the last experiment. I then exposed a clean portion of the plate to the needle without

any other change, and on allowing the discharges to pass, formed the rings just as in the last experiment.

15th. I examined some of the spots with an achromatic microscope, magnifying 200 diameters; I could not, however, discover any feature which the naked eye did not show, or any peculiar molecular state; the polishing scratches on the plate were highly magnified, but the electrized spots only showed more dimly the colours or the lights and shadows which they exhibited to the naked eye.

16th. I took the discharge on a silver plate in vacua of the following gases respectively:—Oxygen, protoxide of nitrogen, deutoxide of nitrogen, carbonic acid, carbonic oxide and olefiant gas.

The first four gases presented nothing remarkable, the plate was oxidated whether positive or negative, as in a vacuum of atmospheric air. In the protoxide of nitrogen the colour of the discharge was a beautiful crimson on both terminals.

In deutoxide of nitrogen a greater tendency to reduction was shown when the plate was negative than in the other three gases, and there was also a tendency to the formation of rings. In carbonic oxide the plate was oxidated when positive, and the oxide reduced when negative, just as with a vacuum of air and hydrogen, but rather more slowly; with a mixture of five volumes of carbonic oxide and one volume of oxygen, the rings were formed very distinctly, particularly if the plate was made negative first, and then positive. The luminous spot on the plate, when positive in this gas, was coloured green.

When the plate was negative in olefiant gas it darkened, showing the rings of colour produced by thin plates, and very distinct from the other rings of which I have spoken. After a short time a pulverulent deposit was formed on the plate, giving brilliant sparks or stars of light which were not shown by any other gas.

This deposit was too minute for analysis, but I have no doubt, from the gas used and the appearances presented, it was carbon.

I have given in the above experiments the conditions under which they succeeded best; but upon repetition, although the exact volumes of gases and other conditions were carefully attended to, they sometimes required a slight alteration to succeed, variations taking place from causes which I could not detect; thus it was sometimes necessary to add a little more hydrogen, sometimes a little more oxygen or air, to alter slightly the state of attenuation in the gas, &c.

The necessarily varying condition of the battery, and the state of the contact breaker, slight impurities in the gases or on the surface of the plates would be quite sufficient to account for these irregularities. I mention them for the guidance of any one who may wish to repeat the experiments; a very little practice will enable any electrician to have the results at his command. When there is too great a proportion of air or oxygen, oxidation takes place at both poles; when too much hydrogen, reduction takes place at both; and to effect oxidation or reduction by reversing the

direction of the discharge, an intermediate condition is requisite; so if the gas be not sufficiently attenuated the oxidation is too rapid, and the plate too much corroded to bring out the effects clearly; if too much attenuated, too long a time is required and the effect is feeble and indistinct.

I have above selected all the experiments which I consider material in this, I believe, new class of phenomena. The spots produced by electrical discharges, both on conducting bodies and on electrics, have been before noticed and experimented on, one class by PRIESTLEY*, and another class by KARSTEN† and others, but as far as I am aware no distinct electro-chemical action in dry gases, depending upon the antithetic state of the terminals and presenting a definite relation of the chemical to the electrical actions in gaseous media, has been pointed out. I now proceed to consider the relation which these results bear to other electrical phenomena.

As may be gathered from my opening remarks, the experiments above detailed appear to me to furnish a previously deficient link in the chain of analogy connecting dielectric induction with electrolysis. The only satisfactory rationale which I can present to my own mind of these phenomena is the following. The discharges being interrupted (as is evident from the nature of the apparatus, and may be easily proved by agitating a mirror near them and regarding their reflected images in the moving mirror), the gaseous medium is polarized anterior to each discharge, and polarized not merely physically, as is generally admitted, but chemically, the oxygen or anion being determined to the positive terminal or anode, and the hydrogen or cation being determined to the negative terminal or cathode; at the instant preceding discharge there would then be a molecule or superficial layer of oxygen or of electro-negative molecules in contact with the anode, and a similar layer of hydrogen or of electro-positive molecules in contact with the cathode, in other words, the electrodes in gas would be polarized as the electrodes in liquid are. The discharge now takes place, by which the superficial termini of metal or of oxide, as the case may be, are highly ignited or brought into a state of chemical exaltation at which their affinities can act; the anode thus becomes oxidated, and the cathode, if an oxide, reduced. I have elsewhere‡ shown strong reasons for assuming that the electric or voltaic discharge, the moment polarity is subverted, may be regarded as an intensely heated state of the electrodes, and of the intermedium across which it passes; and my present explanation is perfectly consistent with and derivable from my previous views of the disruptive discharge.

Two other theories might be proposed to account for the phenomena I am considering; the one, that the disruptive discharge itself is analogous to the electrolytic, and that the oxygen and hydrogen are reciprocally transferred by the discharge itself; this would not, I think, be consistent with the generally known facts connected with the discharge, and is entirely ineffectual in explaining the experiments

* History of Electricity, 2nd edition, p. 624.

† Archives de l'Electricité, vol. ii. p. 647; vol. iii. p. 310.

‡ Philosophical Transactions, 1847, pp. 10, 16, 21. Correlation of Physical Forces, p. 50, 2nd edition.

2nd and 3rd, where either the positive or negative terminal can be made either to oxidate or reduce, according to the nature of the chemical medium present, while these experiments are entirely in accordance with, and the results of them flow as a necessary consequence of, the view first advanced. The other theory which may be advanced is, that by dielectric induction the gases may be bodily separated, a layer, not molecular, but corporeal or voluminous, if I may be allowed these expressions, of oxygen being developed on the side next the anode, and one of hydrogen next the cathode, the gas intervening between the terminals being thus divided, as it were, into two halves: this would certainly be a most curious phenomenon, but I believe it to be so inconsistent with the vast mass of accumulated facts in electrical science, and likely to have produced in cosmical phenomena so many results which, if existing, must long ere this have been detected, that I will not do more than advert to it.

I have adopted the views which I have first stated as being the least removed from ordinary theories or modes of regarding electrical phenomena, and because in the present instance I can present the phenomena in no other way which is in the least degree satisfactory to my own mind, while this view to me well accounts for them. Assuming then for the present this view, we get a close approximation, I may say an identity of the state of polarization in gaseous non-conducting dielectrics, and in electrolytes anterior respectively to discharge or to electrolysis.

Faraday observes, Experimental Researches, 1164, "In an electrolyte induction is the first state, and decomposition the second." My present experiments show, I believe, that in induction across gaseous dielectrics there is a commencement, so to speak, of decomposition, a polar arrangement not merely of the molecules, irrespective of their chemical characters, but a chemical alternation of their forces, the electro-negative element being determined or directed, though *not travelling* in one direction, and the electro-positive in the opposite direction.

This arrangement is only evidenced at present, as it is in electrolysis, by the action at the polar extremities or termini of the dielectric; possibly future reasearches may show, by the action of polarized light, by magnetism or some other means of analysis, that the polarity extends, as we theoretically believe it does, through the whole intervening matter.

In the Experiment No. 5 with oxygen and excess of nitrogen, reduction takes place by the effect of negative electricity and heat, at least there seems every reason from analogy to believe that the effect of the nitrogen is only negative, protecting the plate from oxygen, or at furthest catalytic, aiding the reduction as sulphuric acid aids the electrolysis of water. Upon the state of association of the gases in what is generally called mixture, I venture an opinion with the greatest diffidence. I have always inclined to the opinion that the difference between physical admixture, as it is termed, of gases and chemical union, is one of degree, and the views of Dalton ever presented to my mind grave difficulties*. My present results seem to me in favour of

* Philosophical Transactions, 1843, p. 112.

the chemical view, as otherwise we can scarcely imagine electricity as effecting in the instances given a merely physical separation; it may indeed be said that there is composition and decomposition produced by the same discharge, but this is very difficult to conceive, and can hardly apply to the cases of oxygen with nitrogen and of carbonic oxide.

In the experiments I have detailed, the flame or visible effect of the electric discharge coincided with the chemical effect; when the plate was positive, a small globule of flame of a purple colour was visible on the part of the plate attacked, and a bluish flame extended over an inch or more of the needle. When the plate was negative, a wider and less defined disc of blue flame extended over the part of the plate opposed to the positive point, like a splash of liquid thrown upon it, and a pencil of light appeared on the point. Sometimes, but not always, this flame avoided the oxidated portion, probably from its inferior conducting power; and when this was the case reduction took place in a much slighter degree, or not at all; sometimes, and I observed this particularly with bismuth, the flame attached itself to the oxidated portion, and then reduction immediately followed. Here, as in all the electrical phenomena that I can call to mind, we get the visible effects of electricity associated with physical changes in the matter acting, changes of state in the terminals, polarization of the intervening medium, or both*. These experiments furnish additional arguments for the view which I have long advocated, which regards electricity as force or motion, and not as matter or a specific fluid†.

The chemical polarity of gases shown, as I believe, in this paper, associates itself with an experiment which I made known in a lecture at the London Institution in the year 1843‡, and which was subsequently verified by Mr. Gassiot§ with more perfect apparatus than I possessed, viz. that when discs of zinc and copper are closely approximated, but not brought in contact, and then suddenly separated, effects of electrical tension are exhibited, the one disc making the electroscope diverge with positive, and the other with negative electricity, showing that the effects ascribed by Volta to contact can be produced without contact, and by mere approximation, the intermediate dielectric being polarized, or a radiation analogous, if not identical, with that which produces the images of Moser taking place from plate to plate.

The present experiments also associate themselves with the gas battery, where, though an electrolyte is used as the means of making the action continuous, or producing what is called current electricity, the initiating effect is gaseous polarity, the films of gas in contact with the respective plates of platinum having antithetic chemical and electrical states.

* Gases at present believed to be elementary, probably undergo a *quasi* chemical polarization by electricity; thus portions of oxygen are changed to ozone, &c. See a recent paper by MM. Fremy and E. Becquerel, Comptes Rendus, Paris, March 15.—Note added to the Proof, W. R. G.

† Printed Lecture at the London Institution, 1842, p. 28. Correlation of Physical Forces, p. 48.

‡ Literary Gazette, 1843, p. 39.

§ Philosophical Magazine, October, 1844.

The results detailed in Experiment 13th, appear to open a new field of research. PRIESTLEY observed concentric circles produced by the electrical discharge from a powerful Leyden battery, which he describes as consisting of minute cavities and globules of fused metal*. In my experiments there is an alternation of oxidation and reduction, a medium capable of producing both being present; the lateral effect and complementary colours have to my mind something closely resembling the phenomena of interference in light, although from the polar character of the force, it is difficult to imagine any precisely analogous condition of electricity. The discharge taking place from different parts of the needle and extending from its point to a considerable distance over its surface, would give different lengths for the lines of polarization and discharge to the different parts of the disc on the silver plate affected by the discharge; and assuming electricity to be propagated by undulations, there would be interference; but instead of alternations of light and darkness we get alternations of positive and negative electricity. The ring of polished metal between the central spot and the exterior ring, quite distinguishes these rings from the ordinary colours of thin plates, *i. e.* colours, the annular succession of which depends only on the different thicknesses of the film; here doubtless the colours of the oxidated portions are colours of thin plates. Experiment 14 shows clearly that the action by which the polished ring is formed is a polar action of the discharge, and not a mere absence of action.

When the plate is negative, the effect is, as I have observed, less marked and more uncertain; but in this case it should be recollected that the visible discharge issues from the point, and does not extend, or extends to a very small degree, over the surface of the needle.

If the phenomena were such that the central portion were always clear, while around it was one, and one only circle of oxide, it might be accounted for by the hypothesis, that the lines of polarization and discharge between a point and flat surface, assume the form of a hollow cone; but a cone of negative bounded by cones of positive action, still gives the idea of some lateral fits or phases of undulation.

The high rarefaction of the medium by the discharge, and its intermitting character, might occasion pulsations by the inrushing of the surrounding gas, and thus vacua in circles might be formed at the places where the action of oxidation is rendered null; but this view is, I think, inadmissible; it does not account for the effects obtaining only in certain mixtures, it does not account for the reducing action, the plate being positive, and presents other difficulties. The point involved in Experiments 13 and 14, though not perhaps the least valuable one given in this paper, presents apparently a wide field of inquiry; I therefore will not further dilate on it at present, and hope to make it the subject of future investigation.

* History of Electricity, 2nd edition, p. 624.

December 27, 1851.

Postscript, April 24th.

I may, I trust, be permitted to add to this paper one or two experiments on the subject last discussed. Assuming that the alternations of oxidation and reduction were produced by interference in consequence of the discharge proceeding from successive points of the terminal or terminals, a difference of effect might be anticipated if the electricity passed from a point only, and not from a line as was the case in Experiment 13. I therefore sealed a platinum wire $\frac{1}{80}$th of an inch in diameter into a piece of glass tubing, and then ground the extremity to a flat surface, so that the section only of the wire was exposed; this wire was placed opposite, and at 0·07 of an inch distance from the polished silver plate, in a mixture of one volume of oxygen with five volumes of hydrogen attenuated until the barometer stood at half an inch; discharges from the secondary coil were then passed, the plate being positive, and a round dark spot of oxide formed represented at fig. 8; the platinum sealed in glass was then removed and the steel needle substituted for it, all else, viz. plate, gas, barometer height, &c., being the same: the system of rings represented at fig. 9 was now produced.

Another experiment was made, directed to the same point: a wire of copper 0·04 inch diameter, and a thread of glass of the same diameter were attached by sealing-wax at their extremities in a horizontal position 0·025 of an inch from different parts of a silver plate, being insulated from the silver by the wax interposed at the extremities. The gaseous mixture and barometric height being the same as in the last experiment, and the silver plate made positive, when the platinum wire sealed in glass was brought near the plate, and the discharges passed, a spot similar to fig. 8 was formed; but when the coated point of platinum was brought over the copper wire at 0·02 inch distance, a figure consisting of two separated semicircles was formed, having spots in the bisection of the chords, as shown at fig. 10, the portion between the spots and the semicircular line of oxide being of polished silver. With the glass thread the effect was the same, but produced with greater difficulty and not so well defined.

In many repetitions of these experiments which I have made, I have invariably produced the alternately polished and oxidated rings from the bare wire, and have not procured them from the coated wire, except to a very slight degree, and under certain circumstances, which, as far as I could trace, were as follows:—

1st. When the extremity of the wire was very near the plate, so that it had a sensible magnitude with reference to the intervening space, a slight formation of minute rings could be detected at the commencement of the experiment.

2nd. When the experiment was long continued, or when the coated platinum wire had been used for previous experiments, a set of rings, not consisting of an alternation of oxidated and polished rings, but of annuli of different degrees of oxidation, were formed.

When the experiment is continued for some time, a dark deposit is formed on the glass around the extremity of the platinum wire, giving an extended conducting surface; and this may be the reason why such rings are formed, though these rings, in all the cases which I have observed, differ broadly from the rings formed by the bare needle or wire, not having the interposed spaces of perfectly bright silver; and in all the cases the difference of effect produced by the coated and the bare wire is very marked; in by far the greater number of experiments, when proper precautions are taken, not the slightest formation of rings takes place with the coated wire; with the bare wire, in the gaseous mixture last mentioned, I have always seen them formed.

Thus there are three systems of rings which may be formed by the discharge. First, rings such as those seen in the ordinary cases of thin plates; these I have only observed with olefiant gas, though probably there are many other conditions in which they may be produced. Secondly, rings formed by the superposition of layers of oxides, possibly arising from the fact that at certain definite periods portions of the plate become by oxidation inferior conductors, and other portions are attacked, and being at a different distance undergo a different molecular change by oxidation. Thirdly, and to me far the most interesting set of phenomena are presented by the rings alternately bright and oxidated, showing effects of oxidation and reduction by the same current on the same plate, and which only take place in certain gaseous mixtures, of which, up to this time, one volume oxygen+five volumes hydrogen is the most efficient which I have obtained.

I cannot at present see any better mode of explaining these phenomena than by regarding them as analogous to the phenomena of interference in light, though doubtless if this be a right view, the very different modes of action of light and electricity would present very numerous phenomenal distinctions. Alternations of opposite polar electrical actions in the discharges passing in the same direction are, I think, very clearly shown in these experiments, and this appears to me a result worthy of attention.

Though acquainted with NOBILI's beautiful experiments on the formation of coloured rings by deposition in electrolyzed liquids, yet as I was working on gases it did not occur to me to refer to his memoirs*; I have done so since making the experiments given in this Postscript, and find that with regard to the rings so formed by electrolysis, he suggests interference as a possible explanation.

The dark space in the discharge to which FARADAY has called attention, may possibly be connected with these phenomena. I have observed, that in a well-exhausted receiver containing a small piece of phosphorus, the discharge is throughout its course striated by transverse non-luminous bands, presenting a very beautiful effect, and a yellow deposit, which, as far as I have yet examined it seems to be allotropic phosphorus, is deposited on the plate of the air-pump and on the neigh-

* Ann. de Ch. et de Phys. vol. xxxiv.

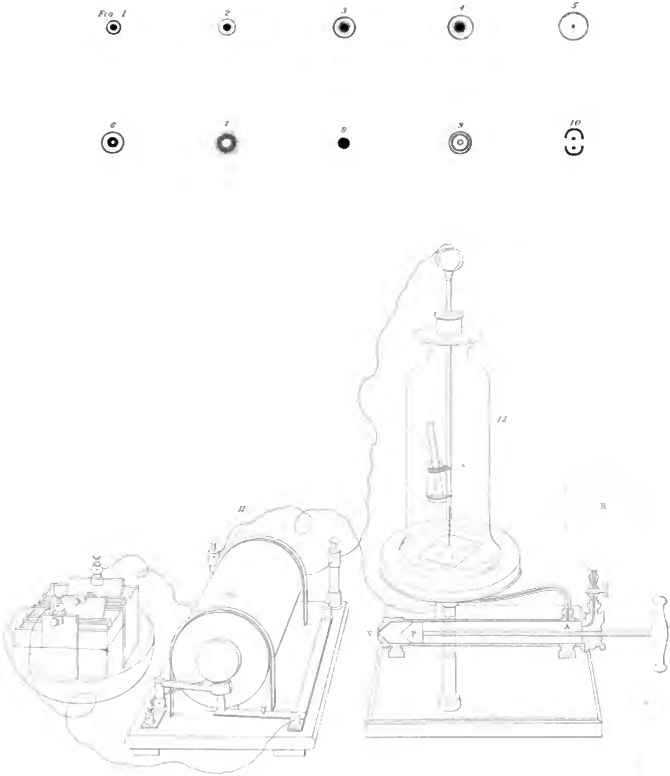
Fig. 1
2
3
4
5
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11
12
B
V
P
A

bouring substances; to show this effect well the needle should be positive and the plate negative, and the distance between them about an inch.

I could dilate much further on these experiments, but have already trespassed perhaps too far for a Postscript. Variations in the form of the terminals, in the nature of the gas, vapour, or gaseous mixture, in the density of the gas, in the intensity and quantity of the discharge, in the nature of the plate, &c. will occur to those who may feel inclined to repeat these experiments, and if I am not over-sanguine, promise results of much interest.

Description of Plate.

PLATE VIII.

Figures 1 to 10 show the spots and rings in the order referred to: it should be observed that printed figures give but a very imperfect notion of the actual effects.

Fig. 11 is the coil apparatus, the contact breaker being in front.

Fig. 12. The air-pump, of a construction which I proposed many years ago, and have found most useful for electrical or chemical experiments on gases.

P. An imperforate piston, with a conical end, which, when pressed down, fits accurately the end of the tube, the apex touching the valve V, which opens outwards.

A. Aperture for the air to rush from the receiver when the piston has been drawn beyond it.

B. Bladder containing the gas to be experimented on.

The piston-rod works air-tight in a collar of leathers, and the operation of the pump will be easily understood without further description.

If it be required to examine the gas after experiment, a bladder, or tube leading to a pneumatic trough, can be attached at the extremity over the valve V.

VIII. *On Periodical Laws discoverable in the mean effects of the larger Magnetic Disturbances.*—No. II. *By* Colonel Edward Sabine, *R.A., Treas. and V.P.R.S.*

Received March 18,—Read May 6, 1852.

In a former paper presented to the Royal Society in January 1851, bearing the same title as the present, I submitted to the Society the evidence afforded by the principal disturbances of the Magnetic Declination at Toronto and Hobarton in the years 1843, 1844 and 1845, of the existence of periodical laws by which their occurrence and mean effects appeared to be regulated. At the close of that paper I expressed the intention of submitting, on some future occasion, the results of a similar investigation into the periodical laws which might be expected to show themselves in like manner in the disturbances of the other two magnetic elements, viz. the Inclination and the Total Force.

Having since had occasion to examine the disturbances of the Declination at the same two stations in the three succeeding years 1846, 1847 and 1848, I have had the satisfaction of finding that the observations of these years confirm every deduction which I had ventured to make from the analysis of the disturbances of the former period; whilst new and important features have presented themselves in the comparison of the frequency and amount of the disturbances in *different years*, apparently indicating the existence of a *periodical variation*, which, either from a causal connection (meaning thereby their being possibly joint effects of a common cause), or by a singular coincidence, corresponds precisely both in period and epoch, with the variation in the frequency and magnitude of the solar spots, recently announced by M. Schwabe as the result of his systematic and long-continued observations. As facts and collocations of this description are of particular interest at the present moment, from their bearing on inquiries in which physical philosophers are engaged, I have deemed it better to communicate them at once, in the form of a second paper on the disturbances of the Declination, than to await the completion of the investigation into the laws of the disturbances of the Inclination and of the Total Force, for which I have not yet been able to command the necessary leisure.

The method pursued in examining the laws of the Declination-disturbances in 1846, 1847 and 1848, is the same as that adopted for the three preceding years and described in my former paper. The observations having been made hourly, every hourly observation which was found to differ by a certain prescribed amount* from the mean value of the Declination in the same month and at the same hour was separated from

* Philosophical Transactions, Part I. 1851, p. 127.

the rest, and a body of *disturbed observations* was thus collected; of which the recognised characteristic was simply that they were the disturbances of largest amount occurring in the whole period. The number of observations thus separated in the period commencing July 1, 1843 and ending July 1, 1848*, was at Toronto 3940, and at Hobarton 3469, being respectively 1 in 9·43 at Toronto and 1 in 10·55 at Hobarton of the whole number of hourly observations. The disturbed observations being next distributed into the several hours, months, and years in which they had occurred, their numbers and aggregate values in each particular hour, month and year, were ascertained. They were then divided into easterly and westerly deflections, and the same process of distribution was gone through with each of the two divisions. The *mean* hourly, monthly, and yearly numbers and aggregate values in the whole period were then taken as the respective units, and the ratios to these units computed for each of the hours, months and years; whereby the relations, whether of numbers or of aggregate values in different hours, different months, and different years, were shown. The numerical values of this analysis will be found in the second volumes of the Toronto and Hobarton Observations, and the ratios only are stated here, as it is only the relation *inter se* of the numbers and values, and not the absolute numbers and values themselves, which are required in this discussion.

I shall proceed to discuss separately,—1st, the Inequality or variation in the occurrence and aggregate values of the disturbed observations in *different hours of the day and night*; 2nd, *in different months of the year*; and 3rd, *in different years.*

I. Inequality or variation in the number and aggregate values of the disturbed observations in *different hours.*

This examination may be most conveniently made, by separating the disturbed observations at once into easterly and westerly disturbances, and classing together,—1st, easterly disturbances at Toronto and westerly at Hobarton; and 2nd, westerly at Toronto and easterly at Hobarton: an appropriate classification, as the stations are in opposite hemispheres, and one which will be seen to be justified by the characteristics which they respectively present.

The following Table exhibits the ratios of the numbers and aggregate values, at the different hours, in the five years from July 1843 to July 1848, of the easterly disturbed observations at Toronto and westerly at Hobarton, to the mean hourly number and aggregate value of the same taken as the respective units.

* The period for which the disturbances were examined in my former paper, was from January 1843 to December 1845 inclusive, making three complete years. For the present paper I have taken from July 1843 to June 1848 inclusive, making five complete years, of which 2½ are part of the period previously examined, and 2½ are new.

Table I.

Hours of local astronomical time.	Toronto.		Hobarton.		Hours of local astronomical time.
	Numbers.	Values.	Numbers.	Values.	
h					h
18	0·86	0·69	0·54	0·44	18
19	0·83	0·61	0·73	0·53	19
20	0·82	0·62	0·86	0·70	20
21	0·81	0·59	0·73	0·55	21
22	0·82	0·60	0·77	0·55	22
23	0·81	0·65	0·80	0·62	23
0	0·61	0·45	0·87	0·65	0
1	0·57	0·44	0·87	0·64	1
2	0·44	0·34	0·92	0·71	2
3	0·38	0·33	0·76	0·56	3
4	0·52	0·43	0·73	0·56	4
5	0·54	0·50	0·67	0·52	5
6	0·78	0·94	0·66	0·72	6
7	1·10	1·17	0·94	1·04	7
8	1·31	1·58	1·23	1·31	8
9	1·88	2·47	1·54	1·79	9
10	1·67	1·95	1·72	1·96	10
11	1·61	1·66	1·72	2·31	11
12	1·59	1·57	1·71	2·05	12
13	1·49	1·60	1·48	1·72	13
14	1·17	1·30	1·23	1·52	14
15	1·18	1·25	1·11	1·26	15
16	1·10	1·29	0·82	0·84	16
17	1·09	0·98	0·54	0·47	17

On examining this Table, we perceive,—

1st. That in this division of the disturbed observations, namely, those which are easterly at Toronto and westerly at Hobarton, there is a well-marked and systematic variation dependent on hour in the numbers of the disturbed observations and in their aggregate values. The most obvious feature is, that at both stations there are fewer disturbances, and their aggregate values are less in the hours of the day than in those of the night; and as the stations differ from each other nine hours in longitude, the dependence of the variation is on the hours of *local* and not on those of *absolute* time. If we divide the twenty-four hours at each station into four equal divisions according to the hours of local time, viz. 6 to 11 A.M., noon to 5 P.M., 6 P.M. to 11 P.M., and midnight to 5 A.M., the 1st and 2nd divisions being those of the day, and the 3rd and 4th those of the night, we find the mean ratios to be as follows,—the mean hourly number, and the mean hourly aggregate value in the whole period being still considered as the respective units.

Table II.

Hours of local time.		Toronto.		Hobarton.	
		Numbers.	Values.	Numbers.	Values.
Day ...	6 to 11 A.M. inclusive	0·82	0·63	0·74	0·57
	Noon to 5 P.M. inclusive ...	0·51	0·41	0·80	0·61
Night	6 P.M. to 11 P.M. inclusive...	1·40	1·63	1·30	1·52
	Midnight to 5 A.M. inclusive	1·27	1·33	1·15	1·31

2nd. With this marked and very striking correspondence in the diurnal variation of this branch of the larger disturbances at Toronto and Hobarton, both as respects frequency of occurrence and comparative value, we notice minor distinctive features, which, considering the number of years embraced in the inquiry and the systematic mode of observation pursued, may claim to be regarded as indications of persistent rather than of accidental differences. Thus 9 P.M. is at Toronto the hour of maximum frequency and value; both which maxima take place at Hobarton at 11 P.M., or two hours later. This feature is a well-marked one at both stations, and particularly in the aggregate values. In the period of the occurrences of the minimum of frequency and of value there is also a systematic difference, the period at Toronto being 2 and 3 P.M., and at Hobarton 5 and 6 A.M. The features of minimum are however less distinctly marked than those of maximum. As both the numbers and the aggregate values diminish concurrently, it is obvious that the minimum is ascribable chiefly to the diminished *frequency* of the disturbances at those hours; at Toronto 2 and 3 P.M. have decidedly the fewest easterly disturbances, and at Hobarton 5 and 6 A.M. as decidedly the fewest westerly.

It may be useful occasionally to bring into notice, concurrently with the variations of the numbers and aggregate values, the variation of the *average* values of the disturbed observations. The average values at the several hours are the quotients obtained by dividing the aggregate values by the numbers, and the average value in the twenty-four hours constitutes the unit of the ratios which show the variation at the different hours. These ratios, for the easterly disturbances at Toronto and westerly at Hobarton, are contained in Table III.

Table III.

Hours of local astronomical time.	Toronto. Easterly.		Hobarton. Westerly.		Hours of local astronomical time.	Toronto. Easterly.		Hobarton. Westerly.	
h					h				
18	0·85		0·87		6	1·26		1·16	
19	0·78		0·77		7	1·15		1·18	
20	0·80	0·81	0·85	0·81	8	1·23	1·23	1·14	1·22
21	0·77		0·80		9	1·39		1·23	
22	0·78		0·75		10	1·23		1·20	
23	0·85		0·82		11	1·09		1·43	
0	0·78		0·77		12	1·05		1·28	
1	0·81		0·77		13	1·13		1·23	
2	0·83	0·87	0·82	0·79	14	1·17	1·11	1·30	1·21
3	0·89		0·77		15	1·12		1·18	
4	0·89		0·82		16	1·23		1·09	
5	0·99		0·80		17	0·95		1·20	

It is here seen that the average value has a similar law of variation to that of the numbers and aggregate values: it is uniformly less during the hours of the day than in the hours of the night; and has a maximum at Toronto at 9 P.M. and at Hobarton at 11 P.M. The epoch of minimum is not strongly marked at either station.

We now pass to the westerly disturbances at Toronto and easterly at Hobarton,—and in Table IV. the ratios of the numbers and aggregate values are arranged opposite to the respective hours; the mean hourly number and aggregate value being taken as the respective units.

TABLE IV.

Hours of local astronomical time.	Toronto.		Hobarton.		Hours of local astronomical time.
	Numbers.	Values.	Numbers.	Values.	
h					h
18	1·17	1·53	0·94	1·02	18
19	1·27	2·16	1·33	1·53	19
20	1·46	1·87	1·41	1·58	20
21	1·39	1·69	1·41	1·41	21
22	1·40	1·33	1·30	1·27	22
23	1·27	1·04	1·32	1·24	23
0	1·10	0·89	1·23	1·14	0
1	0·89	0·66	1·38	1·26	1
2	0·75	0·64	1·38	1·32	2
3	0·99	0·74	1·35	1·40	3
4	0·96	0·76	1·30	1·39	4
5	0·91	0·74	1·16	1·32	5
6	0·75	0·63	1·01	1·16	6
7	0·63	0·51	0·64	0·62	7
8	0·61	0·46	0·45	0·40	8
9	0·60	0·48	0·29	0·32	9
10	0·48	0·54	0·36	0·28	10
11	0·66	0·52	0·88	0·74	11
12	0·73	0·81	0·74	0·62	12
13	0·99	1·10	0·64	0·55	13
14	1·21	1·08	0·70	0·63	14
15	1·23	1·06	0·86	0·85	15
16	1·38	1·25	1·09	1·07	16
17	1·19	1·51	0·67	0·87	17

On examining this Table, we perceive that in this division of the larger disturbances at Toronto and Hobarton, viz. those which are westerly at Toronto and easterly at Hobarton, there is also a very marked and systematic variation dependent upon the hours of local time, but the correspondence of the variation at similar hours of local time at the two stations is not so complete as in the former case. At Hobarton, the contrast both in frequency and aggregate value is still between the hours of the day and those of the night; at Toronto it is between the hours from noon to midnight, and those from midnight to noon. At Hobarton, the *nodal* hours, if they may be so called, are, both in the westerly and easterly disturbances, about 6 A.M. and 6 P.M.; at Toronto they are in the easterly disturbances, about 6 A.M. and 6 P.M.; but in the westerly, six hours different, or about noon and midnight. At Hobarton, in the westerly disturbances the ratios are greater than unity in the night hours, or from 6 P.M. to 6 A.M., and less than unity in the day hours, or from 6 A.M. to 6 P.M.; whilst in the easterly disturbance the converse takes place, the ratios being greater than unity in the day hours, and less than unity in the night hours. At

Toronto, in the easterly disturbances the ratios are greater than unity in the night hours, or from 6 P.M. to 6 A.M., and less than unity in the day hours, or from 6 A.M. to 6 P.M.; but in the westerly disturbances the ratios are greater than unity from midnight to noon, and less than unity from noon to midnight. Dividing the twenty-four hours into four six-hourly divisions as before, we have the mean ratios as follows:—

TABLE V.

Hours of local time.		Toronto.		Hobarton.	
		Numbers.	Values.	Numbers.	Values.
Day ...	6 to 11 A.M. inclusive	1·33	1·60	1·29	1·34
	Noon to 5 P.M. inclusive ...	0·93	0·74	1·30	1·31
Night	6 P.M. to 11 P.M. inclusive...	0·62	0·52	0·61	0·59
	Midnight to 5 A.M. inclusive	1·12	1·13	0·78	0·77

By means of this Table we may perceive more immediately, and therefore perhaps more distinctly, the respects in which the westerly disturbances at Toronto and the easterly at Hobarton agree, and those in which they differ. They agree in the ratios of frequency and aggregate value being above unity from 6 A.M. to noon, and below unity from 6 P.M. to midnight. They differ in the ratios being at Toronto above, and at Hobarton below unity, from midnight to 5 A.M., and at Toronto below, and at Hobarton above unity, from noon to 5 P.M.

The hours of maximum, 7 and 8 A.M., are the same at both stations; as are the hours of minimum, 8 to 10 P.M.

The ratios of the *average* values in this division of the disturbances are as follows:—

TABLE VI.

Hours of local astronomical time.	Toronto. Westerly.	Hobarton. Easterly.	Hours of local astronomical time.	Toronto. Westerly.	Hobarton. Easterly.
h			h		
18	1·33	1·09	6	0·84	1·14
19	1·75	1·17	7	0·84	0·98
20	1·31	1·12	8	0·78	0·87
21	1·40	1·01	9	0·82	1·09
22	0·97	0·98	10	1·14	0·77
23	0·84	0·96	11	0·81	0·85
0	0·84	0·93	12	1·13	0·85
1	0·76	0·90	13	1·14	0·87
2	0·88	0·96	14	0·91	0·90
3	0·78	1·04	15	0·88	0·98
4	0·81	1·06	16	0·93	0·98
5	0·84	1·14	17	1·31	1·33

Here also there is a partial accord and a partial difference between the stations. The ratios are above unity at both stations from 5 to 9 A.M. inclusive; they are also above unity at Toronto at midnight, at 10 P.M. and 1 A.M., and at Hobarton from

3 to 6 P.M. inclusive. On the whole, the average magnitude of a disturbance is greatest at both stations from 5 to 7 A.M.

The preponderance of easterly or westerly aggregate values at the different hours, shows the direction in which the magnet was deflected by the disturbances at the respective hours from the position in which it would have been found had they not occurred; and the ratio of preponderance shows the relative magnitude of the deflection. If we take the aggregate values of the westerly disturbances at Toronto and of the easterly at Hobarton at the different hours as the respective units at those hours, and compute the ratios which the easterly at Toronto and westerly at Hobarton at the same hours bear to them respectively, we obtain, as in the following Table, the ratios which at Toronto the aggregate values of the easterly disturbances bear to the westerly, and at Hobarton the westerly to the easterly, at the different hours of the day and night. When the ratios at Toronto are below unity, the mean deflection of the north end of the magnet at that hour is to the west, and when above unity, to the east. At Hobarton, the ratios which are less than unity indicate easterly deflections, and when above unity, westerly deflections.

Table VII.

Hours of local astronomical time.	Toronto. Ratios of easterly aggregate values to westerly.	Hobarton. Ratios of westerly aggregate values to easterly.	Hours of local astronomical time.	Toronto. Ratios of easterly aggregate values to westerly.	Hobarton. Ratios of westerly aggregate values to easterly.
h			h		
18	0·52	0·61	6	1·75	0·86
19	0·32	0·48	7	2·64	2·32
20	0·38	0·61	8	3·93	4·59
21	0·36	0·54	9	5·96	7·87
22	0·52	0·60	10	4·18	9·77
23	0·72	0·69	11	3·71	4·33
0	0·58	0·79	12	2·24	4·58
1	0·76	0·71	13	1·67	4·83
2	0·61	0·75	14	1·38	3·35
3	0·49	0·55	15	1·36	2·07
4	0·66	0·56	16	1·18	1·09
5	0·78	0·55	17	0·75	0·75

We have in this Table unmistakeable evidence of a variation, depending on the hour of local time, in the magnetic direction occasioned by the disturbances, and of a correspondence in the phenomena at the two stations indicative of a common law. During the hours of the day, or from 5 A.M. to 5 P.M. at Toronto and Hobarton, the deflection of the north end of the magnet occasioned by the disturbances is to the west at Toronto, and to the east at Hobarton. A little before 6 P.M. at Toronto, and a little after 6 P.M. at Hobarton, the deflections at both stations pass through zero, (or the undisturbed position of the magnet,) into deflections of the opposite character, becoming easterly at Toronto, and westerly at Hobarton. The magnitude of those deflections rapidly augments to a maximum, which is reached at Toronto at 9 P.M., and at Hobarton an hour later, from which hour it progressively diminishes

until between 4 and 5 A.M., when the deflections at both stations again pass through zero to a maximum of westerly deflection at Toronto, and of easterly at Hobarton, which occur at the same hour, 7 A.M., at both stations.

The mean ratios in each of the four divisions of the twenty-four hours are as follows:—

TABLE VIII.

Hours of local time.		Aggregate values.	
		Toronto. Easterly to Westerly.	Hobarton. Westerly to Easterly.
Day ...	6 A.M. to 11 A.M. inclusive...	0·44	0·59
	Noon to 5 P.M. inclusive ...	0·65	0·65
Night	6 P.M. to 11 P.M. inclusive...	3·70	4·96
	Midnight to 5 A.M. inclusive	1·43	2·78

Passing now for the moment, and in this particular case only, from ratios to *absolute values*, I have placed in the following Table the arc-values of the deflections of the north end of the magnet at Toronto and Hobarton, at the different hours, caused by the disturbed observations, and taken on a daily average during the whole period of observation.

TABLE IX.

Hours of local astronomical time.	Mean diurnal deflection caused by the disturbed observations.		Hours of local astronomical time.	Mean diurnal deflection caused by the disturbed observations.	
	Toronto.	Hobarton.		Toronto.	Hobarton.
h	′	′	h	′	′
18	0·24 West.	0·06 East.	6	0·14 East.	0·02 East.
19	0·48 West.	0·13 East.	7	0·28 East.	0·13 West.
20	0·37 West.	0·10 East.	8	0·46 East.	0·23 West.
21	0·33 West.	0·11 East.	9	0·78 East.	0·35 West.
22	0·21 West.	0·09 East.	10	0·56 East.	0·40 West.
23	0·10 West.	0·07 East.	11	0·46 East.	0·40 West.
0	0·13 West.	0·04 East.	12	0·33 East.	0·36 West.
1	0·05 West.	0·07 East.	13	0·24 East.	0·29 West.
2	0·08 West.	0·05 East.	14	0·14 East.	0·24 West.
3	0·12 West.	0·10 East.	15	0·13 East.	0·15 West.
4	0·08 West.	0·09 East.	16	0·08 East.	0·01 West.
5	0·05 West.	0·10 East.	17	0·13 West.	0·03 East.

The analogy of the two stations is generally so close as to give a greater importance than might otherwise be ascribed to the principal feature of difference in the diurnal progression, namely, that the nocturnal easterly deflection at Toronto precedes the westerly at Hobarton throughout by about an hour; it commences earlier, reaches its maximum earlier, and diminishes earlier.

II. Inequality or Variation in the numbers and values of the disturbed observations in *different months.*

The following Tables X. and XI. show the ratios of the numbers and aggregate values of the disturbed observations at Toronto and Hobarton in different months to the mean monthly number and aggregate value taken as the respective units.

TABLE X.—Toronto.

Months.	Easterly.		Westerly.		Easterly and Westerly combined.	
	Numbers.	Values.	Numbers.	Values.	Numbers.	Values.
January	0·60	0·59	0·71	0·60	0·65	0·59
February	0·69	0·78	0·81	0·80	0·74	0·79
March	1·02	1·09	1·02	0·98	1·02	1·04
April	1·20	1·31	1·27	1·38	1·24	1·34
May	1·10	1·03	0·99	0·94	1·05	0·99
June	0·88	0·77	0·75	0·52	0·82	0·66
July	1·22	1·12	1·01	0·88	1·12	1·01
August	1·42	1·37	1·29	1·08	1·36	1·24
September	1·65	1·63	1·47	1·74	1·57	1·69
October	1·01	1·15	1·09	1·19	1·04	1·17
November	0·66	0·64	0·85	0·83	0·75	0·73
December	0·55	0·53	0·74	1·05	0·64	0·77

TABLE XI.—Hobarton.

Months.	Westerly.		Easterly.		Westerly and Easterly combined.	
	Numbers.	Values.	Numbers.	Values.	Numbers.	Values.
January	1·82	1·54	1·65	1·62	1·74	1·58
February	1·16	1·05	1·21	1·16	1·18	1·10
March	1·04	1·11	1·14	1·11	1·08	1·12
April	1·02	1·18	1·10	1·26	1·05	1·22
May	0·53	0·51	0·62	0·65	0·57	0·57
June	0·37	0·32	0·32	0·30	0·35	0·31
July	0·47	0·54	0·50	0·51	0·49	0·53
August	0·78	0·73	0·86	0·84	0·82	0·78
September	1·14	1·50	1·35	1·29	1·24	1·41
October	1·23	1·27	1·24	1·22	1·24	1·25
November	1·11	0·95	0·79	0·73	0·97	0·86
December	1·30	1·29	1·23	1·29	1·27	1·29

It is obvious, on the mere inspection of these Tables, that there is a systematic variation in the numbers and aggregate values of the disturbances in the different months; and as at both stations the easterly and westerly ratios, separately considered, differ little, in the characters which they assign to the variation, from the ratios of the two combined, we may fix our attention chiefly on the two final columns of each table.

The most distinctly marked feature is that the disturbances are less frequent, and have a less aggregate value in November, December, January and February at Toronto,

and in May, June, July, and August at Hobarton than in the other eight months of the year. As we have before seen that in the *hours* of their occurrence the disturbances appear to be governed by a law depending on *local* hours, so here, we recognise *local* effects depending on the period of the year, and possibly the influence of *local* seasons (since we are scarcely yet in a condition to discriminate ás to causes). The mean monthly ratios in the different seasons are shown in the following Table, in which it will be remembered that November, December, January and February, are the winter months at Toronto, and May, June, July and August, at Hobarton.

TABLE XII.

Stations.	Winter.		Summer.		Spring and Autumn.	
	Numbers.	Values.	Numbers.	Values.	Numbers.	Values.
Toronto	0·70	0·72	1·08	0·96	1·22	1·31
Hobarton ...	0·56	0·56	1·28	1·21	1·16	1·24

It is seen that the mean ratios, both of frequency and of aggregate value, are much less in the winter months at both stations than in the months of summer or o spring and autumn, or, otherwise stated, least at Toronto in the months when th sun is in the southern signs, and least at Hobarton in the months when he is in th northern signs.

If we compare the ratios of the numbers in the different months with those of the aggregate values in the same months, we perceive that the *average* value of a disturbed observation is greater at both stations in the winter than in the summer months, and that it is greatest in the intermediate or equinoctial months. In these respects, and also in the circumstance of the minimum ratio, both of numbers and aggregate values, being in the midwinter month, (January at Toronto, and June at Hobarton,) the two stations agree. They differ 1° in the relative amount of the mean ratios in the months of summer and in those of spring and autumn, the mean ratios being decidedly greater at Toronto in spring and autumn than in summer, whilst at Hobarton there is no such marked difference: and 2° in the character of the progression which the ratios at the two stations indicate in the different months: at Toronto there are two decided minima, one in the midwinter, and the other in the midsummer, with a progressive increase on either side of the respective minima to April and September, which are the months of maximum disturbance: at Hobarton there is but one decided minimum, which is in the midwinter month; whilst in January, which is the month of midsummer, there is as decided a maximum in the ratios, both of numbers and values. At the same time there is so far an agreement with Toronto, that there is a tendency at Hobarton towards secondary maxima in April and September.

The points of accordance and points of difference thus noticed, are precisely the

same as were deduced in my former paper from the observations in 1843, 1844 and 1845. From the confirmation which has now been derived from the observations in the following years, 1846, 1847 and 1848, they may perhaps claim to be regarded as the exponents of persistent natural laws.

III. Variation in the numbers and aggregate values of the disturbed observations *in different years.*

Table XIII. exhibits the ratios of the numbers and aggregate values of the disturbed observations at Toronto and Hobarton in the different years, to the average annual number and aggregate value respectively*.

TABLE XIII.

Years.	Numbers.		Values.		Years.
	Toronto.	Hobarton.	Toronto.	Hobarton.	
1843.........	0·68	0·52	0·55	0·48	1843.
1844.........	0·76	0·81	0·73	0·82	1844.
1845.........	0·72	0·72	0·62	0·67	1845.
1846.........	1·31	1·09	1·26	1·03	1846.
1847.........	1·19	1·36	1·40	1·44	1847.
1848.........	1·37	1·50	1·43	1·60	1848.

On the first aspect of this Table, two features of principal interest present themselves; first, there is a considerable variation in the numbers and values of the disturbed observations in different years; and second, there is a remarkable correspondence in the variation in different years at the two stations.

Before we proceed to consider the first of these features, which is obviously one of great importance, it may be desirable, in reference to the correspondence at the two stations, to state more precisely than has been done previously, the degree of simultaneity at the two stations of the observations from which the conclusions are derived.

The weeks of observation commenced at both stations at midnight on the Sundays, and terminated an hour before midnight on the Saturdays. As these epochs were of *local* time, the week of observation commenced at Hobarton at 3^h, and at Toronto at 18^h of Göttingen time of the same astronomical day; and terminated at Hobarton at 2^h, and at Toronto at 17^h of Göttingen time. There were consequently fifteen hours at the commencement of each week (3^h to 17^h Göttingen) in which observations were made at Hobarton without simultaneous observations at Toronto; and fifteen hours at the termination of each week in which observations were made at Toronto without simultaneous observations at Hobarton. There were also nine hours in every week, between the conclusion of the one week at Toronto and the

* As 1843 and 1848 are only half years, the ratios are taken to the half average annual number and half average aggregate value in the five years.

commencement of the next at Hobarton (18^h at Göttingen to 2^h of the following day), in which no observations were made at either observatory.

In a week of seven days there are 168 hours, and deducting nine, there remain 159 hours in each week in which observations were made at one or other of the two observatories, from which the ratios in Table XIII. are derived. Deducting twice fifteen or thirty hours from the 159, we have 129 hours in each week in which observations were made simultaneously at the two observatories, and thirty hours in which they were made at one or other observatory, but not simultaneously at both. A still further small deduction would require to be made from the simultaneous portion, on account of the Good Fridays and the Christmas days, and of the observations accidentally missed. We may state, therefore, in round numbers, that about four-fifths of the whole number of observations which have contributed to form the ratios in Table XIII. were simultaneous at both stations, and that about one-fifth were not so.

This consideration being premised, we shall be inclined perhaps to regard the accordance in the ratios at the two stations in different years as being quite as near as could be expected, even on the extreme supposition which the case will admit, namely, that of *all* disturbances being *general.* That they are so *for the most part* at Toronto and Hobarton, may be concluded from the circumstance, that by far the greater part of the disturbances which form the subject of discussion in this paper, occurred *on the same days at the two stations.* This may be seen by comparing the Tables in the Abstracts prefixed to the second volumes of the Toronto and Hobarton Observations, in which the 3940 largest disturbances at Toronto, and the 3469 at Hobarton, are placed in separate tables, showing the day and hour of the occurrence, together with the direction and amount of each, for the purpose of facilitating their intercomparison, and of aiding generally in comparisons of a similar nature between the observations at these stations and the observations which by concerted arrangement may have been made simultaneously with them at other observatories. In all such comparisons the modifying influence of hours and periods of the year shown in the first and second sections of this paper, must be kept in view; and it must also be remembered that the evidence of the *general* character of these magnetic affections, which may be afforded by the comparison of the observations of the Declination alone, may be expected to be greatly strengthened when the disturbances of the Inclination and of the Total Force shall have been subjected to a similar process. The evidence furnished by a single element must necessarily be partial and incomplete.

Recurring now to the ratios in Table XIII., and directing our attention to the *character* of the inequality which they show to have existed in the amount of disturbance in different years, the facts which present themselves most obviously and unquestionably to our notice are, that in the years 1843, 1844 and 1845, the ratios were uniformly *considerably less than unity*, and that in the years 1846, 1847 and 1848, they were as uniformly *considerably greater than unity.* The mean ratios

in the three first years are, at Toronto 0·69, and at Hobarton 0·68; whilst in the three last years they are at Toronto 1·33, and at Hobarton 1·34. Facts of such remarkable character, evidenced by the independent and concurrent testimony of so large a body of observations at stations so widely distant from each other, seem to be well deserving the consideration of magnetical physicists; more particularly of those who are disposed to regard thermometrical differences as the cause of the periodical and other magnetic variations. The ratios of disturbance in the years 1846, 1847 and 1848, were nearly *twice as great* as in the years 1843, 1844 and 1845. Did there occur any notable differences of either local or general temperature, or thermometrical peculiarities of any description, in the years in question, to which variations of such magnitude in the ratios of magnetic disturbance can be ascribed, or with which they can be connected?

We should not however derive all the advantage which an examination of the ratios in Table XIII. seems suited to afford to those who desire to obtain an insight into the character of the variations they represent, were we to overlook the still more remarkable fact which they manifest, of a general, and with a single exception, uninterrupted *progressive increase* in the amount of disturbance from a minimum in 1843 to a maximum in 1848.

The interruption is in 1845, when the ratios, both of numbers and values, are less than in 1844. This interruption of the perfect continuity of the progression occurs alike at both stations; it is not of large amount, and is the sole exception to an otherwise continuous increase in the amount of disturbance during the years comprehended in this investigation.

The accordance with which the ratios at Toronto and Hobarton indicate this progression, is scarcely less remarkable than are the facts which they combine to indicate. It is indeed difficult to regard results so strikingly correspondent in any other light, than as independent and mutually corroborative measures of the same general phenomenon; and to view the inconsiderable differences between the ratios of the several years at the two stations as due either to accidents of observation, or to the want of strict simultaneity in all cases which has already been described. In such case a combination of the ratios obtained in opposite hemispheres would perhaps present a not improbable approximate view of the general variation in the amount of disturbance in the different years, occasioned in both hemispheres by the class of phenomena under notice. It is contained in the following Table.

TABLE XIV.—Mean of the ratios at Toronto and Hobarton.

Years.	Numbers.	Values.
1843.........	0·60	0·52
1844.........	0·78	0·78
1845.........	0·72	0·65
1846.........	1·20	1·15
1847.........	1·28	1·42
1848.........	1·43	1·51

The variation in the amount of disturbance in the different years presented in this Table, has certainly far more the aspect of a *periodical inequality*, than of what may be called for distinction's sake, *accidental* variation. The character, with the single exception already noticed, is that of an increase systematically progressive between the years 1843 and 1848. But the existence of a periodical inequality of this nature, affecting at the same time, and in the same manner, parts of the globe most remote from each other, would be a circumstance of such extreme importance in theoretical respects, that we are bound to receive the facts which may appear to indicate it with the utmost caution, and to await the confirmation it may obtain from contemporaneous observations at other stations. The magnetic disturbances present as well-marked and as notable features over the greater part of Europe as they do at Toronto and Hobarton; and there exist, or there should exist at those European observatories which have professed to adopt and carry out the system of observation proposed by the Royal Society, hourly or two-hourly observations, not only contemporaneous but simultaneous with those which have been discussed in this paper, and which, if examined, should yield corresponding conclusions, if the phenomena be general. In magnetical no less than in astronomical observations, the work of an observatory is but partially performed, until the observations have been subjected to processes of reduction, and their bearing on the points of theory for which they were instituted has been examined and shown.

Pending such confirmation, the general progressive increase in the amount of disturbance at Toronto and Hobarton, between the years 1843 and 1848, derives great additional interest and importance from its apparent connection with an equally remarkable progressive increase which took place at the same two stations, in the magnitude of the diurnal range of the Declination in the same years. The Tables in which the hourly observations at Toronto and Hobarton are recorded in the volumes of those observatories, exhibit for each month the mean monthly diurnal variation: the extreme east and west positions of the magnet at any two hours in these monthly means, show the mean magnitude or average range of the diurnal variation in that month. The subjoined Tables XV. and XVI. contain the mean magnitudes or ranges in the four months constituting the respective seasons, and in the twelve months constituting the year, in each year from 1843 to 1848.

Table XV.—Mean monthly diurnal range of the Declination at Toronto.

Years.	Winter.	Spring and Autumn.	Summer.	Mean in the whole year.
	November, December, January, February.	March, April, September, October.	May, June, July, August.	
1843.........	5′·64	9′·36	11′·70	8′·90
1844.........	5·70	8·74	12·17	8·87
1845.........	5·73	9·15	13·36	9·41
1846.........	6·33	9·21	12·27	9·27
1847.........	7·28	10·08	13·84	10·40
1848.........	9·48	11·04	15·82	12·11

TABLE XVI.—Mean monthly diurnal range of the Declination at Hobarton.

Years.	Winter.	Spring and Autumn.	Summer.	Mean of the whole year.
	May, June, July, August.	September, October, March, April.	November, December, January, February.	
1843.........	′ 4·50	′ 7·80	′ 10·16	′ 7·66
1844.........	4·30	8·45	10·77	7·84
1845.........	4·39	8·61	12·16	8·39
1846.........	5·10	9·50	12·58	9·06
1847.........	5·38	10·97	13·43	9·93
1848.........	7·09	10·67	14·14	10·63

We perceive by Tables XV. and XVI. that a generally and almost uniformly progressive increase took place at Toronto and Hobarton in the mean monthly range of the diurnal variation of the Declination between 1843 and 1848, contemporaneously with the increase in the amount of the disturbances produced by the variations which are of less regular occurrence and have distinct phenomenal laws. This coincidence appears to afford a more direct and decided indication of a causal connection subsisting between the two classes of phenomena than any which has previously presented itself.

It might be supposed that an increased amount of disturbance occurring in any year from the last named causes, *i. e.* the disturbances, might have a *direct* effect in increasing the diurnal range of the Declination; and such undoubtedly must be the case on individual days; but when *mean* values are in question, as is the case here, the difference in the mean monthly diurnal range, when the disturbed observations are retained, or when they are omitted, is scarcely sensible. The following Table contains the mean monthly diurnal range in the different years at Hobarton, when the 3469 disturbances of largest amount have been withdrawn. When compared with the values in Table XVI., where the disturbances are retained, the difference is seen to be wholly insignificant.

TABLE XVII.—Mean monthly diurnal range of the Declination at Hobarton, the 3469 disturbances of largest amount being omitted.

Years.	Winter.	Spring and Autumn.	Summer.	Mean of the whole year.
	May, June, July, August.	March, April, September, October.	November, December, January, February.	
1843.........	′ 4·49	′ 7·80	′ 10·67	′ 7·65
1844.........	4·19	8·29	10·62	7·70
1845.........	4·37	8·57	12·13	8·36
1846.........	4·59	9·26	12·76	8·87
1847.........	5·18	10·50	13·80	9·83
1848.........	7·02	10·53	14·19	10·58

Facts so remarkable as those presented in Tables XV. and XVI., showing in the course of six years a progressive increase in the range of the diurnal variation taken

from the monthly means of the observations in the several months, from 8′·90 to 12′·11 at one station, and from 7′·66 to 10′·63 at another station separated from the former by nearly half the surface of the globe, might naturally have created an expectation that they would prove to be independent and corresponding measures of a general phenomenon. Fortunately, in the case of the diurnal range, we have not to wait, as we have in the case of the disturbance-progression, for a confirmation of its extension to Europe. In a recent number of Poggendorff's Annalen, 1851, No. 12, December 23 (which only reached the author of this paper when the greater part of it was already written), Dr. Lamont has published a Table of the mean monthly range of the diurnal variation of the Declination at Munich, from 1841 to 1850 inclusive, from which he also has been led to infer the probable existence of a periodical inequality, having its epoch of minimum in 1843·5, and of maximum in 1848·5. The mean range of the diurnal variation in monthly periods at Munich in the years discussed in this paper, is stated in Dr. Lamont's communication to have been as follows:—

1843	7′·15
1844	6·61
1845	8·13
1846	8·81
1847	9·55
1848	11·15

The years which Dr. Lamont infers from the Munich observations to have been those which include the half-period of the inequality of the diurnal range, (or that portion of the period which is comprised between the epochs of minimum and maximum,) are precisely the same years over which my discussion of the disturbances has extended, and from which I have been led to infer the probable existence of a periodical inequality in those phenomena also, having the very same epochs of minimum and maximum. Dr. Lamont confines himself entirely to the diurnal inequality of the Declination, leaving untouched the subject of the disturbances (or, as they are more usually termed in Germany, the magnetic storms).

Whether the progressive increase so distinctly marked in the two classes of phenomena between the years 1843 and 1848 be or be not the result of causes which have a periodical action, in a cycle which may be either of regular or of variable duration, the fact of the progressive increase being concurrent in both classes is of no slight importance. It tends to indicate a causal connection subsisting between the disturbances and the diurnal variation; which latter, in addition to the laws which point directly or indirectly to the sun as the origin of its every-day phenomena, has other phases which mark unmistakeably, and at stations very variously situated in geographical and magnetical respects, the equinoxes as epochs of periodical change. The investigation, of which the results are contained in this paper, has shown that the disturbances have also a law of diurnal action, depending like that of the regular

diurnal variation on the hours of local time, but with different hours of maxima and minima; it has also shown, generally, that there is an influence connected with the period of the year on the frequency of occurrence of the disturbances of principal magnitude, affecting the aggregate effects of the disturbing causes; but it has not yet succeeded in tracing a definite epoch of semiannual change in the disturbances with the same precision as in the diurnal variation, of which the phenomena have been so much longer and so much more minutely studied, and in which the epoch of change in the phase depending on the earth's revolution in its orbit has been distinctly traced to the very days of the equinoxes*.

The progressive augmentation of the range of the diurnal variation between 1843 and 1848, is quite as conspicuous in the respective winters at Toronto and Hobarton as in their summers: the ratio of the increment of the range is even somewhat greater in winter than in summer. This has an important bearing when we regard the diurnal variation as divided into two portions, one depending on the earth's revolution on its axis, and the other on the earth's revolution in its orbit.

The extreme range of the periodical inequality of the diurnal variation from 1843 to 1848, *i.e.* the difference between the ranges of the diurnal variation in 1843 and 1848, is less at Toronto and Hobarton than the difference in the mean winter and summer ranges in any single year, *i.e.* than the inequality due to the position of the sun with reference to the equator.

That the progressive increase in the mean monthly diurnal range from 1843 to 1848, was not confined at Toronto and Hobarton to the Declination only, but took place likewise in the diurnal variations of the Inclination and Total Force, is shown in the subjoined Tables XVIII. and XIX., XX. and XXI., which appear to require no further explanation, as they are arranged on the same plan as Tables XV. and XVI.

Table XVIII.—Mean monthly diurnal range of the Inclination at Toronto.

Years.	Winter.	Spring and Autumn.	Summer.	Mean in the whole year.
	November, December, January, February.	March, April, September, October.	May, June, July, August.	
1843.........	′ 1·26	′ 1·40	′ 1·50	1·39
1844.........	0·78	1·39	1·59	1·25
1845.........	0·88	1·35	1·57	1·27
1846.........	1·09	1·59	1·92	1·53
1847.........	1·43	2·22	1·98	1·88
1848.........	1·73	2·17	2·18	2·03

* Magnetical and Meteorological Observations at the Cape of Good Hope, vol. i. p. 15, &c.; and at Toronto, vol. ii. Plate II. figs. 6 and 7.

TABLE XIX.—Mean monthly diurnal range of the Total Force at Toronto, in parts of the Force.

Years.	Winter.	Spring and Autumn.	Summer.	Mean in the whole year.
	November, December, January, February.	March, April, September, October.	May, June, July, August.	
1843.........	·00028	·00040	·00040	·00036
1844.........	·00027	·00059	·00039	·00042
1845.........	·00032	·00038	·00046	·00039
1846.........	·00020	·00059	·00077	·00052
1847.........	·00048	·00064	·00052	·00055
1848.........	·00061	·00075	·00054	·00063

TABLE XX.—Mean monthly diurnal range of the Inclination at Hobarton.

Years.	Winter.	Spring and Autumn.	Summer.	Mean in the whole year.
	May, June, July, August.	September, October, March, April.	November, December, January, February.	
1843.........	1′·43	1′·68	1′·58	1′·56
1844.........	1·39	1·76	1·72	1·62
1845.........	1·25	1·78	2·42	1·85
1846.........	1·62	2·04	2·36	2·01
1847.........	1·54	2·45	2·70	2·23
1848.........	1·92	2·40	3·27	2·53

TABLE XXI.—Mean monthly diurnal range of the Total Force at Hobarton, in parts of the Force.

Years.	Winter.	Spring and Autumn.	Summer.	Mean in the whole year.
	May, June, July, August.	September, October, March, April.	November, December, January, February.	
1843.........	·00057	·00043	·00049	·00050
1844.........	·00029	·00033	·00038	·00033
1845.........	·00031	·00031	·00036	·00033
1846.........	·00033	·00038	·00039	·00037
1847.........	·00031	·00040	·00049	·00040
1848.........	·00032	·00039	·00063	·00045

We perceive by these Tables that the indications of a periodical inequality are not less general in magnetical respects, *i. e.* in their extension to all the magnetical elements, than in their manifestation in parts of the globe most remote from each other. In the Inclination, the progressive increase is as distinctly marked as in the Declination; and if it is somewhat less so in the Total Force, this may with some probability be ascribed to the less perfect instrumental means from which the diurnal variation of that element is chiefly derived. In our present ignorance of the physical agency by which the periodical magnetic variations are produced, the possibility of the discovery of some cosmical connection which may throw light on a subject as

yet so obscure, should not be altogether overlooked. As the sun must be recognised as at least the *primary* source of all magnetic variations which conform to a law of local hours, it seems not unreasonable that in the case of other variations also, whether of irregular occurrence or of longer period, we should look in the first instance to any periodical variation by which we may learn that the sun is affected, to see whether any coincidence of period or epoch is traceable. Now the facts of the *solar spots*, as they have been recently made known to us by the assiduous and systematic labours of Schwabe, present us with phenomena which appear to indicate the existence of some periodical affection of an outer envelope, (the photosphere,) of the sun; and it is certainly a most striking coincidence, that the period, and the epochs of minima and maxima, which M. Schwabe has assigned to the variation of the solar spots, are absolutely identical with those which have been here assigned to the magnetic variations. In the third volume of Kosmos, page 402 (English translation, vol. iii. pp. 291 and 292), Baron von Humboldt has published a tabular abstract supplied by M. Schwabe, of the results of that gentleman's observations of the solar spots from 1826 to 1850; from which M. Schwabe has derived the conclusion, that "the numbers in the Table leave no room to doubt that, at least from the years 1826 to 1850, the solar spots have shown a period of about ten years, with maxima in 1828, 1837 and 1848, and minima in 1833 and 1843." "In almost all the years except those of the minima," M. Schwabe says, "I have observed large spots visible to the naked eye,—I mean spots whose diameters are above 50″:" and 1847 and 1848 are enumerated amongst the years in which the largest spots appeared. M. Schwabe's Table is as follows:—

Table XXII.

Years.	Groups of spots.	Days free from spots.	No. of days of observation.
1826.........	118	22	277
1827.........	161	2	273
1828.........	225	0	282
1829.........	199	0	244
1830.........	190	1	217
1831.........	149	3	239
1832.........	84	49	270
1833.........	33	139	267
1834.........	51	120	273
1835.........	173	18	244
1836.........	272	0	200
1837.........	333	0	168
1838.........	282	0	202
1839.........	162	0	205
1840.........	152	3	263
1841.........	102	15	283
1842.........	68	64	307
1843.........	34	149	312
1844.........	52	111	321
1845.........	114	29	332
1846.........	157	1	314
1847.........	257	0	276
1848.........	330	0	278
1849.........	238	0	285
1850.........	186	2	308

M. Schwabe has not been able to derive from the indications of the thermometer or barometer any sensible connection between climatic conditions and the number of spots. The same remark would of course hold good in respect to the connection of climatic conditions with the magnetic inequalities, as their periodical variation in different years corresponds with that of the solar spots. But it is quite conceivable that affections of the gaseous envelope of the sun, or causes occasioning those affections, may give rise to sensible *magnetical* effects at the surface of our planet, without producing sensible *thermic* effects.

It may be confidently anticipated that so remarkable a coincidence in the degree of energy with which the causes producing obscurations in the luminous disc of the sun, and those producing the magnetic variations at the surface of our planet, appear to have acted in the different years between 1843 and 1848, will receive due attention at those observatories which, by their more permanent character, are more particularly adapted for the investigation of problems requiring several years for their solution.

As the physical agency by which the phenomena are produced is in both cases unknown to us, our only resource for distinguishing between accidental coincidence and causal connection seems to be *perseverance in observation*, until either the inferences from a possibly too limited induction are disproved, or until a more extensive induction has sufficed to establish the existence of a connection, although its precise nature may still be imperfectly understood. For such continued investigation we must look to those observatories which are permanent in their institution; and in this particular problem, to those especially which combine astronomical and magnetical research. The hourly observations at the British Colonial Observatories, which, combined with M. Schwabe's observations of the sun in Germany, have led to the discovery of the existence of the coincidence during the years 1843–1848, ceased in 1848, having accomplished the special objects for which they were instituted. There are yet remaining for analysis, in reference to the disturbance-variations, the hourly observations at Hobarton in 1841 and 1842, and the 2-hourly at Toronto in the same years, which will show whether the aggregate values of the disturbances were greater in those years than in 1843, as they should have been in conformity with a periodical inequality having 1843 as a minimum epoch.

Woolwich, March 16, 1852.

Postscript, May 24th, 1852.

The interval which has elapsed between the presentation and the printing of this paper has enabled me to complete the examination of the disturbances at Toronto from the commencement of the observations in 1841, and to subjoin the results. The observations were made two-hourly from January 1841 to June 1842 inclusive, and hourly from July 1842 to July 1848 inclusive: all those which differed 3·6 or upwards from the mean or normal Declination at the same hour in the same month have been separated from the others, and constitute the body of disturbed observations submitted to examination. It appears desirable to state, in addition to the ratios, the numbers and aggregate values of the disturbed observations in each year, dividing them into easterly and westerly disturbances: this is done in the following Table:—

Table XXIII.

Years.	Disturbed observations.					
	Numbers.			Aggregate values.		
	Easterly.	Westerly.	Total.	Easterly.	Westerly.	Total.
1841. Twelve months: observations two-hourly	282	288	570	1865'	1750'	3615'
1842. {January to June, two-hourly / July to December, hourly}	327	279	606	1947	1623	3570
1843. Twelve months; hourly	268	204	472	1515	1133	2648
1844. Ditto Ditto	327	269	596	2162	1692	3854
1845. Ditto Ditto	298	269	567	1761	1544	3305
1846. Ditto Ditto	547	484	1031	3655	3002	6657
1847. Ditto Ditto	532	409	941	3620	3804	7424
1848. January to June, hourly	288	250	538	2185	1609	3794

It will be observed that in this Table the values from 1843 to 1847 inclusive, are strictly intercomparable, those years being complete years of hourly observation; whereas in 1841 and in part of 1842 the observations were only two-hourly, and in 1848 the observations were limited to the first six months. In order therefore to render the numbers and aggregate values in Table XXIII. readily comparable by the eye, it may be convenient to double those of 1841 and 1848, and to augment those of 1842 in the proportion of 4 to 3: this is done in the following Table:—

Table XXIV.

Years.	Numbers.			Aggregate values.		
	Easterly.	Westerly.	Total.	Easterly.	Westerly.	Total.
1841.	564	576	1140	3730'	3500'	7230'
1842.	436	372	808	2596	2164	4760
1843.	268	204	472	1515	1133	2648
1844.	327	269	596	2162	1692	3854
1845.	298	269	567	1761	1544	3305
1846.	547	484	1031	3655	3002	6657
1847.	532	409	941	3620	3804	7424
1848.	576	500	1076	4370	3217	7587

Assuming the existence of a periodical inequality, such as has been described in the paper to which this Postscript is annexed, we may take the means of the numbers and values from 1843 to 1848 inclusive, (constituting approximately the half period of the supposed inequality,) as the units of ratios, which will show with more convenient simplicity the variation of the numbers and aggregate values in the different years. These are contained in Table XXV.

TABLE XXV.

		Numbers.			Aggregate values.		
		Easterly.	Westerly.	Total.	Easterly.	Westerly.	Total.
Units.		424·7	355·8	780·5	2847′	2399′	5246′
Ratios.	1841.	1·33	1·62	1·46	1·31	1·46	1·38
	1842.	1·03	1·05	1·04	0·91	0·90	0·91
	1843.	0·63	0·57	0·60	0·53	0·47	0·50
	1844.	0·77	0·76	0·76	0·76	0·71	0·73
	1845.	0·70	0·76	0·73	0·62	0·64	0·63
	1846.	1·29	1·36	1·32	1·28	1·25	1·27
	1847.	1·25	1·15	1·21	1·27	1·59	1·42
	1848.	1·36	1·40	1·38	1·54	1·34	1·45

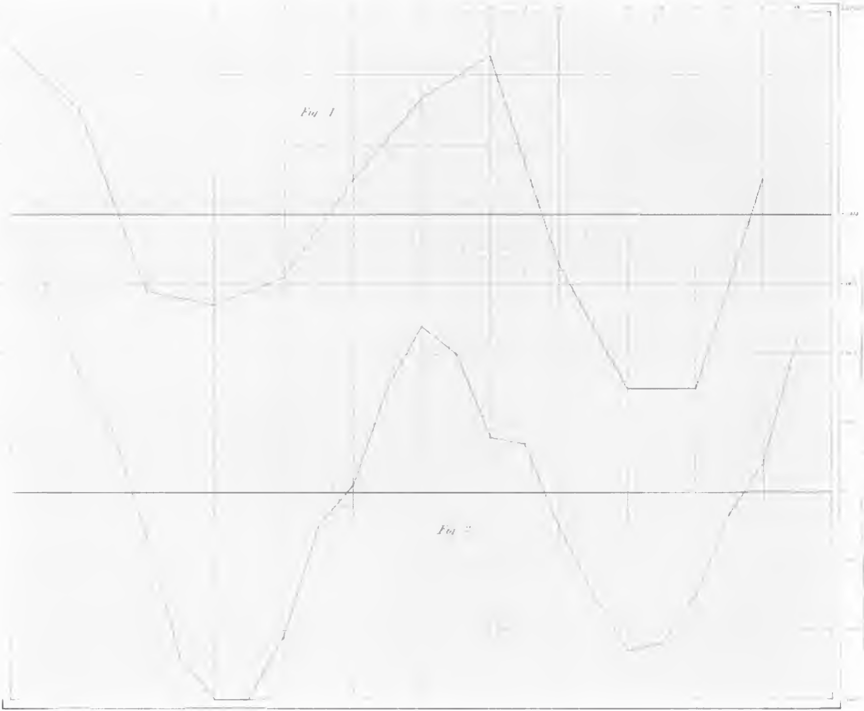
It is seen by this Table that the ratios, both of numbers and values, in the years 1841 and 1842 are greater than in 1843, and that 1843 is in every respect, whether of numbers or values, easterly or westerly, the minimum of the whole series. The ratios of the numbers and values in 1841 are greater than in 1842, presenting the same appearance of a periodical inequality previous to 1843, as in the years subsequent to that date.

It may be desirable to subjoin also in one view the inequality in the range of the Diurnal Variation at Toronto and Hobarton from 1841 to 1851 inclusive, and at Munich from 1841 to 1850 inclusive, the latter being taken from the Number of POGGENDORFF's Annalen, already referred to.

TABLE XXVI.

Years.	Toronto.				Munich.			Hobarton.			
	Nov. Dec. Jan. Feb.	Mar. Apr. Sept. Oct.	May, June, July, Aug.	Mean of the whole year.	Winter.	Summer.	Mean of the whole year.	Nov. Dec. Jan. Feb.	Mar. Apr. Sept. Oct.	May, June, July, Aug.	Mean of the whole year.
1841.	6′·67	9′·46	12′·38	9′·50	5′·12	10′·53	7′·82	11′·13	8′·77	4′·94	8′·28
1842.	5·67	8·87	11·48	8·67	5·07	9·09	7·08	10·56	8·14	4·55	7·75
1843.	5·64	9·36	11·70	8·90	4·70	9·59	7·15	10·16	7·80	4·50	7·66
1844.	5·70	8·74	12·17	8·87	4·44	8·79	6·61	10·77	8·45	4·30	7·84
1845.	5·73	9·15	13·36	9·41	5·89	10·37	8·13	12·16	8·61	4·39	8·39
1846.	6·33	9·21	12·27	9·27	6·08	11·55	8·81	12·58	9·50	5·10	9·06
1847.	7·28	10·08	13·84	10·40	7·13	11·98	9·55	13·43	10·97	5·38	9·93
1848.	9·48	11·04	15·82	12·11	7·85	14·44	11·15	14·14	10·67	7·09	10·63
1849.	8·25	12·25	14·80	11·77	8·06	13·21	10·64	11·32	8·28	4·79	8·13
1850.	8·01	10·90	13·74	10·88	7·61	13·27	10·44	11·32	9·50	4·89	8·57
1851.	7·01	10·82	12·61	10·15	......	......	......	8·54	6·85	4·57	6·65

CURVES OF THE LUNAR ATMOSPHERIC TIDE



Scale ·001 of an Inch of Barometric Pressure to 0·1 of an Inch

The curve rising denotes an

J. Basire sc.

IX. *On the Lunar Atmospheric Tide at Singapore.*

By Captain C. M. ELLIOT, *Madras Engineers, F.R.S.*

Received December 18, 1851.—Read March 11, 1852.

AT the commencement of the year 1847, a paper by Colonel SABINE, R.A., V.P.R.S., was read before the Royal Society on the Lunar Atmospheric Tide at St. Helena. The influence of the moon upon the barometer, although small in amount, was shown in a very striking and decided manner; for after eliminating the regular diurnal variation, the differences arranged in lunar tables showed a decided maximum, both at the superior and inferior culmination of the moon, and a decided minimum at its rising and setting.

The effect which the moon's position, relatively to the meridian of the place, had upon the barometric pressure, was publicly noticed, about the middle of 1842, by Captain LEFROY, R.A., who appears to have had his attention directed to it from the first establishment of the observatory at St. Helena.

On the receipt of Colonel SABINE's paper, I was anxious to ascertain if the fact of the moon's influence, so clearly and decidedly shown at St. Helena, could be similarly proved by the Singapore barometric observations.

I therefore determined, before leaving England for India, to proceed upon the plan adopted by Colonel SABINE, and in order that a comparison might be made between the results at Singapore and at St. Helena, have copied to a considerable extent the form of the different lunar tables drawn up by him in his very valuable paper.

The observatory at Singapore was in latitude 1° 18′ 32″ north, and longitude 103° 56′ 30″ east of Greenwich. The cistern of the barometer was a few feet above high-water mark: the barometer was made by NEWMAN: the diameter of the tube $=0{\cdot}532$ inch. The observations of the barometer, by my assistants, were made at every two hours during the whole of 1841, the early part of 1842, and that of 1843; during the rest of the time, to the close of 1845, at every hour.

In making out the Tables, showing the moon's effect upon the barometer, I have only taken complete astronomical days, from noon to noon; and as Sundays were omitted, and as the observations commenced at midnight on Sunday and terminated at 11 P.M. on Saturday, the broken portions of Monday and of Saturday have not been taken into consideration; the mean, however, of the entire month has been assumed to be identical with the mean of these complete days.

The Barometer Tables, corrected to 32°, having been made out, the mean monthly

height of the barometer at each hour was deducted from the height given by observation at the corresponding hours during the month; if the height of the barometer, at any hour of observation, was less than the mean height at that hour, the difference was put down with the sign −; if greater, with the sign +. By this process the diurnal variation was eliminated. The residual quantities were then arranged in Tables, and the observation corresponding the nearest in time to the moon's culmination being marked for each day, the whole were again re-arranged in lunar tables as follows:—

The moon's superior culmination was assumed as 0 hour of lunar time, and the differences corresponding to that hour placed in the first column; the remaining columns were similarly formed.

The means of the sums of these differences are exhibited in Table I., which consists of two parts, the first part containing the barometric differences at the lunar hours from the superior to the inferior passage, and the second part, from the inferior to the superior passage; the means for each period of six months are shown in the final column of the second part of the Table.

Table II. shows the differences between the mean results and the several numbers in Table I. It must be borne in mind, that had the sets of weekly observations been complete, the sum of the minus differences would have equalled the plus ones: this, however, is not the case owing to the omissions above noted, and an inequality is thereby produced which occasioned the formation of Table I., in which the range of the mean values is shown, the lowest number being assumed as zero.

The means of the complete years 1841, 1844 and 1845, are shown in Table III.; and as two-hourly observations were taken during the first six months of 1842 and of 1843*, and hourly observations during the latter portions of those years, the results of the first six months of 1842 have been combined with the first six months of 1843, and the hourly observations of the latter halves of these years have been combined for the same reason.

In Table IV. the whole of the two-hourly observations, for a period of twenty-four months, have been added together for a general mean, and similarly the whole of the hourly observations for a period of thirty-six months. The results are exhibited in Plate X., which is drawn to ·001 of barometric pressure to 0·74 of an inch of scale.

Finally, Table V. exhibits the observations of three years combined, so as to show the effect upon the barometer of the moon when similarly situated both in its superior and inferior passage; and in a column in juxtaposition is placed a similar table derived from two years' observation at St. Helena, extracted from Colonel Sabine's valuable paper on the subject; from which it will be observed that the effect produced by the moon upon the barometer at Singapore, in the vicinity of the Equator, is slightly greater than at St. Helena, more distant by 14½ degrees of latitude.

* In 1843, in consequence of a deficiency in the number of assistants.

With respect to the oceanic tide, the difference between rise and fall at spring tides varies from 9 to 12 feet. The establishment of the port is about 10^h 50^m A.M.

Table I.

Mean Barometrical Differences at the several Lunar Hours.

1st Part. From the Superior to the Inferior Passage.

	Mean of each of the Lunar Hours.											
	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
First part of 1841	·00603	....	·00540	....	·00107	....	·00077	....	·00176	....	·00332	
Second part of 1841	·00619	....	·00442	....	·00293	....	·00184	....	·00369	....	·00445	
First part of 1842	·00599	....	·00470	....	·00214	....	·00275	....	·00000	....	00264	
Second part of 1842	·00796	·00685	·00568	·00473	·00315	·00085	·00061	·00000	·00109	·00326	·00355	·00625
First part of 1843	·00372	....	·00411	....	·00191	....	·00201	....	·00311	....	·00369	
Second part of 1843	·00728	·00612	·00502	·00391	·00363	·00135	·00000	·00129	·00138	·00324	·00199	·00278
First part of 1844	·00551	·00533	·00405	·00376	·00304	·00050	·00000	·00030	·00141	·00365	·00500	·00739
Second part of 1844	·00882	·00827	·00702	·00641	·00390	·00173	·00004	·00064	·00128	·00215	·00279	·00351
First part of 1845	·00674	·00440	·00253	·00150	·00113	·00000	·00100	·00025	·00114	·00277	·00305	·00422
Second part of 1845	·00774	·00829	·00611	·00571	·00280	·00199	·00024	·00015	00161	·00341	·00411	·00526

2nd Part. From the Inferior to the Superior Passage.

	Mean of each of the Lunar Hours.												Mean in the six months.
	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	
First part of 1841	·00168	....	·00500	....	·00[illegible]59	....	·00069	....	·00000	....	·00397	....	·00294
Second part of 1841	·00723	....	·00594	....	·00113	....	·00000	....	·00097	....	·00405	....	·00358
First part of 1842	·00431	....	·00552	....	·00286	....	·00178	....	·00092	....	·00490	....	·00321
Second part of 1842	·00807	·00734	·00629	00622	00499	·00409	·00325	·00253	·00313	·00456	·00567	·00690	·00446
First part of 1843	·00301	....	·00505	....	·00278	....	·00000	....	·00052	....	·00103	....	·00261
Second part of 1843	·00372	·00013	·00343	·00500	·00443	·00344	·00178	·00134	·00156	·00265	·00179	·00513	·00303
First part of 1844	·00812	·00692	·00640	·00591	·00371	·00200	·00081	·00006	·00186	·00271	·00335	·00371	·00356
Second part of 1844	·00303	·00456	·00348	·00192	·00100	·000[illegible]8	·00000	·00162	·00322	·00338	·00428	·00615	·003[illegible]4
First part of 1845	·00553	·00111	·00282	·00286	·002[illegible]9	·00110	·00086	·00158	·00116	·00283	·00400	·00530	·00267
Second part of 1845	·00641	·00553	·00333	·00200	·00111	·00014	·00000	·00051	·00043	·00259	·00437	·00646	·00335

Table II.

Numerical Values of the excess or defect of the Barometric Differences at the several Lunar Hours.

1st Part. From the Superior to the Inferior Passage.

	Lunar Hours.											
	0.	1.	2.	3.	4.	5.	6	7.	8.	9.	10.	11.
First part of 1841	+·0031	....	+·0025	....	−·0020	....	−·0022	....	−·0012	....	+·0004	
Second part of 1841	+·0028	....	+·0008	....	−·0006	....	−·0017	....	+·0001	....	+·0010	
First part of 1842	+·0028	....	+·0015	....	−·0011	....	−·0005	....	−·0032	....	−·0006	
Second part of 1842	+·0033	+·0024	+·0012	+·0003	−·0013	−·0036	−·0038	−·0045	−·0034	−·0012	−·0009	+·0018
First part of 1843	+·0011	....	+·0015	....	−·0007	....	−·0006	....	+·0005	....	+·0011	
Second part of 1843	+·0040	+·0029	+·0018	+·0007	+·0004	−·0019	−·0032	−·0019	−·0018	−·0010	−·0012	−·0004
First part of 1844	+·0019	+·0018	+·0005	+·0002	−·0005	−·0031	−·0036	−·0033	−·0021	+·0001	+·0020	+·0038
Second part of 1844	+·0055	+·0049	+·0037	+·0031	+·0006	−·0016	−·0033	−·0027	−·0021	−·0012	−·0006	+·0002
First part of 1845	+·0041	+·0017	−·0002	−·0012	−·0015	−·0027	−·0011	−·0024	−·0015	+·0001	+·0004	+·0016
Second part of 1845	+·0044	+·0049	+·0028	+·0024	−·0006	−·0014	−·0031	−·0032	−·0017	+·0001	+·0008	+·0019

2nd Part. From the Inferior to the Superior Passage.

	Lunar Hours.											
	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.
First part of 1841	+·0017	....	+·0021	....	+·0003	....	−·0022	....	−·0029	....	+·0010	
Second part of 1841 ...	+·0036	....	+·0024	....	−·0024	....	−·0036	....	−·0026	....	+·0005	
First part of 1842	+·0011	....	+·0023	....	−·0003	....	−·0014	....	−·0023	....	+·0017	
Second part of 1842 ...	+·0036	+·0029	+·0018	+·0018	+·0005	−·0004	−·0012	−·0019	−·0013	+·0001	+·0012	+·0024
First part of 1843	+·0004	....	+·0025	....	−·0002	....	−·0026	....	−·0021	....	−·0013	
Second part of 1843 ...	+·0005	+·0009	+·0002	+·0028	+·0012	+·0002	−·0014	−·0019	−·0017	−·0006	−·0014	+·0019
First part of 1844	+·0046	+·0034	+·0028	+·0023	+·0001	−·0016	−·0027	−·0035	−·0017	−·0008	−·0002	+·0002
Second part of 1844 ...	−·0003	+·0012	+·0001	−·0014	−·0023	−·0025	−·0033	−·0017	−·0001	·0000	+·0009	+·0028
First part of 1845	+·0029	+·0014	+·0002	+·0002	·0000	−·0016	−·0018	−·0011	−·0015	+·0002	+·0013	+·0026
Second part of 1845 ...	+·0031	+·0022	·0000	−·0013	−·0022	−·0032	−·0033	−·0028	−·0029	−·0008	+·0010	+·0031

Table III.

Excess or Defect of the Barometric Differences at the several Lunar Hours shown in Annual Means.

1st Part. From the Superior to the Inferior Passage.

	Lunar Hours.											
	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
1841	+·0028	....	+·0016	....	−·0013	....	−·0020	....	−·0005	....	+·0007	
First part of 1842, and first part of 1843 ...	+·0020	....	+·0015	....	−·0009	....	−·0006	....	−·0013	....	+·0003	
Second part of 1842, and second part of 1843...	+·0036	+·0026	+·0015	+·0005	−·0008	−·0027	−·0035	−·0032	−·0026	−·0011	−·0010	+·0007
1844	+·0037	+·0032	+·0021	+·0016	·0000	−·0024	−·0034	−·0030	−·0021	−·0005	+·0007	+·0020
1845	+·0012	+·0033	+·0014	+·0006	−·0010	−·0020	−·0021	−·0028	−·0016	+·0001	+·0006	+·0018

2nd Part. From the Inferior to the Superior Passage.

	Lunar Hours.											
	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.
1841	+·0026	....	+·0022	....	−·0010	....	−·0029	....	−·0027	....	+·0007	
First part of 1842, and first part of 1843 ...	+·0008	....	+·0024	....	−·0003	....	−·0024	....	−·0022	....	+·0002	
Second part of 1842, and second part of 1843...	+·0020	+·0019	+·0010	+·0023	+·0008	−·0001	−·0013	−·0019	−·0015	−·0003	−·0001	+·0002
1844	+·0021	+·0023	+·0013	+·0004	−·0011	−·0020	−·0030	−·0026	−·0009	−·0004	+·0004	+·0015
1845	+·0030	+·0018	+·0001	−·0005	−·0011	−·0024	−·0026	−·0020	−·0022	−·0003	+·0012	+·0029

Table IV.

Excess or Defect of the Barometric Differences at the several Lunar Hours as severally deduced from Hourly and 2-Hourly Observations.

1st Part. From the Superior to the Inferior Passage.

	Lunar Hours.											
	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
Mean of two years.........	+·0024	....	+·0015	....	−·0011	....	−·0013	....	−·0009	....	+·0005	....
Mean of three years	+·0038	+·0030	+·0017	+·0009	−·0006	−·0024	−·0030	−·0030	−·0021	−·0005	+·0001	+·0015

2nd Part. From the Inferior to the Superior Passage.

	Lunar Hours.											
	12.	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.
Mean of two years.........	+·0017	....	+·0023	....	−·0007	....	−·0025	....	−·0025	....	+·0005	
Mean of three years	+·0024	+·0020	+·0008	+·0007	−·0005	−·0015	−·0023	−·0022	−·0015	−·0003	+·0004	+·0022

Table V.

Mean Variation of Barometric Pressure deduced from the Hourly Observations.

Moon's distance from the meridian.	Variations of barometric pressure. At the hours following the meridian passage.		Variations of barometric pressure. At the hours preceding the meridian passage.		Hourly variations. From the observations at the hours following the meridian passage.	Hourly variations. From the observations at the hours preceding the meridian passage.	Mean of three years at Singapore.	Mean of two years at St. Helena.	Moon's distance from the meridian.
h	h in.	in.	h in.	in.	in.	in.	in.	in.	h
0	0 +·0038 12 +·0024	+·0031	0 +·0038 12 +·0024	+·0031	+·0057	+·0057	+·00570	+·00365	0
1	1 +·0030 13 +·0020	+·0025	11 +·0015 23 +·0022	+·0018	+·0051	+·0044	+·00475	+·00336	1
2	2 +·0017 14 +·0008	+·0012	10 +·0001 22 +·0004	+·0002	+·0038	+·0028	+·00330	+·00275	2
3	3 +·0009 15 +·0007	+·0008	9 −·0005 21 −·0003	−·0004	+·0034	+·0022	+·00280	+·00158	3
4	4 −·0006 16 −·0005	−·0005	8 −·0021 20 −·0015	−·0018	+·0021	+·0008	+·00145	+·00110	4
5	5 −·0024 17 −·0015	−·0019	7 −·0030 19 −·0022	−·0026	+·0007	·0000	+·00035	+·00046	5
6	6 −·0030 18 −·0023	−·0026	6 −·0030 18 −·0023	−·0026	·0000	·0000	·00000	·00000	6

X. *Discovery that the Veins of the Bat's Wing (which are furnished with valves) are endowed with rythmical contractility, and that the onward flow of blood is accelerated by each contraction.*

By T. Wharton Jones, *F.R.S., Fullerian Professor of Physiology in the Royal Institution of Great Britain, Ophthalmic Surgeon to University College Hospital, and Corresponding Member of the Society of Biology of Paris, &c. &c.*

Received November 20, 1851.—Read February 5, 1852.

In entering on the investigation of the state of the blood and the blood-vessels in inflammation excited in the web of the Bat's wing, I applied myself, in the first place, to the study of the distribution, structure and endowments of the arteries, capillaries and veins of the part, and of the phenomena of the circulation in them.

I had not observed the circulation under the microscope long, before I was struck with something peculiar in the flow of blood in the veins; I therefore directed my attention to them, and discovered that they contracted and dilated rythmically. Following the veins for some extent in their course, I further discovered them to be provided with valves, some of which completely opposed regurgitation of blood, others only partially.

The cause of the peculiarity in the flow of blood in the veins was thus no longer doubtful, but some continued observation was required before I was able to make out exactly its mode of operation.

The act of contraction of the vein is manifested by progressive constriction of its caliber and increasing thickness of its wall; the relaxation of the vessel, by a return to the former width of caliber and thickness of wall.

The rythmical contractions and dilatations of the veins are, in the natural state, continually going on; but sometimes with greater, sometimes with less rapidity, and sometimes to a greater, sometimes to a less extent. The average number of contractions in a minute, I have found to be ten. I have on some occasions counted only seven or eight, and on other occasions as many as twelve or thirteen. Most usually, the numbers were nine and eleven. The supervening dilatations take place rather more quickly than the contractions. The amount of constriction of one of the larger veins—one about $\frac{1}{300}$dth or $\frac{1}{400}$dth of an inch in width when dilated—at each contraction of its walls, may be put down at a fourth or fifth of its whole width when in a state of dilatation; I have sometimes estimated it at nearly a third, sometimes at not more than a sixth.

The contractions *centrad* and *distad* of a valve appeared to be simultaneous, as did also the dilatations.

The smaller veins, those of the first and second order, proceeding from the radicles, contract, but not in a very marked manner, and are destitute of valves.

During contraction, the flow of blood in the vein is accelerated. On the cessation of the contraction, the flow is checked, and a tendency to regurgitation of the blood takes place, which brings the valves into play. Where the valves are perfect, the backward movement of the blood is at once stopped by their closure; but where the valves are not complete, the blood regurgitates more or less freely*. But this check to the onward flow of the blood is usually only for a moment or two. Already, even while the vein is in the act of again becoming dilated, the onward flow of blood recommences and goes on, though comparatively slowly, until dilatation is completed and contraction supervenes; whereupon acceleration of the flow takes place as before.

It is to be observed, that in determining the flow of blood in the veins (the phenomena of which I have now described), the action of the heart is concerned as well as the contractions of the veins themselves. It appears to be the heart's action which maintains the onward flow of blood during the dilatation of the vein, whilst it is the contraction of the vein, coming in aid of the heart's action, which causes the acceleration. Sometimes the *vis a tergo* is sufficient to keep up a pretty steady flow in the veins, this being only accelerated at each contraction of these vessels.

The check to the flow of blood in the veins takes place at the completion of the contraction or commencement of the dilatation. The number of checks observable in a minute, therefore, corresponds with the number of contractions. In one case, while an assistant marked the time by a seconds' watch, I observed that a complete valve checked the tendency to regurgitation nine times in a minute; and on counting the number of contractions of the same vessel, I found them also nine in a minute. In another case, eleven checks and eleven contractions were counted; and so on repeatedly. Though I quote these little experiments, I would remark that, after some practice in the observation, the eye is quite able to take in at one glance the succession and relations of the two phenomena.

The valves of the veins are composed sometimes of but a single flap, sometimes of two. In the situation of a valve, and *centrad* of the insertion of its flaps, the veins present the usual dilatations or sinuses corresponding to the sinuses of Valsalva at the origin of the pulmonary artery and aorta. These sinuses are best seen when the valve happens to present its flaps edgeways to the observer.

Valves are found close to the entrance of a large branch, but distad of it (Plate IV. fig. 2). They are also found at intermediate parts of the veins (fig. 1). Tracing the

* Sometimes, as for example, into a venous branch with an incomplete valve, a retrograde flow of blood takes place from a large vein, at the moment this latter is contracting and propelling its blood onwards.—May 7, 1852.

veins from radicles to trunks, the first valves I have noticed were at the junction of the second order of veins to form the third.

In watching the circulation, it is interesting to observe the backward eddy of blood-corpuscles into the sinuses of the valves, when the blood issues from the narrow valvular opening into the wide part of the vein beyond (fig. 1).

In structure, the valves are seen to be a reduplication of the clear innermost coat of the vein, with sometimes a pretty evident layer of fibrous tissue intervening.

Each vein is closely accompanied by an artery, a nerve only intervening. The average diameter of a vein is to that of its accompanying artery as about 3 to 2.

The contractility of the arteries is altogether different in its nature from that of the veins. It is *tonic contractility, not rythmical.* On the application of pressure over an artery, this vessel may be seen to become constricted, sometimes even to temporary obliteration of its caliber, and that uniformly throughout some extent of its course, both above and below the point where the pressure was applied; or, the constriction is greater or less at intervals, so that the vessel presents a varicose appearance. This tonic contraction of the arteries of the Bat's wing does not take place quite so quickly as the same phenomenon in the Frog's web, and, ordinarily, continues a longer time*.

The pulsation of a vein so affects its accompanying artery as to push the latter, as a whole, to and fro. That the movement of the artery referred to is really owing to this cause, and not to any pulsation or rythmical contraction and dilatation of its own walls, is evident from this, that the movements are synchronous with the contractions and dilatations of the vein, and that *both sides of the artery move in the same direction*, not approximating and receding from each other, so as to constrict or dilate the caliber, as in the case of the vein.

I have not been able to observe unequivocal evidences of tonic contractility of veins in addition to their rythmical contractility. When pressure is, at the same time, applied over the vein as well as the artery, the vein is not found to become tonically constricted in the same manner as the artery, upward and downward. At the place where the vein was pressed on, a mechanical indentation of its wall may perhaps be seen. And in addition to this, there may often be observed an appearance of great and abrupt constriction. This appearance, however, is not owing to contraction of the walls of the vein, but to a deposit of a viscid-looking grayish granular lymph within the vessel at the place, obstructing its channel and narrowing the stream of blood (figs. 3 and 4). On watching, I have seen portions of this deposit detached and carried away by the stream of blood, with corresponding enlargement of the channel, and again an additional deposit with renewed narrowing of the stream. When the pressure has been considerable, I have seen the vein become for a time wholly obstructed by the deposit. A similar deposit of lymph takes place in the artery. In one case, I observed that the artery, at the place pressed on, was

* When a frog under examination struggles, the arteries of the web are seen to become constricted. I have observed the same thing in the web of the Bat's wing and the ear of a white rabbit.

actually not so much constricted as above and below, though, on account of the narrowness of the stream of blood from the presence of the lymphy deposit, it appeared as much so at first sight (fig. 4).

Having subjected the web to the galvanic influence from a single pair of plates, I found all the smaller arteries of the part in a state of considerable tonic constriction, but the larger arteries constricted in a less degree. The effect of galvanism on the veins appeared to be to render their rythmical contractions somewhat more brisk, they having been previously rather languid. On cutting a vein across, I did not observe tonic constriction of it, any more than in the Frog.

After the application of a drop of Vinum opii to the web, the veins were found dilated as well as the arteries, and their rythmical contractions appeared to be suspended.

It has been stated by an authority not liable to err, that, on mechanical irritation, both artery and vein of the Bat's web gradually contract and close, and, by and by, dilate wider than before. And, again, that in Bats, contraction of veins is quite as well marked as that of arteries.

These statements, it will be observed, imply tonic contractility of the veins.

Notwithstanding my attention has been repeatedly directed to the point, I have not, as previously stated, been able to observe unequivocal evidences of tonic contractility of veins, in addition to their rythmical contractility. For this reason, I cannot help venturing on the supposition that Mr. Paget* must have made his statements either from a hasty and imperfect observation of the proper rythmical contractions of the veins; or, seeing that in rythmical contraction of the veins the constriction is never to closure, like that of the arteries, under some such misapprehension as to the nature of the vessel observed, as he certainly must have laboured under when he supposed that arteries and veins of the second and third order open directly into each other without any intermedium of capillaries.

The arteries and their subdivisions anastomose freely with each other, forming a network all through the web, the meshes of which go on to diminish towards the free margin. Each artery and each subdivision of an artery is closely accompanied by a vein; and these veins, like the arteries they accompany, anastomose with each other. But it is to be remarked that nowhere do the arteries and veins directly communicate. The only communication is the usual one through the medium of capillaries. The capillaries, the walls of which are destitute of contractility, receive the blood from small arterial twigs which arise from the arterial network, and return it to the venous radicles which open into corresponding veins. These arterial twigs, capillaries and venous radicles, form networks within the meshes of the great vascular network, and a looped network at the margin of the web† (Plate V.).

The observations recorded in the preceding pages were made principally with

* Lectures on Inflammation at the Royal College of Surgeons in 1850.

† I shall have occasion to treat of this point more in detail in a paper on the state of the blood and the blood-vessels in inflammation of the web of the Bat's wing.

one-eighth of an inch object-glass, and the two lowest eyepieces, affording magnifying powers of 370 and 550 diameters.

The web of the wing was stretched out on the object plate, wetted on both sides with water, and covered with a thin plate of glass at the spot to be examined.

Appendix to the Foregoing Paper.

Received December 11, 1851,—Read February 5, 1852.

In consequence of the dark pigment in the cells of the epidermis of the web of the Bat's wing, the structure of the vessels cannot be well made out except by dissection.

A small piece of the web containing vessels being detached and disposed in a drop of water, under the simple microscope, the two layers of skin may be readily torn from each other with needles, and the artery and vein with their accompanying nerve, which lies between the two, separated in one bundle.

In pieces cut out of a web which had been dried, the bundle of vessels and nerve was, after tearing away the skin, left surrounded by a sheath of cellular and elastic fibres disposed longitudinally; but in pieces cut out from the living web and directly examined, this sheath was always detached along with the skin, and the vessels with their accompanying nerve at once laid bare (see Plate IV. fig. 5.).

Both artery and vein are seen to have a middle coat of circularly disposed muscular fibres; but the appearance of the fibres is different in the two vessels.

The fibres of the vein are about $\frac{1}{3600}$dths of an inch broad, pale, grayish, semitransparent and granular-looking. In general aspect they very much resemble the muscular fibres of the lymphatic hearts of the frog. In none of the muscular fibres of the vein, however, did I detect an unequivocal appearance of transverse marking.

The fibres of the middle coat of the artery are not so pale-looking as those of the middle coat of the vein, are clearer, and exhibit a more strongly marked contour.

Second Appendix.

Received May 10,—Read May 13, 1852.

From a microscopical examination of the blood-vessels and circulation in the ears of the Long-eared Bat, I have ascertained that, different from what I discovered to be the case in the wings, the veins of the ears are unfurnished with valves, and are not endowed with rythmical contractility, and that the onward flow of blood in them is consequently uniform. I ought, perhaps, to qualify the statement that the veins of the ear are not endowed with rythmical contractility, by saying, that I think I noticed a very slight tendency to it, here and there in a vein, but so slight as not to have the smallest effect on the flow of blood.

This observation regarding the ear of the Bat illustrates how that the heart's action is sufficient of itself for the circulation of the blood in the body generally; but that being sufficient for that only, the supplementary force of rythmical con-

tractility of veins, supported by the presence of valves, is called forth to promote the flow of blood in the wings, which, on account of their extent, are, as regards their circulation, in a considerable degree, though not entirely, beyond the sphere of the heart's influence.

I may take this opportunity to mention that I have also found the veins of the mesentery of the Mouse destitute of rythmical contractility.

Explanation of the Plates.

PLATE IV.

Fig. 1. A vein with a complete valve. In this figure an attempt has been made to represent the backward eddy of blood-corpuscles into the sinuses of the valve, at the time the blood issues from the narrow valvular opening into the wide part of the vein beyond.

Fig. 2. A vein with a valve close to the entrance of a large branch.

Fig. 3. An artery and vein over which pressure had been applied. The artery is seen constricted at intervals, above and below the place of pressure *a*. The vein is not constricted, but there is seen, at the place where the pressure was applied, a grayish granular deposit of lymph within the vessel, giving rise to the appearance of constriction by narrowing the stream of blood.

Fig. 4. Representation of the case in which the artery, at the place pressed on, was actually not so much constricted as above and below; though, on account of the narrowness of the stream of blood from the presence of the lymphy deposit, it appeared as much so at first sight.

In this case the channel of the vein was much narrowed by a deposit of lymphy matter on either side within the vessel.

Fig. 5. This represents an artery (*a*) and a vein (*v*), with an accompanying nerve (*n*) lying between the two, as seen with a magnifying power of about 370 diameters; immediately after being separated, by dissection under the simple microscope, from a small piece cut out of the living web. The cellular sheath was detached along with the two layers of skin.

The artery is at one place tonically constricted, and there the middle coat is seen to be thicker.

The difference in the general aspect of the fibres of the middle coats of the two vessels may be recognised.

PLATE V.

This represents a portion of the vascular network of the web of the Bat's wing, as seen under a low magnifying power. The arteries are observed to anastomose with each other, and the veins with each other; but nowhere are the arteries and veins seen to communicate directly. The only communication, it is to be observed, is through the medium of the capillaries.

Fig. 1.

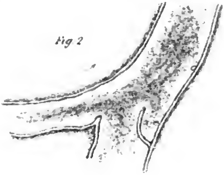


Fig. 2.

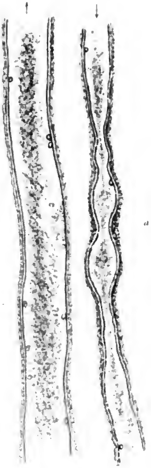


Fig. 3.

Fig. 4.

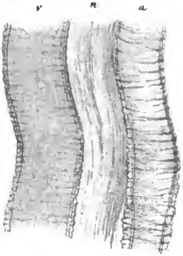
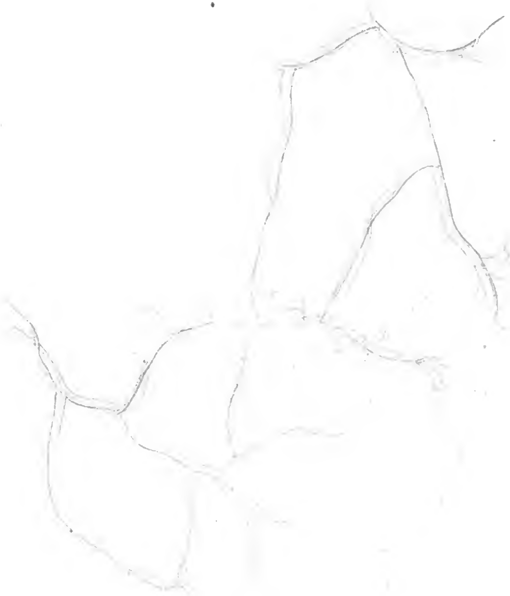


Fig. 5.

J. Basire lith.



XI. *Experimental Researches in Electricity.—Twenty-ninth Series.* By Michael Faraday, *Esq., D.C.L., F.R.S., Fullerian Prof. Chem. Royal Institution, Foreign Associate of the Acad. Sciences, Paris, Ord. Boruss. Pour le Mérite, Eq., Memb. Royal and Imp. Acadd. of Sciences, Petersburgh, Florence, Copenhagen, Berlin, Göttingen, Modena, Stockholm, Munich, Bruxelles, Vienna, Bologna, &c. &c.*

Received December 31, 1851,—Read March 25 and April 1, 1852.

§ 35. *On the employment of the Induced Magneto-electric Current as a test and measure of Magnetic Forces.*

3177. THE proposition which I have made to use the induced magneto-electric current as an experimental indication of the presence, direction and amount of magnetic forces (3074.), makes it requisite that I should also clearly demonstrate the principles and develope the practice necessary for such a purpose; and especially that I should prove that the amount of current induced is precisely proportionate to the amount of lines of magnetic force intersected by the moving wire, in which the electric current is generated and appears (3082, 3109.). The proof already given is, I think, sufficient for those who may repeat the experiments; but in order to accumulate evidence, as is indeed but proper in the first announcement of such a proposition, I proceeded to experiment with the magnetic power of the earth, which presents us with a field of action, not rapidly varying in force with the distance, as in the case of small magnets, but one which for a given place may be considered as uniform in power and direction; for if a room be cleared of all common magnets, then the terrestrial lines of magnetic force which pass through it, have one common direction, being that of the dip, as indicated by a free needle or other means, and are in every part in equal proportion or quantity, *i.e.* have equal power. Now the force being the same everywhere, the proportion of it to the current evolved in the moving wire is then perhaps more simply and directly determined, than in the case where, a small magnet being employed, the force rapidly changes in amount with the distance.

¶ i. *Galvanometer.*

3178. For such experimental results as I now propose to give, I must refer to the galvanometer employed and the precautions requisite for its proper use. The instrument has been already described in principle (3123.), and a figure of the conductor which surrounds the needles, given. This conductor may be considered as a square copper bar, 0·2 of an inch in thickness, which passes twice round the plane of vibration of each of the needles forming the astatic combination, and then is continued outwards and terminates in two descending portions, which are intended to dip into

cups of mercury. As both the needles are within the convolutions of this bar, an indicating bristle or fine wire of copper is fixed parallel to, and above them upon the same axis, and this, in travelling over the usual graduated circle, shows the place and the extent of vibration or swing of the needles below. The suspension is by cocoon silk, and in other respects the instrument is like a good ordinary galvanometer.

3179. It is highly important that the bar of copper about the needles should be perfectly clean. The vertical zero plane should, according to the construction, be midway between the two vertical coils of the bar, fig. 1; instead of which the needle at first pointed to the one side or the other, being evidently attracted by the upright portions of the bar. I at first feared that the copper was magnetic, but on cleaning the surface carefully with fine sand-paper, I was able to remove this effect, due no doubt to iron communicated by handling or the use of tools, and the needle then stood truly in a plane equidistant from the two coils, when that plane corresponded with the magnetic meridian.

Fig. 1.

3180. The connexions for this galvanometer (3123, 3133.) were all of copper rod or wire 0·2 of an inch in diameter; but even with wires of this thickness the extent of the conductors should not be made more than is necessary; for the increase from 6 to 8, 10 or 12 feet in length, makes a considerable difference at the galvanometer, when electric currents, low in intensity, are to be measured. It is most beautiful to observe in such cases the application of Ohm's law of currents to the effects produced. When the connexions were extended to a distance, straight lengths of wire with dropping ends were provided, and these by dipping into cups of mercury completed the connexion and circuit. The cups consisted of cavities turned in flat pieces of wood. The ends of the connecting rods and of the galvanometer bar were first tinned, and then amalgamated; after which their contact with the mercury was both ready and certain. Even where connexion had to be made by contact of the solid substances, I found it very convenient and certain to tin and amalgamate the ends of the conductors, wiping off the excess of mercury. The surfaces thus prepared are always ready for a good and perfect contact.

3181. When the needle has taken up its position under the earth's influence, and the copper coil is adjusted to it, the needle ought to stand at true zero, and appears so to do. When that is really the case, equal forces applied in succession on opposite sides of the needle (by two contrary currents through the coil for instance) ought to deflect the needle *equally* on both sides, and they do so. But sometimes, when the needle appears to stand at zero, it may not be truly in the magnetic meridian; for a little torsion in the suspension thread, even though it be only 10° or 15° (for an indifferent needle), and quite insensible to the eye looking at the magnetic needle, does deflect it, and then the force which opposes the swing of the needle, and which stops and returns the needle towards zero (being due both to the torsion and the earth's force), is not equal on the two sides, and the consequence is, that the extent of

swing in the two directions is not equal for equal powers, but is greater on one side than the other.

3182. I have not yet seen a galvanometer which has an adjustment for the torsion of the suspending filament. Also, there may be other causes, as the presence about a room, in its walls and other places, of unknown masses of iron, which may render the forces on opposite sides of the instrument zero unequal in a slight degree; for these reasons it is better to make *double observations*. All the phenomena we have to deal with, present effects in two contrary directions. If a loop pass over the pole of a magnet (3133.), it produces a swing in one direction; if it be taken away, the swing is in the other direction; if the rectangles and rings to be described (3192.) be rotated one way, they produce one current; if the contrary way, the other and contrary current is produced. I have therefore always in measuring the power of a pole or the effect of a revolving intersecting wire made many observations in both directions, either alternately or irregularly; have then ascertained the average of those on the one side, and also on the other (which have differed in different cases from $\frac{1}{30}$th to $\frac{1}{300}$th part), and have then taken the mean of these averages as the expression of the power of the induced electric current, or of the magnetic forces inducing it.

3183. Care must be taken as to the position of the instrument and apparatus connected with it, in relation to a fire or sources of different temperatures, that parts which can generate thermo-currents may not become warmed or cooled in different degrees. The instrument is exceedingly sensible to thermo-electric currents; the accidental falling of a sun-beam upon one of two connecting mercury cups for a few moments disturbed the indications and rendered them useless for some time.

3184. In order to ascertain practically, *i. e.* experimentally, the comparative value of degrees in different parts of the scale or graduation of this instrument and so to render it a measurer, the following trials were made. A loop like that before described (3133.), fig. 2, was connected with the galvanometer by communications which removed the loop 9 feet from the instrument, and it was then fixed. A compound bar-magnet consisting of two plates, each 12 inches long, 1 inch broad, and 0·5 in thickness, was selected of such strength as to lift a bunch of clean iron filings, averaging 45 grains at either extremity. Blocks were arranged at the loop, so that this magnet, held in a vertical position, could have one end passed downwards through the loop until the latter coincided with the equator of the magnet (3191.); after which it could be quickly removed and the same operation be repeated at pleasure. When the magnet was thus moved, the loop being unconnected (at one of the mercury cups) with the galvanometer, there was no sensible change of place in the needles; the direct influence of the magnet at this distance of 9 feet being too small for such an effect.

Fig. 2.

3185. It must be well understood, that, in all the observations made with this instrument, the *swing* is observed and counted as the effect produced, unless otherwise expressed. A constant current in an instrument will give a constant and continued

deflection, but such is not the case here. The currents observed are for short periods, and they give, as it were, a blow or push to the needle, the effect of which, in swinging the needle, continues to increase the extent of the deflection long after the current is over. Nevertheless the extent of the swing is dependent on the electricity which passed in that brief current; and, as the experiments seem to indicate, is simply proportional to it, whether the electricity pass in a longer or a shorter time (3104.), and notwithstanding the comparative variability of the current in strength during the time of its continuance.

3186. The compound bar being introduced once into the loop and left there, the swing at the galvanometer was observed and found to be 16°; the galvanometer needle was then brought to zero, and the bar removed, which gave a reverse current and swing, and this also was 16°. Many alternations, as before described, gave 16° as the mean result, *i. e.* the result of one intersection of the lines of force of this magnet (3102.). In order to comprehend the manner in which the effect of two or more intersections of these lines of force were added together, it should be remembered that a swing of the needle from right to left occupied some time (13 seconds); so that one is able to introduce the magnet into the loop, then break the electric circuit by raising one end of the communicating wire out of the mercury, remove the magnet, which by this motion does nothing, restore the mercury contact, and reintroduce the magnet into the loop, before a tenth part of the time has passed, during which the needles, urged by the first impulse, would swing. In this way two impulses could be added together, and their joint effect on the needle observed; and, indeed, by practice, three and even four impulses could be given within the needful time, *i. e.* within one-half or two-thirds of the time of the full swing; but of course the latter impulses would have less power upon the needles, because these would be more or less oblique to the current in the copper coil at the time when the impulses were given. There can be no doubt, that, as regarded the currents induced in the loop by the magnet, they would be equal on every introduction of the same magnet.

3187. Proceeding in this way I obtained results for one, two, three, and even four introductions with the same magnet.

One introduction	15°
Two introductions	31·25
Three introductions	46·87
Four introductions	58·50

Here the approximation to 1, 2, 3, 4 cannot escape observation*; and I may remark,

* See note to (3189.) $\sin \frac{15}{2} = \sin 7^{\circ}\ 30' = \cdot 130526 \qquad \cdot 130526$

$$\sin \frac{31\cdot25}{2} = \sin 15\cdot625 = \sin 15\ 37\cdot5 = \cdot269200 \qquad \frac{269200}{2} = \cdot134600$$

$$\sin \frac{46\cdot87}{2} = \sin 23\cdot435 = \sin 23\ 26\cdot1 = \cdot3976818 \qquad \frac{3976818}{3} = \cdot1328606$$

$$\sin \frac{58\cdot50}{2} = \sin 29\cdot25 = \sin 27\ 15 = \cdot4886212 \qquad \frac{4886212}{4} = \cdot1221553$$

that, whilst observing the place attained at the end of a swing which is retained only for an instant, some degree of error must creep in; and that that error must be greatest, in the first number, where it falls altogether upon the unit of comparison than in the other observations, where only one-half or one-third of it is added to a half or a third of the whole result. Thus, if we halve the arc for two introductions of the pole, it gives 15°·625; if we take the third of that for three introductions, it gives 15°·61;—numbers which are almost identical, so that if the first number was increased by only 0°·6, the proportion would be as 1, 2 and 3. The reason why the fourth, which is 14°·625, is less, may perhaps be referred to the cause already assigned, namely, the declination distance of the needle from the coil when that impulse was given (3186.).

3188. In order to avoid in some degree this case, and to compare the degrees at the beginning of the scale, which are most important for the comparison of future experiments with one another, I took one of the bars of the compound magnet employed above (3184.). The results were as follows:—

One introduction	8°
Two introductions	15·75
Three introductions	23·87
Four introductions	31·66

which numbers are very closely as 1, 2, 3 and 4. If we divide as before, we have 8°, 7°·87, 7°·95, 7°·91; so that if only 0·09 be subtracted from the first observation, or 8°, it leaves that simple result *.

3189. Hence it appears, that in this mode of applying and measuring the magnetic powers, the number of degrees of swing deflection are for small arcs nearly proportional to the magnetic force which has been brought into action on the moving wire †.

* $\sin\frac{8}{2} = \sin 4^\circ = \cdot0697565 \qquad \cdot0697565$

$\sin\frac{15\cdot75}{2} = \sin 7\cdot875 = \sin 7^\circ\, 52'\cdot5 = \cdot1370123 \qquad \frac{1370123}{2} = \cdot0685061$

$\sin\frac{23\cdot87}{2} = \sin 11\cdot935 = \sin 11\; 56\cdot1 = \cdot2068019 \qquad \frac{2068019}{3} = \cdot0689340$

$\sin\frac{31\cdot66}{2} = \sin 15\cdot83 = \sin 15\; 49\cdot8 = \cdot2727840 \qquad \frac{2727840}{4} = \cdot0681960$

† Mr. Christie has recalled my attention to a paper in the Philosophical Transactions, 1833, p. 95, in which he has investigated, at p. 111, &c., the effect of what may be called magneto-electric impulses in deflecting the magnetic needle. He found that the velocity of the projection of the needle, which is a measure of the force acting upon it at the instant of its moving, will be proportional to the sine of half the arc of swing. My statement, therefore, would as a general expression be erroneous; but for small arcs the results as given by it are not far from the truth. The error does not interfere with the general reasoning and conclusions of the paper; and as the numbers are the results of experiment, which, though made with a first and therefore rough apparatus, were still made with some care, and are expressed simply as deflections, I prefer their appearance as they are rather than in an altered state. Mr. Christie has been so kind as to give me the true expression of force for many of the cases, and I have inserted the results as foot-notes where the cases occur.—Jan. 26, 1852.

3190. I have found the needles very constant in their strength for days and weeks together. By care, the constancy of their state for a day is easily secured; and that is all that is required in comparative experiments. Those which I have in use weigh with their axis and indicating wire 9 grains; and when out of the copper coil vibrate to and fro once in 26 seconds.

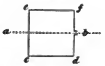
3191. With this instrument thus examined, I repeated most of the experiments with loops formerly described (3133. &c.), with the same results as before. It was also ascertained that the equator of a regular bar-magnet was the place at which the loop should be arrested, to produce the maximum action; and that if it came short of, or passed beyond that place, the final result was less. Employing a magnet 12 inches long, when the loop passed

2·3 inches over the pole the deflection was	5°·91
4·1 inches over the pole the deflection was	7·50
5·1 inches over the pole the deflection was	7·74
6·1 inches over the pole the deflection was	8·16
8·0 inches over the pole the deflection was	7·75
9·0 inches over the pole the deflection was	6·50

¶ ii. *Revolving Rectangles and Rings**.

3192. The form of moving wire which I have adopted for experiments with the magnetic forces of the earth (3177.), is either that of a rectangle or a ring. If a wire rectangle (fig. 3) be placed in a plane, perpendicular to the dip and then turned once round the axis *a b*, the two parts *c d* and *e f* will twice intersect the lines of magnetic force within the area *c e d f*. In the first 180° of revolution the contrary direction in which the two parts *c d* and *e f* intersect those lines, will cause them to conspire in producing one current, tending to run round the rectangle (161) in a given direction; in the following 180° of revolution they will combine in their effect to produce a contrary current; so that if the first current is from *d* by *c e* and *f* to *d* again, the second will be from *d* by *f e* and *c* to *d*. If the rectangle, instead of being closed, be open at *b*, and the ends there produced be connected with a commutator, which changes sides when the rectangle comes into the plane perpendicular to the dip, *i. e.* at every half revolution, then these successive currents can be gathered up and sent on to the galvanometer to be measured. The parts *c e* and *d f* of the

Fig. 3.

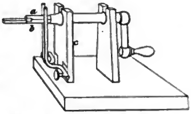


* A friend has pointed out to me that in July 1832, Nobili made experiments with rotating rings or spirals subject to the earth's magnetic influence; they were subsequent to and consequent upon my own experiments upon swinging wires (171, 148.) and revolving globes (160.) of January 1832; but he extended the considerations to the *thickness* of the wire; the *diameter* of the spirals and the *number* of the spirals dependent upon the *length* of the wire. The results (tabulated) will be found in vol. i. page 244, &c. of the Florence edition of his Mémoires.—March 1, 1852.

rectangle may be looked upon simply as conductors; for as they do not in their motion intersect any of the lines of force, so they do not tend to produce any current.

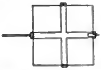
3193. The apparatus which carries these rectangles, and is also the commutator for changing the induced currents, consists of two uprights, fixed on a wooden stand, and carrying above a wooden horizontal axle, one end of which is furnished with a handle, whilst the other projects, and is shaped as in fig. 4. It may there be seen, that two semi-cylindrical plates of copper *a b* are fixed on the axle, forming a cylinder round it, except that they do not touch each other at their edges, which therefore leave two lines of separation on opposite sides of the axle. Two strong copper rods, 0·2 of an inch in diameter, are fixed to the lower part of the upright *c*, terminating there in sockets with screws for the purpose of receiving the ends of the rods proceeding from the galvanometer cups (3180.): in the other direction the rods rise up parallel to each other, and being perfectly straight, press strongly against the curved plates of the commutator on opposite sides: the consequence is, that, whenever in the rotation of the axle, the lines of separation between the commutator plates arrive at and pass the horizontal plane, their contact with these bearing rods is changed, and consequently the direction of the current proceeding from these plates to the rods, and so on to the galvanometer, is changed also. The other or outer ends of the commutator plates are tinned, for the purpose of being connected by soldering to the ends of any rectangle or ring which is to be subjected to experiment.

Fig. 4.



3194. The rectangle itself is tied on to a slight wooden cross (fig. 5), which has a socket on one arm that slides on to and over the part of the wooden axle projecting beyond the commutator plates, so that it shall revolve with the axle. A small copper rod forms a continuation of that part of the frame which occupies the place of axle, and the end of this rod enters into a hole in a separate upright, serving to support and steady the rectangle and its frame. The frames are of two or three sizes, so as to receive rectangles of 12 inches in the side, or even larger, up to 36 inches square. The rectangle is adjusted in its place, so that it shall be in the horizontal plane when the division between the commutator plates is in the same plane, and then its extremities are soldered to the two commutator plates, one to each. It is now evident, that when dealing with the lines of force of the earth, or any other lines, the axle has only to be turned until the upright copper rods touch on each side at the separation of the commutator plates, and then the instrument adjusted in position, so that the plane of the ring or rectangle is perpendicular to the direction of the lines of force which are to be examined, and then any revolution of the commutator and intersecting wire will produce the maximum current which such wire and such magnetic force can produce. The lines of terrestrial magnetic force are inclined at an angle

Fig. 5.



of 69° to the horizontal plane. As, however, only comparative results were required, the instrument was, in all the ensuing experiments, placed in the horizontal plane, with the axis of rotation perpendicular to the plane of the magnetic meridian; under which circumstances no cause of error or variation was introduced into the results. As no extra magnet was employed, the commutator was placed within 3 feet of the galvanometer, so that two pieces of copper wire 3 feet long and 0·2 of an inch in thickness, sufficed to complete the communication. One end of each of these dipped into the galvanometer mercury cups, the other ends were tinned, amalgamated, introduced into the sockets of the commutator rods (3193.), and secured by the pinching screw (fig. 4).

3195. When a given length of wire is to be disposed of in the form best suited to produce the maximum effect, then the circumstances to be considered are contrary for the case of a loop to be employed with a small magnet (39. 3184.), and a rectangle or other formed loop to be employed with the lines of terrestrial force. In the case of the small magnet, *all* the lines of force belonging to it are inclosed by the loop; and if the wire is so long that it can be formed into a loop of two or more convolutions, and yet pass over the pole, then twice or many times the electricity will be evolved that a single loop can produce (36.). In the case of the earth's force, the contrary result is true; for as in circles, squares, similar rectangles, &c. the areas inclosed are as the squares of the periphery, and the lines of force intersected are as the areas, it is much better to arrange a given wire in one simple circuit than in two or more convolutions. Twelve feet of wire in one square intersects in one revolution the lines of force passing through an area of nine square feet, whilst if arranged in a triple circuit, about a square of one foot area, it will only intersect the lines due to that area; and it is thrice as advantageous to intersect the lines within nine square feet once, as it is to intersect those of one square foot three times.

3196. A square was prepared, containing 4 feet in length of copper wire 0·05 of an inch in diameter; it inclosed one square foot of area, and was mounted on the commutator and connected in the manner already described (3194.). Six revolutions of it produced a swing deflection of 14° or 15°, and twelve quick revolutions were possible within the required time (3104.). The results of *quick* and *slow* revolutions were first compared. Six slow revolutions gave as the average of several experiments 15°·5 swing. Six moderate revolutions gave also an average of 15°·5; six quick revolutions gave an average of 15°·66. At another time twelve moderate revolutions gave an average of 28°·75, and twelve quick revolutions gave an average of 31°·33 swing. As before explained (3186.), the probable reason why the quick revolutions gave a larger result than the moderate or slow revolutions is, that in slow time the later revolutions are performed at a period when the needle is so far from parallel with the copper coil of the galvanometer, that the impulses due to them are less effectually exerted. Hence a small or moderate number of revolutions and a quick motion is

best. The difference in the extreme case is less than might have been expected, and shows that there is no practical objection in this respect to the method proposed of experimenting with the lines of magnetic force.

3197. In order to obtain for the present an expression of the power of the earth's magnetic force by this rectangle, observations were made on both sides of zero, as already recommended (3182.). Nine moderately quick direct revolutions (*i. e.* as the hands of the clock) gave as the average of many experiments 23°·87, and nine reverse revolutions gave 23°·37; the mean of these is 23°·62 for the nine revolutions of the rectangle, and therefore 2°·624 per revolution. Now the six quick revolutions (3196.) gave 15°·66, which is 2°·61 per revolution, and the twelve quick revolutions gave 31°·33, which is also 2°·61 per revolution; and these results of 2°·624, 2°·61, and 2°·61, are very much in accordance, and give great confidence in this method of investigating magnetic forces*.

3198. A rectangle was prepared of the same length (4 feet) of the same wire, but the sides were respectively 8 and 16 inches (fig. 6), so that when revolving the intersecting parts should be only 8 inches in length instead of 12. The area of the rectangle was necessarily 128 square inches instead of 144. This rectangle showed the same difference of quick and slow rotations as before (3196.). When nine direct revolutions were made, the result was 20°·87 swing. Nine reverse revolutions gave an average of 20°·25 swing; the mean is 20°·56, or 2°·284 per revolution. A third rectangle was prepared of the same length and kind of wire, the sides of which were respectively 8 and 16 inches long (fig. 7), but now so revolved that the intersecting parts were 16 inches, or twice as long as before; the area of the rectangle remained the same, *i. e.* 128 inches. The like effect of slow and quick revolutions appeared as in the former cases (3196. 3198.). Nine direct revolutions gave as the average effect 20°·75; and nine reverse revolutions produced 21°·375; the mean is 21°·06, or 2°·34 per revolution.

Fig. 6.

Fig. 7.

3199. Now 2°·34 is so near to 2°·284, that they may in the present state of the investigation be considered the same. The little difference that is evident, was, I suspect, occasioned by centrifugal power throwing out the middle of the longer intersecting parts during the revolution. The coincidence of the numbers shows, that the variation in the arrangement of the rectangle and in the length of the parts of the wires intersecting the lines of magnetic force, have had no influence in altering the result, which, being dependent alone on the number of lines of force intersected, is the same for both; for the area of the rectangles is the same. This is still further shown by comparing the results with those obtained with the square. The area in

* $\sin\frac{15·66}{2} = \sin 7°·83 = \sin 7° 49'·8 = ·1362343$ $\qquad \frac{1362343}{6} = ·0227057$

$\sin\frac{23·62}{2} = \sin 11·81 = \sin 11\ 48·6 = ·2047069$ $\qquad \frac{2047069}{9} = ·0227474$

$\sin\frac{31·33}{2} = \sin 15·665 = \sin 15\ 40 = ·2700403$ $\qquad \frac{2700403}{12} = ·0225034$

that case was 144 square inches, and the effect per revolution $2^\circ{\cdot}61$. With the long rectangles the area is 128 square inches, and the mean of the two results is $2^\circ{\cdot}312$ per revolution. Now 144 square inches is to 128 square inches as $2^\circ{\cdot}61$ is to $2^\circ{\cdot}32$; a result so near to $2^\circ{\cdot}312$ that it may be here considered as the same; proving that the electric current induced is directly as the lines of magnetic force intersected by the moving wire*.

3200. It may also be perceived that no difference is produced when the lines of force are chiefly disposed in the direction of the motion of the wire, or else, chiefly in the direction of the length of the wire; *i. e.* no alterations are occasioned by variations in the *velocity* of the motion, or of the length of the wire, provided the amount of lines of magnetic force intersected remains the same.

3201. Having a square on the frame 12 inches in the side but consisting of copper wire 0·1 of an inch in thickness, I obtained the average result of many observations for one, two, three, four and five revolutions of the wire.

One revolution gave	7°	equal to	7°	per revolution.
Two revolutions gave	13·875	equal to	6·937	per revolution.
Three revolutions gave	21·075	equal to	7·025	per revolution.
Four revolutions gave	28·637	equal to	7·159	per revolution.
Five revolutions gave	37·637	equal to	7·527	per revolution.

These results are exceedingly close upon each other, especially for the first 30°, and confirm several of the conclusions before drawn (3189. 3199.) as to the indications of the instrument, the amount of the curves, &c.†

* Oblong rectangles of 128 square inches area give a mean of $20^\circ{\cdot}81$ (3198.). The rectangle of 144 square inches gave a mean of $23^\circ{\cdot}62$ (3197.).

$$\sin\frac{20{\cdot}81}{2}=\sin 10^\circ{\cdot}405=\sin 10^\circ\ 24'{\cdot}3={\cdot}1806049$$

$$\sin\frac{23{\cdot}62}{2}=\sin 11{\cdot}81\ =\sin 11\ 48{\cdot}6={\cdot}2047069$$

$$\frac{128}{144}=\frac{8}{9}\qquad \begin{aligned}{\cdot}1806049\times 9&=1{\cdot}6254441\\ {\cdot}2047069\times 8&=1{\cdot}6376552\end{aligned}\qquad \text{Or thus: }\begin{aligned}\frac{{\cdot}1806049}{8}&={\cdot}0225756\\ \frac{{\cdot}2047069}{9}&={\cdot}0227452\end{aligned}$$

†

	Differences.	
$\sin\frac{7}{2}=\sin 3^\circ{\cdot}30\ ={\cdot}0610485$		${\cdot}0610485$
	·0597381	
$\sin\frac{13{\cdot}875}{2}=\sin 6{\cdot}9375=\sin 6^\circ\ 56'{\cdot}25={\cdot}1207866$		$\frac{{\cdot}1207866}{2}={\cdot}0603933$
	·0620924	
$\sin\frac{21{\cdot}075}{2}=\sin 10{\cdot}5375=\sin 10\ 32{\cdot}25={\cdot}1828790$		$\frac{{\cdot}1828790}{3}={\cdot}0609596$
	·0644329	
$\sin\frac{28{\cdot}637}{2}=\sin 14{\cdot}3185=\sin 14\ 19{\cdot}11={\cdot}2473119$		$\frac{{\cdot}2473119}{4}={\cdot}0618279$
	·0752595	
$\sin\frac{37{\cdot}637}{2}=\sin 18{\cdot}8185=\sin 18\ 49{\cdot}11={\cdot}3225714$		$\frac{{\cdot}3225714}{5}={\cdot}0645142$

3202. At another time I compared the effect of equable revolutions with other revolutions very irregular in their rates, the motion being sometimes even backwards and continually differing in degree by fits and starts, yet always so that within the proper time a certain number of revolutions should have been completed. The rectangle was of wire 0·2 of an inch thick; the mean of many experiments, which were closely alike in their results, gave for two smooth, equable revolutions 17°·5, and also for two irregular uncertain revolutions the same amount of 17°·5.

3203. The relation of the current produced to the mass of the wire was then examined; a relation, which has been investigated on a former occasion by loops and small magnets (3133.)*. For the present purpose two other equal squares were prepared, each a foot in the side, but the copper wire of which they consisted was respectively 0·1 and 0·2 of an inch in diameter; so that with the former rectangle they formed a series of three, having the same size, shape and area, but the masses of the moving wire increasing in the proportion of one, four and sixteen. When the rectangle of 0·1 wire was employed, six direct revolutions gave an average result of 41°·75, and six to the left gave 46°·25; the mean of the two is 44°, and this divided by 6 gives 7°·33 as the deflection per revolution. Again, three direct revolutions gave 20°·12, and three reverse revolutions 23°·1; the mean being 21°·61, and the deflection per revolution 7°·20. This is very close to the former result with six revolutions, namely 7°·33, and is a large increase upon the effect of the rectangle of wire 0·05 in diameter, namely 2°·61; nevertheless, it is not as 4 : 1; nor could such a result be expected, inasmuch as the mass of the chief conductor remained the same (3137.). When the results are compared with those made with like wires in the form of loops, they are found to be exceedingly close; in that case the results were as 16° to 44°·4 (3136.), which would accord with a ratio in the present case of 2°·61 to 7°·26; and it is as 2°·61 to 7°·242, almost identical.

3204. The average of the direct and reverse revolutions is seen above to differ considerably, *i. e.* up to 4° and 5° in the higher case. This does not indicate any error in principle, but results simply from the circumstance, that when the needles were quiescent in the galvanometer, they stood a little on one side of zero (3182.). I did not wish to adjust the instrument at the time, as I was watching for spontaneous alterations of the zero place, and prefer giving the numbers as they came out in the graduation, to any pen-and-ink correction of the notes.

3205. The third square of 0·2 wire gave such large swings, that I employed only a small number of revolutions. Three direct revolutions gave an average of 25°·58; three reverse revolutions gave 28°·5; the mean is 27°·04, and the amount per revolution 9°·01. Again, two direct revolutions gave 17°·5; two reverse revolutions gave 18°; the mean is 17°·75, and the amount per revolution 8°·87; the mean of the two final

* See a corresponding investigation by Christie. Philosophical Transactions, 1833, p. 120.

results is 8°·94, and is again an increase on the effect produced by the preceding rectangle of wire, only half the diameter of the present. This thickness of wire was also employed formerly as a loop (3136.); and if we compare the results then obtained with the present results, it is remarkable how near they approach to each other; a circumstance which leads to great confidence in the principles and practice of both forms of examination. When wires having masses in the proportion of 1 : 4 and 16 were employed as loops, the currents indicated by the galvanometer were as 1·00, 2·77, and 3·58; now that they are employed as rectangles subject to the earth's magnetic power, they are as 1·00, 2·78, and 3·45*.

3206. I formed a square, 12 inches in the side, of four convolutions of copper wire 0·05 of an inch in diameter; the single wire which formed it was consequently 16 feet long. Such a rectangle will, in revolving, intersect the same number of lines of magnetic force as the former rectangle made with wire 0·1 in diameter (3203.); there will also be the same mass of wire intersecting the lines, but, as a conductor, the first wire has in respect of diameter, only one-fourth the conducting power of the second; and then, to increase the obstruction, it is four times as long. Six direct revolutions gave an average result of 20°·6, and six reverse revolutions 19°·7; the mean is 20°·15, and the proportion per revolution 3°·36. With the other rectangle having equal area and mass, but a single wire (3203.), the result per revolution was 7°·26; being above, though near upon twice as much as in the present case. Hence for such an excellent conducting galvanometer as that described (3123. 3178.), the moving wire had better be as one single thick wire rather than as many convolutions of a thin one. If it be, under all variations of circumstances, the same wire for the same area, then, of course, two or more convolutions are better than one.

3207. It was to be expected, however, that the thin wire rectangle would produce a current of more *intensity* than that in the thick wire, though less in quantity; and to prove this point experimentally, I connected the two rectangles in succession with RUHMKORFF's galvanometer (3086.), having wire only $\frac{1}{133}$th of an inch in diameter. That of the single thick wire now gave only 1°·66 of swing for twelve revolutions of the rectangle, or 0°·138 per revolution; whilst the other of four convolutions of thin

* $\sin \frac{27·04}{2} = \sin 13°·52 = \sin 13° 31'·2 = ·2337848$ $\frac{·2337848}{2} = ·0779283$. The square 12 inches side, of wire 0·05 in diameter, gave for six revolutions (3196. 3197.) ·0227057 as $\sin \frac{1}{2}$ A for one revolution. A like square of wire 0·10 in diameter gave for five revolutions (3021.) $\frac{·3225714}{2} = ·06451428$ as $\sin \frac{1}{2}$ A for one revolution. A like square of wire 0·20 in diameter gave ·0779283 as $\sin \frac{1}{2}$ A for one revolution

$$\frac{·06451428}{·0227057} = 2·841. \qquad \frac{·0779283}{·0227057} = 3·432.$$

wire gave for twelve revolutions 7°·33, or 0°·61 per revolution. Now the needles of the two instruments were not very different in weight and other circumstances, so that without pretending to an accurate comparison, we may still perceive an immense falling-off in both cases, due to the obstruction of the fine wire in the Ruhmkorff's galvanometer; for the thick wire it is from 7°·26 to 0°·138, and for the thin wire from 3°·36 to 0°·610. Still the thin wire rectangle has lost far less proportionately in power than the other; and by this galvanometer is above four times greater in effect than the rectangle of thicker wire. Of the thick wire effect less than a *fiftieth* passes the fine wire galvanometer, all the rest is stopped; of the fine wire effect more than ten times this proportion, or between a fourth and a fifth (because of the higher intensity of the current), surmounts the obstruction presented by the instrument. The quantity of electricity which really passes through the fine wire galvanometer is of course far less than in the proportion indicated above. The thick wire coil makes at the utmost four convolutions about the needles, whereas in the fine wire coil there are probably four hundred or more; so that the electricity which really travels forward as a current, is probably not a hundredth part of that which would be required to give an equal deflection in the thick wire galvanometer. Such a circumstance does not disturb the considerations with respect to the relative intensity of the magneto-electric currents from the two rectangles, which have been stated above.

3208. A large square was now constructed of copper wire 0°·2 of an inch in diameter. The square was 36 inches in the side, and therefore consisted of 12 feet of wire, and inclosed an area of 9 square feet; it was attached to the commutator by expedients, which, though sufficient for the present, were not accurate in the adjustments. It produced a fine effect upon the thick wire galvanometer (3178.); for one revolution caused a swing deflection of 80° or more; and when its rotation was continuous the needles were permanently deflected 40° or 50°. It was very interesting to see how, when this rectangle commenced its motion from the horizontal plane, the current increased in its intensity and then diminished again, the needles showing, that whilst the first 10° or 20° of revolution were being passed, there was very little power exerted in them; but that when it was towards, or near the 90°, the power was great; the wires then intersecting the lines of force nearly at right angles, and therefore, with an equal velocity, crossing the greatest number in a given time. It was also very interesting, by the same indications, to see the two chief impulses (3192.) given in one revolution of the rectangle. Being large and massive in proportion to the former wires, more time was required for a rotation than before, and the point of *time* or *velocity* of rotation became more essential. One rotation in a second was as much as I could well produce. A speed somewhat less than this was easy, convenient and quick enough; it gave for a single revolution near 80°, whilst a revolution with one-half or one-third the velocity, or less, gave only 60°, 50°, or even smaller amounts of deflection.

3209. Observations were now made on the measurement of one rotation having an easy quick velocity. The average of fifteen observations to the right, which came very near to each other, was 78°·846; the average of seventeen similar observations to the left was 78°·382; and the mean of these results, or 78°·614, I believe to be a good first expression for this rectangle. On measuring the distances across after this result, I found that in one direction, *i. e.* across between the intersecting portions of wire, it was rather less than 36 inches; having therefore corrected this error, I repeated the observations and obtained the result of 81°·44. The difference of 2°·83, I believe to be a true result of the alteration and increase of the area on making it more accurately 9 square feet; and it is to me an evidence of the sensibility and certainty of the instrument.

3210. As the two impulses upon the needles in one revolution (3208.) are here sensibly apart in time, and as the needle has as evidently and necessarily left its first place before the second impulse is impressed upon it, so, that second impulse cannot be so effectual as the first. I therefore observed the results with half a revolution, and obtained a mean of 41°·37 for the effect. This number evidently belongs to the first of the two impulses of one revolution; and if we subtract it from 81°·44, it gives 40°·07 as the value of the second impulse under the changed place of the needle. This difference of the two impulses of one revolution, namely 41°·37 and 40°·07, is in perfect accordance with the results that were to be expected.

3211. The square of this same copper wire, 0·2 in thickness, employed on a former occasion (3205.), had an area of one square foot, so that then the lines of force affected or affecting the moving wire, were one-ninth part of what they are in the present case: the effect then was 8°·94 per revolution. If, in comparing these cases, we take the ninth part of 81°·44, it gives 9°·04; a number so near the former, that we may consider the two rectangles as proving the same result, and at the same time the truth of the statement, that the magneto-electric current evolved is as the amount of lines of force intersected. A ninth part of the result with the large rectangle (78°·614), before its area was corrected, is 8°·734; so that the one is above and the other below the amount of the 12-inch rectangle. As that was not very carefully adjusted, nor indeed any of the arrangements made as yet with extreme accuracy, I have little doubt that with accurately adjusted rectangles the results would be strictly proportional to the areas*.

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* The 9 square feet rectangle gave 81°·44 sin $\frac{81\cdot 44}{2}$ = sin 40·72 = sin 40° 43′·2 = ·6523630: or taking 41°·37 for the half revolution for $\frac{1}{2}$ A (3210.) sin 41°·37 = sin 41° 22′·2 = ·6609190, which divided by nine give ·073435 as the force per square foot. The 1 square foot rectangle of like wire (3205.) gave ·07714, or ·07793 as the force of one revolution; the first of which is ·00370 more than $\frac{1}{9}$ of the measure of the effect of the large square; the difference being about $\frac{1}{29}$ of ·07714, or the whole force of one revolution.

3212. The moving wire, in place of being formed into a rectangle, may be adjusted as a ring; and then the advantage is obtained of the largest area which a given length of wire can inclose, and therefore for a uniform wire, the obstruction to the induced current, as respects its conduction, is the least. Small rings of one or several convolutions will probably be very valuable in the examination of small and local magnets under different circumstances. One consisting of ten spirals of copper wire $0^{\circ}{\cdot}032$ of an inch in diameter, containing 49 inches, in a ring about 1·5 inch in diameter, gave but small results under the earth's influence; but when brought near a horseshoe magnet told, in its effects for every difference in distance or in position. A single ring 4 inches in diameter, being made of a convolution of copper wire 0·2 in thickness, was employed with the earth's magnetic force as before; it gave as the average of six revolutions many times repeated $5^{\circ}{\cdot}995$, or $0^{\circ}{\cdot}999$ per revolution. For twelve revolutions it gave a mean of $12^{\circ}{\cdot}375$, or $1^{\circ}{\cdot}031$ per revolution*; the mean of the two results with such different numbers of revolutions being 1°. Another ring, consisting of twenty-six convolutions of copper wire 0·04 of an inch in diameter, was constructed and had a mean diameter of 3·6 or 3·7 inches; it contained 300 inches in length of wire. So the masses of the metal in the two rings are nearly the same, but the latter wire is singly only $\frac{1}{25}$th of the mass of the former. It gave for twelve revolutions a mean of $6^{\circ}{\cdot}25$, or $0^{\circ}{\cdot}52$ per revolution. With the earth's power and the thick wire galvanometer, it gave therefore little more than half the result of the single thick wire ring. We know from former considerations (3206.), that if the 300 inches had been made into one single ring, it would have given a very high effect compared to the present.

3213. The application of the principle of the moving wire in the form of a revolving rectangle, makes the investigation of *conducting* power, and the results produced by difference in the nature of the *substance*, or in diameter, *i. e. mass*, or in *length*, very easy; and the obstruction offered by those parts, which moving not across but parallel to the lines of force (3071.), have no exciting action but perform the part of conductors merely, might be greatly removed by making them massive. They might be made to shift upon the axle so as to bear adjustment for different lengths of wires, and the commutator might in fact be made to a large extent a general instrument.

3214. In looking forward to further applications of the principle of the moving wire, it does not seem at all unlikely that by increased delicacy and perfection of the instrument, by increased velocity, by continued motion for a time in one direction and then reversal of the revolution with the reversal of the direction of the swing, &c., it may be applied with advantage hereafter to the investigation of the earth's magnetic force in different latitudes and places. To obtain the maximum

* $\sin \frac{5{\cdot}995}{2} = \sin \overset{\circ}{2}{\cdot}9975 = \sin \overset{\circ}{2}\ \overset{\prime}{59}{\cdot}85 =$ $\qquad$ ·0522925

$\sin \frac{12{\cdot}375}{2} = \sin 6{\cdot}1875 = \sin 6\ \ 11{\cdot}25 = {\cdot}1077825$ $\qquad$ $\frac{{\cdot}1077825}{2} = {\cdot}0538912.$

effect, the axis of rotation must be perpendicular to the lines of force, *i. e.* the dip. It would even be possible to search for the *direction* of the lines of force, or the dip, by making the axis of rotation variable about the line of dip, adjusting it in two directions until there was no action at the galvanometer, and then observing the position of the axis; a double commutator would be required corresponding to the lines of adjustment, but that is of very simple constrnction.

§ 36. *On the amount and general disposition of the Forces of a Magnet when associated with other magnets.*

3215. Prior to further progress in the experimental development by a moving wire of the disposition of the lines of magnetic force pertaining to a magnet, or of the physical nature of this power and its possible mode of action at a distance, it became quite essential to know what change, if any, took place in the amount of force possessed by a perfect magnet, when subjected to other magnets in favourable or adverse positions; and how the forces combined together, or were disposed of, *i. e.* generally, and in relation to the principle already asserted and I think proved, that the power is in every case definite under those different conditions. The representation of the magnetic power by *lines of force* (3074.), and the employment of the moving wire as a test of the force (3076.), will I think assist much in this investigation.

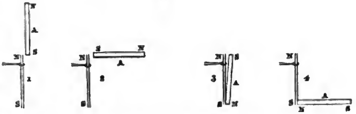
3216. For such a purpose an ordinary magnet is a very irregular and imperfect source of power. It not only, when magnetized to a given degree, is apt by slight circumstances to have its magnetic power diminished or exalted, in a manner which may be considered for the time, permanent; but if placed in adverse or favourable relations to other magnets, frequently admits of a considerable temporary diminution or increase of its power externally, which change disappears as soon as it is removed from the neighbourhood of the dominant magnet. These changes produce corresponding effects upon the moving wire, and they render any magnet snbject to them unfit for investigation in relation to definite power. Unchangeable magnets are, therefore, required, and these are best obtained, as is well known, by selecting good steel for the bars, and then making them exceedingly hard; I therefore procured some plates of thin steel 12 inches long and 1 inch broad, and making them as hard as I could, afterwards magnetized them very carefully and regularly, by two powerful steel bar-magnets, shook them together in different and adverse positions for a little while, and then examined the direction of the forces by iron filings. Small cracks and irregularities were in this way detected in several of them; but two which were very regular in the disposition of their forces were selected for further experiment, and may be distinguished as the subjected magnets D and E.

3217. These two magnets were examined by the moving loop precisely in the manner before described (3133.), *i. e.* by passing the loop over one of the poles, observing the swing, removing it, and again observing the swing and taking an average of many results; the process was performed over both poles at different times. The loop

contained 7·25 inches in length of copper wire 0·1 of an inch in diameter, and was of course employed in all the following comparative experiments; the distance of the loop and magnets from the galvanometer was 9 feet. For one passage over the pole either on or off, *i. e.* for one intersection of the lines of force of the magnet D, the galvanometer deflection was 8°·36. For one intersection of the lines of force of the other bar E, the deflection was 8°·78. The two bars were then placed side by side with like poles together, and afterwards used as one magnet; their conjoined power was 16°·3, being only 0°·84 less than the sum of the powers of the two when estimated separately. This indicates that the component magnets do affect, and in this position reduce, each other somewhat; but it also shows how small the effect is as compared with ordinary magnets (3222.).

3218. The compound magnet D E (3217.) was now subjected to the close action of another magnet, sometimes under adverse, and at other times under favourable conditions; and was examined by the loop as to the sum of its power (not the direction) under these circumstances. For this purpose it was fixed, and another magnet A brought near, and at times in contact with it, in the positions indicated by the figure 8; the loop in each case being applied many times to D E, that a correct average of its power might be procured. The dominant magnet A was much the stronger of the two, having the power indicated by a swing deflection of 25°·74.

Fig. 8.



3219. When the relative position of the magnets was as at 1, then the power of D E was 16°·37; when as at 2, the power was 16°·4; when as at 3, it was 18°·75; and when as at 4, it was 17°·18. All these positions are such as would tend to raise, by induction, the power of the magnet D E, and they do raise it above its first value, which was 16°·3; but it is seen at once how little the first and second positions elevate it; and even the third, which presents the most favourable conditions, only increases the power 2°·45, which falls again in the fourth position.

3220. Then the dominant magnet A was placed in the same positions, but with the ends reversed, so as to exert an adverse or depressing influence; and now the results with D E were as follows:—

Position 1	15°·37
Position 2	15·68
Position 3	15·37
Position 4	16·06

All these are a little below the original force of D E, or 16°·3, as they ought to be, and show how slightly this hard bar-magnet is affected.

3221. A soft iron bar, now applied in the first, second and third positions instead of the magnet A, raised D E to the following values respectively, 16°·24, 16°·43, and 18°.

3222. When an ordinary bar-magnet was employed instead of the hard magnet D E, great changes took place. Thus a bar B, corresponding to bar A in size and general character, was employed in place of the hard magnet. Alone, B had a power of 14°·83, but when associated adversely with A, as in position 3 (3218.), its power fell to 7°·87, being reduced nearly one-half. This loss was chiefly due to a coercion internally, and not to a permanent destruction of the state of magnet B; for when A was removed, B rose again to 13°·06. When B was laid for a few moments favourably on A and then removed, it was found that the latter had been raised to a permanent external action of 15°·25.

3223. A very hard steel bar 6 inches long, 0·5 broad and 0·1 in thickness, given to me by Dr. Scoresby, was magnetized and then found, by the use of the loop, to have a value at my galvanometer of 6°·88 (3189.). It was submitted in position 2 to a compound bar-magnet like D E, having a power of 11°·73, or almost twice its own force, but whether in the adverse or the favourable position, its power was not sensibly altered. When submitted in like manner to a 12-inch bar-magnet having a force of 40°·21, it was raised to 7°·53, or lowered to 5°·87, but here the dominant magnet had nearly six times the power of the one affected.

3224. The variability of soft steel magnets, both in respect of their *absolute* degree of excitation or charge, and also of the disposition of the force externally and internally, when their degree of excitation may for the time be considered as the same, is made very manifest by this mode of examination; and the results agree well with our former knowledge in this respect. It is equally manifest, that hard and invariable magnets are requisite for a correct and close investigation of the disposition and characters of the magnetic force. A common soft bar-magnet may be considered as an assemblage of hard and soft parts, disposed in a manner utterly uncertain; of which some parts take a much higher charge than others, and change less under the influence of external magnets; whilst, because of the presence of other parts within, acting as the keeper or submagnet, they may seem to undergo far greater changes than they really do. Hence the value of these hard and comparatively unchangeable magnets which Scoresby describes.

3225. From these and such results, it appears to me, that with perfect, unchangeable magnets, and using the term *line of force* as a mere representant of the force as before defined (3071. 3072.), the following useful conclusions may be drawn.

3226. Lines of force of different magnets in favourable positions to each other coalesce.

3227. There is no increase of the total force of the lines by this coalescence; the section between the two associated poles gives the same sum of power as that of the section of the lines of the invariable magnet when it is alone (3217.). Under these

circumstances there is, I think, no doubt that the external and internal forces of the same magnet have the same relation and are equivalent to each other, as was determined in a former part of these Researches (3117.); and that therefore the equatorial section, which represents the sum of forces or lines of forces passing through the magnet, remains also unchanged (3232.).

3228. In this case the analogy with two or more voltaic batteries associated end to end in one circuit is perfect. Probably some effect, correspondent to *intensity* in the case of the batteries, will be found to exist amongst the magnets.

3229. The increase of power upon a magnetic needle, or piece of soft iron placed between two opposite, favourable poles, is caused by concentration upon it of the lines which before were diffused, and not by the addition of the power represented by the lines of force of one pole to that of the lines of force of the other. There is no more power represented by all the lines of force than before; and a line of force is not more powerful because it coalesces with a line of force of another magnet. In this respect the analogy with the voltaic pile is also perfect.

3230. A line of magnetic force being considered as a closed circuit (3117.), passes in its course through *both* the magnets, which are for the time placed so as to act on each other favourably, *i. e.* whose lines coincide and coalesce. Coalescence is not the addition of one line of force to another *in power*, but their union in one common circuit.

3231. A line of force may pass through many magnets before its circuit is complete; and these many magnets coincide as a case with that of a single magnet. If a thin bar-magnet 12 inches long be examined by filings (3235.), it will be found to present the well-known beautiful system of forces, perfectly simple in its arrangement. If it be broken in half, without being separated, and again examined, the manner in which, from the destruction of the continuity, the transmission of the force at the equator is interfered with, and many of the lines, which before were within are made to appear externally there, is at once evident, Plate IX. fig. 6. Of those lines, which thus become external, some return back to the pole which is nearest to the new place, at which the lines issue into the air, making their circuit through only one of the halves of the magnet; whilst others proceed onward by paths more or less curved into the second half of the magnet, keeping generally the direction or polarity which they had whilst within the magnet, and complete their circuit through the two. Gradually separating the two halves, and continuing to examine the course of the lines of force, it is beautiful to observe how more and more of the lines which issue from the two new terminations, turn back to the original extremities of the bar, fig. 7, and how the portion which makes a common circuit through the two halves diminishes, until the halves are entirely removed from each other's influence, and then become two separate and independent magnets. The same process may be repeated until there are many magnets in place of one.

3232. All this time the amount of lines of force is the same if the fragments of the

bar preserve their full state of magnetism ; *i. e.* the sum of lines of force in the equator of *either* of the new magnets is equal to the sum of lines of force in the equator of the original unbroken bar. I took a steel bar 12 inches long, 1 inch broad and 0·05 of an inch thick, made it very hard, and magnetized it to saturation by the use of soft iron cores and a helix ; its power was 6°·9. I broke it into two pieces nearly in the middle, and found the power of these respectively 5°·94 and 5°·89 ; indicating a fall not more than was to be expected considering the saturated state of the original magnet. When these halves were placed side by side, with like poles together as a compound magnet, they had a joint power of 11°·06, which, though it shows a mutual quelling influence, is not much below the sum of their powers ascertained separately. All this is in perfect harmony with the voltaic battery, where lines of dynamic electric force are concerned. If, as is well known, we separate a battery of 20 pair of plates into two batteries of 10 pair, or 4 batteries of 5 pair, each of the smaller batteries can supply as much dynamic electricity as the original battery, provided no sensible obstruction be thrown into the course of the lines, *i. e.* the path of the current.

3233. When magnets are placed in an adverse position, as neither could add power to the other in the former case, so now each retains its own power ; and the lines of magnetic force represent this condition accurately. Two magnets placed end to end with like poles together are in this relation ; so also are they if placed with like poles together side by side. In the latter case the two acting as one compound magnet, give a system of lines of force equal to the sum of the two separately (3232.), minus the portion which, as in imperfect magnets, is either directed inwards by the softer parts or ceases to be excited altogether.

§ 37. *Delineation of Lines of Magnetic Force by iron filings.*

3234. It would be a voluntary and unnecessary abandonment of most valuable aid, if an experimentalist, who chooses to consider magnetic power as represented by lines of magnetic force, were to deny himself the use of iron filings. By their employment he may make many conditions of the power, even in complicated cases, visible to the eye at once ; may trace the varying direction of the lines of force and determine the relative polarity ; may observe in which direction the power is increasing or diminishing ; and in complex systems may determine the neutral points or places where there is neither polarity nor power, even when they occur in the midst of powerful magnets. By their use probable results may be seen at once, and many a valuable suggestion gained for future leading experiments.

3235. Nothing is simpler than to lay a magnet upon a table, place a flat piece of paper over it, and then, sprinkling iron filings on the paper, to observe the forms they assume. Nevertheless, to obtain the best and most generally useful results, a few particular instructions may be desirable. The table on which the magnet is laid should be quite horizontal and steady. Means should be taken, by the use of thin boards or laths, or otherwise, to block up round the magnet, so that the paper which

is laid over it should be level. The paper should be without any cockle or bend, and perfectly flat, that the filings may be free to assume the position which the magnet tends to give them. I have found well-made cartridge or thin drawing-paper good for the purpose. It should not be too smooth in ordinary cases, or the filings, when slightly agitated, move too freely towards the magnet. With very weak or distant magnets I have found silvered paper sometimes useful. The filings should be clean, *i. e.* free from much dirt or oxide; the latter forms the lines but does not give good delineations. The filings should be distributed over the paper by means of a sieve more or less fine, their quantity being partly a matter of taste. It is to be remembered, however, that the filings disturb in some degree the conditions of the magnetic power where they are present, and that in the case of small magnets, as needles, a large proportion of them should be avoided. Large and also fine filings are equally useful in turn, when the object is to preserve the forms obtained. For the distribution of the latter it is better to use a fine sieve with the ordinary filings than to separate the filings first: a better distribution on the paper is obtained. The filings being sifted evenly on the paper, the latter should be tapped very lightly by a small piece of wood, as a pen-holder; the taps being applied wherever the particles are not sufficiently arranged. The taps must be perpendicularly downwards, not obliquely, so that the particles, whilst they have the liberty of motion, for an instant are not driven out of their places, and the paper should be held down firmly at one corner, so as not to shift right or left; the lines are instantly formed, especially with fine filings.

3236. The designs thus obtained may be fixed in the following manner, and then form very valuable records of the disposition of the forces in any given case. By turning up two corners of the paper on which the filings rest, they may be used as handles to raise the paper upwards from the magnet, to be deposited on a flat board or other plane surface. A solution of one part of gum in three or four of water having been prepared, a coat of this is to be applied equably by a broad camel-hair pencil, to a piece of cartridge paper, so as to make it fairly wet, but not to float it, and after wafting it through the air once or twice to break the bubbles, it is to be laid carefully over the filings, then covered with ten or twelve folds of equable soft paper, a board placed over the paper, and a half-hundred weight on the board for thirty or forty seconds. Or else, and for large designs it is a better process, whilst the papers are held so that they shall not shift on each other, the hand should be applied so as to rub with moderate pressure over all the surface equably and in one direction. If, after that, the paper be taken up, all the filings will be found to adhere to it with very little injury to the forms of the lines delineated; and when dry they are firmly fixed. If a little solution of the red ferroprussiate of potassa and a small proportion of tartaric acid be added to the gum-water, a yellow tint is given to the paper, which is not unpleasant; but besides that, prussian blue is formed under every particle of iron; and then when the filings are purposely or otherwise dis-

placed, the design still remains recorded. When the designs are to be preserved in blue only, the gum may be dispensed with and the red ferroprussiate solution only be used.

3237. It must be well understood that these forms give no indication by their appearance of the relative strength of the magnetic force at different places, inasmuch as the appearance of the lines depends greatly upon the quantity of filings and the amount of tapping; but the direction and forms of the lines are well given, and these indicate, in a considerable degree, the direction in which the forces increase and diminish.

3238. Plate IX. fig. 1, shows the forms assumed about a bar-magnet. On using a little electro-magnet and varying the strength of the current passed through it, I could not find that a variation in the strength of the magnet produced any alteration in the forms of the lines of force external to it. Fig. 2 shows the lines over a pole, and fig. 3 those between contrary poles. The latter accord with the magnetic curves, as determined and described by Dr. Roget and others, with the assumption of the poles as centres of force. The difference between them and those belonging to a continuous magnet, shown in fig. 1, is evident. Figs. 4, 5 show the lines produced by short magnets. In the latter case the magnet was a steel disc about one inch in diameter and 0·05 in thickness. Fig. 6 shows the result when a bar-magnet is broken in half, but not separated. Fig. 7 shows the development of the lines externally at the two new ends as the halves are more and more separated (3231.). Figs. 8, 9 and 10 present the results, with the two halves or new magnets in different positions. Figs. 11, 12, 13 and 14 show the results with disc magnets. Fig. 15 shows the condition of a system of magnetic forces when it is inclosed by a larger one, and is contrary to it. Fig. 16 shows the coalescence of the lines of force (3226.) when the magnets are so placed that the polarities are in accordance.

3239. Fig. 17 exhibits the lines of force round a vertical wire carrying a current of electricity. Whether the wire was thick or thin appeared to make no difference as to the intensity of the forces, the current remaining the same. Fig. 18 represents the lines round two like currents when within mutual influence. Fig. 19 shows the result when a third current is introduced in the contrary direction. Fig. 20 presents the transition to a helix of three convolutions. Fig. 21 indicates the direction of the lines within and outside the end of a cylindrical helix, on a plane through its axis. Fig. 22 presents the effect when a very small soft iron core is within the helix.

3240. Figs. 23 and 24 give an experimental illustration of the principles which I have adopted in relation to atmospheric magnetism and the general cause of the daily variations, &c. (2864. 2917.). A hemisphere of pure nickel presented to me by Dr. Percy, was supported with its flat face uppermost, and a large ring arranged round it to carry paper, which, resting both on the ring and the nickel, could then have iron filings sprinkled and arranged in form on it. The end of a bar-magnet in the same horizontal plane was adjusted about 2 inches from the nickel, and thus the

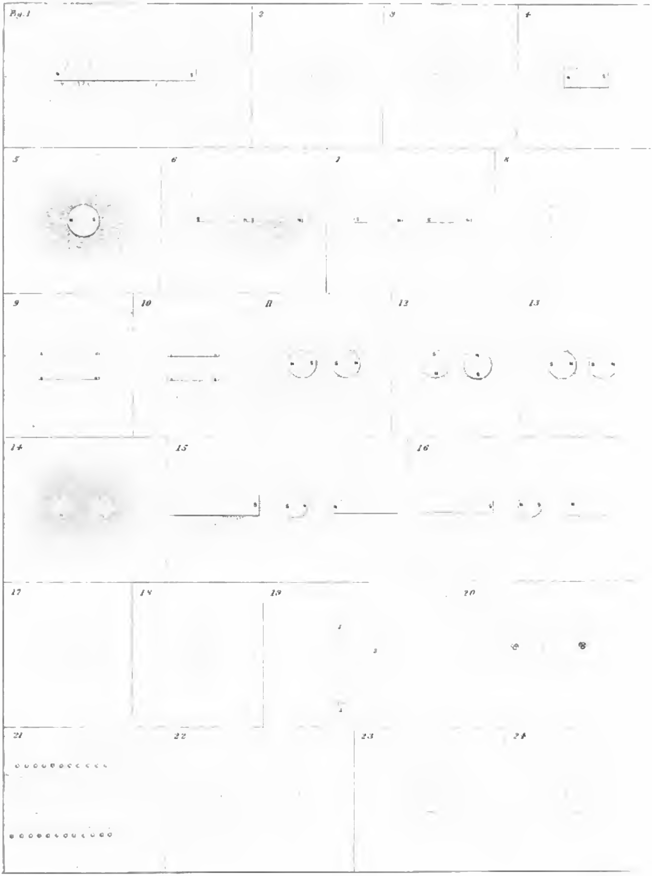


Fig. 1
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forms of the lines of force associated with this pole could be determined over the place of the nickel hemisphere, under different circumstances, or even when it was removed. When the nickel was away, the forms of the lines of force were as in fig. 23; when the nickel was there, they were as in fig. 24. The application of a spirit-lamp to the nickel when in its place, raised its temperature to such a degree (above 600° FAHR.) that it lost its ordinary magnetic condition; and then the forms of the lines of force, as shown by filings, were the same as if the nickel was away. Removing the lamp, I was able to obtain the disposition of filings on successive pieces of paper, and as many as four results, like fig. 23, could be procured before the temperature had sunk so much as to cause the production of lines of force corresponding to fig. 24.

3241. These are exactly the same results with nickel as those I have assumed for the oxygen of the atmosphere. The change in the forms of the lines about the cooling nickel in this experiment are the same changes as those I have figured in the type globe of cooling air (2865. 2874.). Both nickel and oxygen are paramagnetic bodies, and change in the *same direction* by heating and cooling; and as the period of change with oxygen extends through degrees above and below common temperature (2861.), so inflections of the lines of force passing through the atmosphere, correspondent to those of the heating and cooling nickel, *must* take place to some extent. It is seen in the nickel results, that lines of force entirely outside of it do not for that reason continue an undeviating course, but are curved to and fro in consequence of the disposition of other lines within the nickel; a result, which, without reference to either one view or another of the physical action of the magnetic force, must be as true in the oxygen case as in the nickel case, because of the definite character of the magnetic force, whether represented by centres of action or by lines of power.

3242. Whether the amount of the deflection in the case of the atmosphere corresponds with the facts registered by observers, is a question which cannot be answered, I suppose, until we know the effect of very low temperatures upon the magnetic force of the atmosphere. In the nickel experiment the deflection is in places 30° or 40°; in nature the effect to be accounted for is not more than 13 or 14 minutes.

Royal Institution,
December 20, 1851.

XII. *On Symbolic Forms derived from the Conception of the Translation of a Directed Magnitude.* *By the Rev.* M. O'Brien, *M.A., Late Fellow of Caius College, Cambridge, and Professor of Natural Philosophy and Astronomy in King's College, London.*

Received April 21,—Read June 19, 1851.

PART I.

GENERAL INVESTIGATION OF THE SYMBOLIC FORMS.

(1.) THERE can be no doubt, that time and ingenuity have been often wasted in devising systems of notation, and new methods of algebraical representation, which have never proved of any service in advancing the cause of science. It is not surprising, therefore, that symbolical innovations, if they have not the strongest and most obvious reasons to recommend them, are generally received with little favour by mathematicians. At the same time, it must not be forgotten, that the mind has wonderfully enlarged its powers of research by the symbolization of its abstract conceptions, and that the various additions which have been made, from time to time, to mathematical notation, have contributed largely to the progress of physical investigation; witness, for instance, the applications of the negative sign, indices, logarithms, coordinate equations, the differential algorithm, &c.

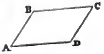
A new notation, or a new application of an old notation, ought, in all cases, to be called for by some want in science, that is, by the existence of some important and often occurring conception for which there is no adequate, or at least no sufficiently general mode of representation. It should be neither artificial nor complicated, but natural and simple: it should also be based on principles of established authority, and framed according to allowed precedents. And, lastly, it should be capable of something more than mere elementary applications, and be recommended by its utility in the higher and more abstruse branches of science.

With these cautions before me, and on these grounds, I venture, in the present paper, to propose a new use of an old notation, which appears to me to supply a want of considerable importance, as I hope to show by the remarkable simplifications which it introduces into many difficult investigations. There is an operation, if I may so call it, of constant occurrence in Geometry and Physics, which consists in the *translation of a directed magnitude*, that is, the parallel motion of a magnitude possessing the property of direction, such, for example, as a force, a velocity, traced line, or the like. This translation, as it may be easily shown, is always an operation

of the kind called "*distributive*," and from its distributiveness all its properties in a great measure follow as necessary consequences. Now this is a fact of great importance to be borne in mind, a thing to which the truly mathematical adage "*when found make a note of*" fully applies. There is no notation actually in use for this purpose; but there are two nearly obsolete signs, which by an easy and natural generalization, and according to the most approved rules of mathematical interpretation, might be made to serve effectually as representatives of the effects produced generally by the translation of a directed magnitude. My design, in what follows, is to show this, and establish the laws according to which these signs are to be used in their enlarged signification. Afterwards I shall endeavour to justify the proposed innovation, if such it is to be considered, by showing its utility in a variety of cases,

(2.) *Instances which suggest the proposed symbolization.*—There are three elementary conceptions which have suggested to my mind the principles which it is the object of this paper to develope, and they will serve here as means of introducing the subject, and furnish the best foundation to build upon. They are the following:—

Fig. 1.



1st. *The generation of surface by the parallel motion of a right line*, of which the simplest instance is a parallelogram ABCD supposed to be generated by the motion of AB, parallel to itself, along AD.—2ndly. *The effect produced on a rigid body by the translation of a force acting upon it* from one point A to another B, the direction of the force remaining unaltered; which effect, as is well known, consists in that peculiar tendency to motion that results from the action of the couple composed of the force at B, and a force equal and opposite to the original force at A.—3rdly. *The effect produced by the translation of a force resulting from the actual motion of its point of application*; which effect is now usually designated by the term *work*. If the point of application be supposed to describe the path ADB, the force all the time acting parallel to its original direction, a certain amount of work is accumulated in consequence of the translation of the force, and this is the effect I allude to.

Fig. 2.

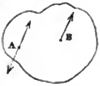
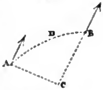


Fig 3.



Now in each of these cases the conception in the mind is that of *the effect produced by the translation of a directed magnitude*; and what is worthy of special remark is this, that in each of these cases the effect alluded to is represented by the *product of two factors, one* being *the translated magnitude*, and *the other the amount of translation* it undergoes. Thus, in the case shown by figure 1, the surface generated is denoted by the product of AB into the perpendicular or *lateral* distance between AB and CD; of which, AB is the translated magnitude, and the perpendicular the amount of translation that takes place *laterally*. In the case shown by figure 2, the effect is also represented by the product of the translated magnitude, that is, the force, into the amount of *lateral* translation. In the case shown by figure 3, if we suppose BC to be the direction of the force produced back-

ward to meet a perpendicular AC let fall from A upon it, the effect produced, that is, the work accumulated, is represented by the product of the translated magnitude (the force, namely,) into the amount of translation which takes place, not laterally, as before, but *longitudinally*, that is, *along* the direction of the force, which longitudinal translation is manifestly CB.

(3.) From these three instances the idea naturally arises of *some necessary connection* between the translation of a directed magnitude and the product of the two factors, the magnitude translated and the amount of translation; or, to say the least, there appears to be some ground for conceiving that the product in question may be the *proper form of notation* for representing the translation. And, secondly, the necessity of *distinguishing* between *lateral* and *longitudinal* translation is clearly indicated, inasmuch as the longitudinal effect is zero in the first two cases, while the lateral effect is zero in the third case. Taking my clue from these suggestions, I shall now proceed to explain my proposed method of notation; observing, that my object is to make it as general as possible consistently with definiteness and utility, and that, for this reason, I shall employ all the generalizations of the elementary algebraical signs which are now admitted by mathematicians.

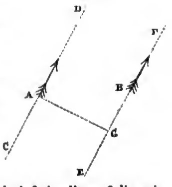
I shall also adopt, to a certain extent, the views of Symbolical Algebra taken by the late Mr. GREGORY, and published by him in several papers, but especially in one read before the Royal Society of Edinburgh on the Foundations of Algebra. I may observe, however, that my proposed method of notation does not assume the correctness of these views, and might be enunciated independently of them; but they appear to my mind to form the most satisfactory theory of Symbolical Algebra.

I. PRELIMINARY DEFINITIONS, STATEMENTS, ETC.

(4.) *Directed Magnitude.*—I use this term to denote any of those magnitudes which we represent graphically by *arrows*; remarking that an arrow represents *three* things, viz. an *origin* or *point of application*, marked by its *feather-extremity*; a particular *magnitude* represented by its *length*; and a particular *direction* shown by the *barb*.

(5.) *Translation, Lateral and Longitudinal.*—"Translation" is the term employed to denote that peculiar and simplest change of position of a rigid body which consists in the parallel and equal motion of all its component particles. I shall use the same term to denote a change of position of a directed magnitude without change of direction, as is shown in fig. 4. The translation therefore of a directed magnitude consists in simple alteration of "*origin*," as from A to B in the figure.

Fig. 4.



But this alteration of origin is of a *twofold* nature, being partly *lateral* and partly *longitudinal*. If CD be the indefinite line of direction

in which the arrow at A lies, and EF that in which the arrow at B lies, the translation from A to B consists of two distinct motions; namely, the shifting of the line of direction from CD to EF, and the shifting of the origin along the line of direction through a space amounting to GB. The former I shall call *lateral* and the latter *longitudinal* translation.

Fig. 5.

If v denote a directed magnitude which is supposed to be translated from A to B, and if u denote the line AB, I shall speak of the translation as *that of v along u*; a proper mode of expression, because every point of the representative arrow v undergoes a motion equivalent in magnitude and direction to u. The translation of v along u is lateral when the angle at A is 90°, and longitudinal when 0°.

(6.) *Distributiveness.*—If $f(x)$ be a function of x which possesses the property expressed by the equation

$$f(x)+f(x')=f(x+x'),$$

it is said to be a *distributive* function of x. If x be any *number* positive, negative, integral or fractional, it may be shown from this equation that

$$f(x)=\mathrm{C}x,$$

C denoting a quantity independent of x, namely the value of $f(x)$ when $x=1$. If x be not a number, but some symbol, whether of specific quantity or operation, the notation Cx has no meaning recognised in ordinary algebra. Hence, following the well-known precedent of indices*, we may generalize the meaning of Cx by assuming it to be the *symbolical form* for denoting *every function, $f(x)$, which is distributive.*

If we further suppose, that $f(x)$, and therefore C, is a distributive function of another independent variable y, we shall find that

$$\mathrm{C}=\mathrm{C}'y \quad \text{and} \therefore \mathrm{C}x=\mathrm{C}'xy.$$

Thus we may, by the same process of generalization, assume C$'xy$ to be the *symbolical form* for denoting *every function* of x and y *which is distributive* with regard to both x and y. C$'$ here is manifestly the value of the function when $x=1$ and $y=1$. Now if we adopt C$'$ to be the unit of the function, as, in fact, we do in many cases of ordinary products, the symbolic form for denoting the function becomes simply xy.

(7.) This appears to me to be the simplest and best method of defining the notation xy in Symbolical Algebra; though I need not avail myself of it here as it is not necessary for my purpose. All I require is some simple notation for denoting a distributive function of two variables; for, as I hope to show, this distributiveness is a characteristic of great importance to be distinctly "noted" in the case of the translation of a directed magnitude. Now there are three different forms in which a product is written in ordinary algebra, viz. xy $x.y$ and $x\times y$: of these, the latter two are now seldom used, and there is no necessity whatever for this redundancy of

* a^x has no meaning, according to its original definition, except x be a positive integer: but we give it a meaning by defining a^x to be the notation for every function $f(x)$ which possesses the property $f(x)f(y)=f(x+y)$.

forms for denoting the same thing. Instead, therefore, of inventing new symbols for representing distributive functions, I shall venture to appropriate the almost obsolete forms $x.y$ and $x\times y$ to the purpose. At the same time it must be borne in mind, that, according to the views of many eminent mathematicians, the product of x and y in Symbolical Algebra may be defined to be *any* distributive function of x and y, and thus the appropriation of $x.y$ and $x\times y$ here proposed is nothing more than a legitimate application of these forms.

(8.) I shall therefore assume $x.y$ and $x\times y$ to be symbolical forms for denoting distributive functions of x and y; in other words, I shall consider $x.y$ and $x\times y$ to be completely defined by the equations

$$x.y+x'.y=(x+x').y \qquad x\times y+x'\times y=(x+x')\times y$$
$$x.y+x.y'=x.(y+y') \qquad x\times y+x\times y'=x\times(y+y'),$$

just as the symbolic form a^m is completely defined by the equation $a^m a^n=a^{m+n}$.

(9.) Whether $x.y$ and $x\times y$ are "*Commutative*" functions of x and y, i. e. whether $x.y=y.x$, and $x\times y=y\times x$, does not appear from these defining equations, and therefore it must be decided by the particular nature of the quantity or operation which each of these forms is assumed to represent.

(10.) *Signification of the sign* $+$. In Symbolical Algebra the sign $+$ may be regarded as simply an abbreviation for the words "*together with*," and thus $u+v$ means simply u "together with" v, or u and v "*put together*." Now these words "*together with*" may be taken in a great variety of senses, as the following examples taken from ordinary algebra show, viz.

$$3+5=8, \qquad 3£+5s.=780d., \qquad 3+4\sqrt{-1}=(2+\sqrt{-1})^2,$$
5 miles east+5 miles west=0, &c. &c.

In the first example $+$ means a "*putting together*" by simple numerical addition; in the second, a "*putting together*" of certain pieces of gold and silver, with reference to a certain conventional value set on them; in the third, a mere symbolical "putting together;" and so on. Hence it is clear that in using the sign $+$ as an abbreviation of the words "*together with*," the precise nature of the "*putting together*" is supposed to be understood in each case. I shall therefore define the notation, $u+v$, to mean, u *and* v *put together in a sense supposed to be understood.*

Now in some cases it is very important that the precise nature of the "*putting together*" denoted by the sign $+$ should be clearly understood, and therefore distinctly specified. This it will be necessary for me to do here with reference to two remarkable significations which have been given to the sign $+$.

(11.) The first is that signification given to $+$ in Symbolical Geometry. If u and v denote two lines of certain magnitudes and drawn in certain directions, then $u+v$ is assumed to denote u and v *put together* as in the figure 6; that is, the *beginning-point* or *origin* of v coinciding with the *end-point* (if I may so use the words) of u. The second signification

Fig. 6.

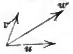
Fig. 7.

is that given to + in Symbolical Mechanics. If u and v denote two forces of certain magnitudes and directions, $u+v$ is assumed to denote u and v *put together* as in figure 7; that is, the *origin* of v coinciding, not with the *end-point*, as before, but with the *origin* of u. The distinction I allude to here is of considerable importance, and requires to be very closely attended to in applying lines to represent forces generally, as will appear. It might be well to distinguish these two significations of + by appropriate terms. I cannot think of any better words, for the purpose, than the two, "*successive*," and "*simultaneous*;" the *putting together* in fig. 6 is manifestly effected by tracing the two lines *in immediate succession*, while that in fig. 7 is a *simultaneous application at the same origin*. I shall therefore call the *putting together* in fig. 6 *successive addition*, and that in fig. 7 *simultaneous addition*.

Fig. 8.

(12.) *Signification of the sign* =. Like +, the sign = denotes *equivalence in a certain sense supposed to be understood*; thus in the example, 3£+5*s*.=780*d*., it denotes equivalence as regards the conventional value of certain coins. In Symbolical Geometry = has reference to the *change of position of the tracing-point* by which lines are supposed to be drawn. Thus if u, v, w denote three traced lines, the equation, $u+v=w$, means, that the tracing of $u+v$ is the same thing as the tracing of w, *so far as the change of position of the tracing-point is concerned*. In this sense it is clear, that w must be the third side of the triangle in fig. 8. In Symbolical Mechanics = has reference to *mechanical effect*. Thus if u, v, w be three forces, the equation, $u+v=w$, means, that *the mechanical effect of* u+v *is the same as that of* w; in other words, it means, that w is the *resultant* of u and v.

Fig. 9.



(13.) *Representation of Forces by Lines*. The suitability of lines to represent forces is obvious enough in ordinary Mechanics, where + has the signification of mere numerical addition; but when we come to Symbolical Mechanics this suitability is no longer a thing to be assumed. A little consideration will show that the question, "Can we assume lines to represent forces generally?" may be stated symbolically as follows, viz. If the lines u and v respectively represent the forces U and V, in magnitude and direction, will $u+v$ also represent U+V in magnitude and direction? if not, the graphical mode of representation becomes inadmissible symbolically. Now, it is clear, by reference to figures 8 and 9, that this question amounts to asking, whether the *Parallelogram of Forces* is true or not? for the peculiar signification of + in Symbolical Geometry makes $u+v$ denote the diagonal of the parallelogram constructed on u and v as sides; whereas the Mechanical signification of + in U+V makes it denote the resultant of U and V.

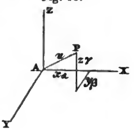
Hence it follows that the *general* representation of forces by lines *assumes* the truth of the Parallelogram of Forces as a necessary condition; and, consequently, any symbolical proof of the Parallelogram of Forces which assumes that lines may be taken generally as representatives of forces, amounts to reasoning in a circle.

My object here, however, in the remarks just made, is to point out the importance of distinctly marking the two significations of the sign $+$.

(14.) In all cases that I shall be concerned with, *successive* and *simultaneous* addition are virtually equivalent, so far as the representation of directed magnitudes by lines is concerned. As regards the *statical effect of forces*, the Parallelogram of Forces shows this: as regards the *dynamical effect*, the Second Law of Motion (I mean NEWTON's 2nd Law) does the same. As regards *velocities* and *displacements* the thing is obvious.

(15.) *Directed Units.* I shall call an arrow of a unity of length (whether it represents a traced line, a force, a velocity, or any other kind of directed magnitude) a "*directed unit.*" I shall always reserve the letters α, β, γ to denote a set of three directed units *at right angles to each other*. Hence, if AX, AY, AZ be three rectangular axes to which α, β, γ are respectively parallel; and if x, y, z denote numerically the three coordinates of any point P; $x\alpha, y\beta, z\gamma$ will be the symbols representing these three coordinates *in magnitude and direction*, inasmuch as $x\alpha$ means x directed units *put together* by *successive* or *geometrical* addition, all in the direction parallel to AX; and so also as regards $y\beta$ and $z\gamma$. Also if u be taken to denote the line AP in magnitude and direction, we have, by *successive* addition,

Fig. 10.



$$u = x\alpha + y\beta + z\gamma.$$

The point P is often called *the point* (xyz), I may therefore speak of it as *the point* (u), inasmuch as u completely defines its position.

If X, Y, Z denote *numerically* three forces parallel to AX, AY, AZ, it is clear that their complete symbolical representatives are $X\alpha$, $Y\beta$, $Z\gamma$. Also, if U denote the resultant of these three forces, we have

$$U = X\alpha + Y\beta + Z\gamma.$$

But here $+$ denotes *simultaneous* addition: we may, however, assuming the truth of the Parallelogram of Forces, regard it as the *successive* $+$, if we please.

(16.) As just observed, I shall always suppose α, β, γ to be a set of *three rectangular directed units*; I shall suppose the same also as regards α', β', γ'; $\alpha'', \beta'', \gamma''$, &c., using the dashes to denote different sets of directed units; but the three in each set are always assumed to be at right angles to each other, unless the contrary be specified.

In speaking of lines as regards magnitude and direction, I shall always use the word "*direction*" as equivalent to "directed unit;" thus I shall call α the "*direction*" of the line $x\alpha$. The *complete symbol* of a line may be therefore described as *its direction multiplied by its magnitude.*

(17.) If r denote the magnitude, and α' the direction of u, $u = r\alpha'$, and therefore, putting for u its value above, we find

$$\alpha' = \frac{x}{r}\alpha + \frac{y}{r}\beta + \frac{z}{r}\gamma,$$

or

$$\alpha' = a\alpha + b\beta + c\gamma,$$

where a, b and c denote the cosines of the angles which α' makes with α, β and γ respectively.

If α' lies in the plane $(\alpha\beta)$, and if θ denote the angle which α' makes with α, this expression becomes

$$\alpha'=\alpha\cos\theta+\beta\sin\theta.$$

(18.) According to the principles of Symbolical Algebra, we have

$$\beta=-^{\frac{1}{2}}\alpha, \qquad \gamma=-^{\frac{1}{2}}\beta, \qquad \alpha=-^{\frac{1}{2}}\gamma, \quad (-^{\frac{1}{2}}=\sqrt{-1}).$$

But it is to be remembered that the sign $-^{\frac{1}{2}}$ here does not denote the same identical operation in these three cases; nor is it necessary that it should, any more than the sign $-$. The true state of the case is this, that $(-^{\frac{1}{2}})\alpha$ is defined by the equation

$$(-^{\frac{1}{2}})(-^{\frac{1}{2}})\alpha=-\alpha;$$

and the general solution of this equation is

$$-^{\frac{1}{2}}\alpha=\beta\cos\theta+\gamma\sin\theta,$$

where θ is perfectly arbitrary. Consequently the extraction of the square root of $-$ gives, not simply two values, positive and negative, as in ordinary extractions of the square root, but an infinite number of values, namely the *whole circle* of directed units at right angles to α.

I shall have no occasion to make any use of the sign $-^{\frac{1}{2}}$, or any reference to the connections just given between α, β and γ, except in some future applications of my method, chiefly in Geometry. The statement just made is intended to show what α, β, γ are with reference to the square roots of $-$ (or -1), and to point out distinctly that α, β, γ are *not* supposed to be *square roots of unity*, but merely *direction-units.*

(19.) *Remarkable signification of $d\alpha$, $d\beta$, $d\gamma$.* This signification I pointed out and made use of in a paper read before the Cambridge Philosophical Society (Nov. 1846), and it may be briefly stated here for the purpose of reference in certain applications of the present method. If α and α' be two directed-units at an indefinitely small angle to each other, we have $\alpha'-\alpha=d\alpha$; but $\alpha'-\alpha$ is the line joining the extremities of α and α', and this line is at right angles to α and α' (ultimately), because α and α' are lines of equal length. Hence $d\alpha$ is the expression for an indefinitely small line *at right angles to α.*

Fig. 11.

This signification of $d\alpha$ is one of great importance in Symbolical Geometry and Mechanics: thus for example, if α, β, γ denote direction-units *fixed* in a rigid body, the angular velocities of the rigid body are represented, in magnitude and direction, by

$$\frac{d\alpha}{dt}, \quad \frac{d\beta}{dt}, \quad \frac{d\gamma}{dt},$$

inasmuch as $d\alpha$, $d\beta$, $d\gamma$ represent, in magnitude and direction, the small angles described in the time t by the extremities of these three direction-units. Of course I mean by the word "*angle*," here, the circular arc which measures it. The importance of this signification of $d\alpha$, $d\beta$, $d\gamma$ will be manifest in many parts of what follows.

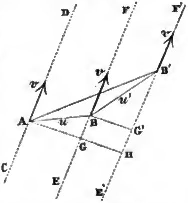
II. Symbolical Representation of the two Effects produced by the Translation of a Directed Magnitude.

(20.) *The Effects symbolized.*—It has been shown that the translation of v *along* u is, generally, of a twofold nature, partly *lateral* and partly *longitudinal*; it is my object to express symbolically the *effects produced*, whether they are geometrical or mechanical effects, by the two kinds of translation. The effect produced by the lateral part of the translation of v along u I shall call *the lateral effect*, and that produced by the longitudinal part, *the longitudinal effect*.

Fig. 12.

(21.) *The two Effects are, each, Distributive Functions of* u *and* v. Let $f(u, v)$ denote the *lateral effect* of the translation of v along u; let u and u' represent the lines AB and BB'; produce the arrows (v) both ways indefinitely to show the *lines of direction* in which v lies in the three parallel positions at A, B, and B'; draw AGH and BG' at right angles to these parallel lines of direction (CD, EF, E'F'). Observe, that v, u and u' are not necessarily in the same plane.

Fig. 13.



Now, as assumed, $f(u, v)$ denotes the effect produced by the shifting of the line CD to the parallel position EF; $f(u', v)$ denotes that by a farther shifting, namely from EF to the parallel position E'F': which two shiftings "*put together*" come to the same thing as one shifting from CD to E'F'. Now since AB' is represented by $u+u'$, the effect of this last-mentioned shifting is denoted by $f(u+u', v)$: we have therefore

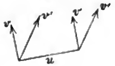
$$f(u, v)+f(u', v)=f(u+u', v),$$

that is, the *lateral effect* of the translation of v along u is a *distributive function as regards* u.

In precisely the same way it may be shown, that the *longitudinal effect* is also a distributive function as regards u. For it is manifest that the three translations, viz. that along u, that along u', and that along $u+u'$, amount respectively to GB, G'B', and HB', as regards longitudinal effect, and we have HB'=GB+G'B'. Whence the conclusion is evident.

It remains to show that both effects are distributive functions *as regards* v *also*; and this is immediately obvious: for the translation of v along u "*together with*" that of v' along u, is the same thing as the translation of $v+v'$ along u; and thus, whether f denote the lateral or longitudinal effect, we have

Fig. 14.



$$f(u, v)+f(u, v')=f(u, v+v').$$

Both effects therefore are distributive functions with respect to v as well as u.

(22.) It is important to observe, here, that the $+$ in $u+u'$ denotes *successive*, while that in $v+v'$ denotes *simultaneous addition*.

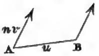
(23.) *Notation adopted to represent the two effects.* I have stated above the reasons why $u.v$ and $u\times v$ may be appropriated to denote, simply, *any* distributive functions of u and v. I have here shown the existence of two such functions, very important to be "noted" symbolically, and to be distinguished from each other. I shall therefore venture farther to employ the notation $u.v$ *exclusively* for the purpose of representing the *lateral effect* of the translation of v along u, and the notation $u\times v$ *exclusively* to represent the *longitudinal effect*.

(24.) As regards the *order of the factors*, I shall always suppose that the *second factor* is the translated magnitude, and the *first factor* the line along which it is translated.

(25.) In using these notations I am not warranted to attribute to them, without proof, any property of an ordinary product, except its distributiveness: for example, I must not put $u.v=v.u$, without investigating whether this equation holds as regards the effects represented by $u.v$ and $v.u$. Nor again, if m and n be any numbers, can I, without proof, put $(mu).(nv)=mn(u.v)$. These points I shall now consider.

(26.) *May Numerical Coefficients, occurring in* u.v *or* u$\times$v, *be brought out and incorporated by actual multiplication?*—Supposing m and n to denote pure numbers, may we put $(mu).(nv)=mn(u.v)$, and $(mu)\times(nv)=mn(u\times v)$? Or, to express the question in words, is the effect produced by the translation of nv along mu equivalent to mn times the effect of the translation of v along u? It is very important to bear in mind, as regards this question, that nv means $v+v+v+$ &c. "*put together*" by *simultaneous* addition; while mu means $u+u+u+$ &c. "*put together*" by *successive* addition (see art. 22). Hence it will not be difficult to show that $u.(nv)=n(u.v)$ in virtue of the distributive property; but, that some additional consideration is requisite to determine whether $(mu).v=m(u.v)$.

Fig. 15.



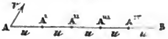
(27.) *First*, as regards $u.(nv)$. The n v's here have the same origin A, and they are translated simultaneously from A to B. This translation is manifestly the same as if each v were translated separately from A to B: and thus it follows that the translation of nv along u is the same thing as n translations of v along u; or, in symbols,

$$u.(nv)=n(u.v),\text{ and } u\times(nv)=n(u\times v).$$

Indeed this is nothing more than a re-assertion of the distributive nature of the translation of v along u, as regards v.

(28.) *Secondly*. In the expression $(mu).v$ the u's have *not* the same origin, but are "*put together*" *successively*, as is represented in fig. 16, making up the line AB (supposing, for a moment, that $m=5$). It is clear, then, that $(5u).v$ means a translation of v from A to B, while $5\ (u.v)$ means five translations of v from A to A'. Hence, before we can decide whether $(mu).v=m(u.v)$, we

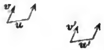
Fig 16.



must determine whether the translations of v from A to A$'$, from A$'$ to A$''$, from A$''$ to A$'''$, &c. are equivalent to each other *in effect*; for, if they are, it makes no difference *in effect* whether we repeat the translation of v from A to A$'$ five times, or simply translate v from A to A$'$, from A$'$ to A$''$, from A$''$ to A$'''$, and so on to B. The question then comes to this—Are we to regard translations as equivalent to each other, in effect, when the magnitudes translated and the lines along which they are translated are respectively equivalent to each other, whether the translations take place *in the same part of space or not?*

(29.) *Fundamental Assumption.*—I am thus led to make the following *Assumption* the basis of my proposed method; viz.—*That parallel and equal translations of parallel and equal magnitudes are equivalent to each other both as regards the lateral and longitudinal effects*; or, symbolically, if u' and v' be respectively parallel and equal to u and v, then

Fig. 17.



$$u'.v'=u.v, \text{ and } u'\times v'=u\times v.$$

This assumption holds true, manifestly, in each of the three *suggesting cases* from which I have taken my start (see art. 2), and I am therefore justified in adopting it, *with the understanding*, of course, that it be shown to hold true, or tacitly admitted, in all cases to which the notation may be applied; or else, should the occasion require it, be abandoned, and, with it, the property expressed by the equation $(mu).v=m(u.v)$.

(30.) Returning to fig. 16, we have, by the Assumption just made,

$$\begin{aligned} \mathrm{AA}'.v=\mathrm{A}'\mathrm{A}''.v=\mathrm{A}''\mathrm{A}'''.v= \&\text{c. } \&\text{c.};\\ \therefore m(u.v)=m(\mathrm{AA}'.v)=\mathrm{AA}'.v+\mathrm{A}'\mathrm{A}''.v+\mathrm{A}''\mathrm{A}'''.v+ \&\text{c.}\\ =(\mathrm{AA}'+\mathrm{A}'\mathrm{A}''+\mathrm{A}''\mathrm{A}'''+ \&\text{c.}).v\\ =(mu).v. \end{aligned}$$

And generally, by what has been proved, we have

$$(mu).(nv)=n\{(mu).v\}=mn(u.v);$$

and, similarly, $$(mu)\times(nv)=mn(u\times v).$$

It appears thus that *numerical coefficients*, occurring in the symbolic forms $u.v$ and $u\times v$, may always be *brought out* and incorporated by actual multiplication *.

(31.) *May the order of the factors* u *and* v *in the symbolic forms* u.v *and* u$\times$v *be changed, or not?*—*First*, as regards $u.v$, may we put $u.v=v.u$? Here I may repeat that the second factor always denotes the translated magnitude, or rather, the representative arrow. Thus the question is—as regards lateral effect, is the translation of the magnitude represented by the *arrow* v along the *line* u equivalent to the translation of that represented by the *arrow* u along the *line* v. This is easily decided as follows.

(32.) It is clear that the *lateral effect* of the translation of a magnitude *in its own*

* It may be shown, in the usual way, that this is true also when m and n are *fractional* or *negative* numbers.

direction is zero: it follows therefore that $u.u=0$, $v.v=0$; also that

$$(u+v).(u+v)=0,$$
$$\therefore\ u.u+u.v+v.u+v.v=0,$$
$$\therefore\ u.v+v.u=0,$$
$$\text{or } v.u=-u.v.$$

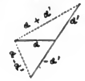
It appears then that $u.v$ and $v.u$ are *equivalent as regards magnitude* but *opposite in sign*.

(33.) *Secondly*, as regards $u\times v$, may we put $v\times u=u\times v$? This question is determined by observing that the *longitudinal effect* of the translation of a magnitude *at right angles to its direction is zero*, as follows.

Let $u=m\alpha$, $v=m'\alpha'$, α and α' being the *directions* (directed units), and m and m' the magnitudes of u and v. Then, by article 30,

$$u\times v=mn(\alpha\times\alpha') \text{ and } v\times u=mn\ (\alpha'.\alpha).$$

Fig. 18.



Now it is clear, from figure 18, that $\alpha+\alpha'$ and $\alpha-\alpha'$ are lines at right angles to each other; therefore

$$(\alpha+\alpha')\times(\alpha-\alpha')=0=(\alpha-\alpha')\times(\alpha+\alpha');$$

therefore, omitting common terms, we find

$$-\alpha\times\alpha'+\alpha'\times\alpha=\alpha\times\alpha'-\alpha'\times\alpha,$$
$$\therefore\ \alpha\times\alpha'=\alpha'\times\alpha.$$

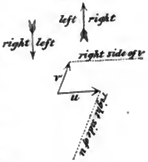
And thus it follows that

$$u\times v=v\times u.$$

It appears then that $u\times v$ and $v\times u$ are *equivalent as regards both magnitude and sign.*

(34.) Thus $u.v$ is *commutative with change of sign*, while $u\times v$ is *simply commutative*. The reason of the change of sign in the former may be easily interpreted as follows. An arrow has two distinct *sides*, which, for the sake of fixing ideas, I may call *right* and *left*, and which may be defined by supposing that I stand *on* the plane of the paper *looking in the direction of the arrow*. Now, referring to fig. 19, it is clear that the translation of v along u is *laterally* a motion *to the right side*, while that of u along v is *to the left*, the translated magnitude in both cases being that with reference to which I speak of right and left. Thus the meaning of the equation $v.u=-u.v$ is obvious.

Fig. 19.



III. Measurement and Summation of Translations.

(35.) *Units of Translation.*—I shall take the translation of a unit *along a perpendicular unit* to be a *unit of lateral translation*; and the translation of a unit *along itself* to be a *unit of longitudinal translation*. Thus (see art. 16) $\alpha.\beta$, $\beta.\gamma$, $\alpha'.\beta'$, &c. are units of lateral translation; and $\alpha\times\alpha$, $\beta\times\beta$, $\alpha'\times\alpha'$, &c. are units of longitudinal translation.

Fig. 20.

(36.) *All Units of Longitudinal Translation are equivalent to each other.* For let α and α' denote any two directed units whatever; then, as above,

$$(\alpha+\alpha')\times(\alpha-\alpha')=0;$$

wherefore, since $\alpha\times\alpha'=\alpha'\times\alpha$, we have

$$\alpha\times\alpha=\alpha'\times\alpha'.$$

In illustration of this result the third *suggesting instance*, that of *Mechanical Work*, may be quoted; inasmuch as work is the effect of longitudinal translation, and all units of work are equivalent to each other, no matter in what directions the working forces act.

(37.) *All Units of Lateral Translation, in the same or in parallel planes, are equivalent to each other.*—Let α, β, α', β' lie in the same plane, and let θ denote the angle which α' makes with α, and therefore that also which β' makes with β (art. 16): then (art. 17)

$$\alpha'=\alpha\cos\theta+\beta\sin\theta$$

$$\beta'=\alpha\cos\left(\theta+\frac{\pi}{2}\right)+\beta\sin\left(\theta+\frac{\pi}{2}\right);$$

$\therefore$, observing that $\alpha.\alpha=\beta.\beta=0$, and $\alpha.\beta=-\beta.\alpha$,

we have

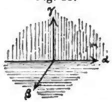
$$\alpha'.\beta'=\alpha.\alpha(\cos^2\theta+\sin^2\theta)=\alpha.\alpha.$$

Hence all units of lateral translation in the same, or in parallel planes, are equivalent to each other.

In illustration of this result, the second *suggesting instance*, that of a *Couple*, may be quoted; for all unit-couples in the same or in parallel planes are equivalent to each other.

(38.) *Directrix.*—Hence, in expressing a unit of lateral translation, it is only necessary to specify a plane parallel to that in which the translation takes place; or, what is better and immediately suggested by the theory of couples, it is only necessary to specify *a line at right angles to the plane of translation.* Such a line I shall, however, designate by the word "*directrix*," not *axis*; because there is no idea of rotation involved in the present theory, translation being a kind of motion essentially different from rotation. I shall assume γ to be the directrix of $\alpha.\beta$ and of all units of lateral translation in planes at right angles to γ; and, generally, I shall define the directrix of any unit of lateral translation to be a directed unit at right angles to the plane of that translation.

Fig. 21.



(39.) But, since $\alpha.\beta=-\beta.\alpha$, it is necessary to distinguish positive from negative translations; and this may be done by giving an appropriate sign to the directrix. I shall therefore assume generally, that $m\gamma$ is the directrix of $m\,\alpha.\beta$, m being any number positive or negative. Thus $-\gamma$ will be the directrix of $-\alpha.\beta$, that is, of $\beta.\alpha$. And, hence, I may adopt the following criterion of

sign. I shall suppose myself standing at right angles to the plane of translation in such a position that the translated magnitude points to the right, while I face the direction in which the translation takes place; and then I shall take an arrow pointing *from foot to head* as the directrix. According to this criterion the letters written underneath the following translations are their respective directrices; viz.

$$\begin{array}{cccccc} \alpha.\beta & \beta.\gamma & \gamma.\alpha & \beta.\alpha & \gamma.\beta & \alpha.\gamma \\ \gamma & \alpha & \beta & -\gamma & -\alpha & -\beta \end{array}$$

(40.) *Measurement of a Translation.*—Let $u.v$, or $u\times v$, be the translation, α the direction of u, $(\alpha\beta)$ the plane of $u.v$, θ the angle which v makes with u, m and n the magnitudes of u and v; then

$$u=m\alpha,\quad v=n(\alpha\cos\theta+\beta\sin\theta);$$
$$\therefore\ u.v=mn\sin\theta(\alpha.\beta),\quad (\text{since } \alpha.\alpha=0).$$

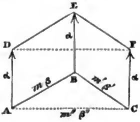
Hence there are $mn\sin\theta$ units in the translation $u.v$: $mn\sin\theta$, therefore, is the numerical magnitude of $u.v$, and its directrix is $(mn\sin\theta)\gamma$.

Again, $u\times v=mn\cos\theta(\alpha\times\alpha)$, (since $\alpha\times\beta=0$). Hence the numerical magnitude of $u\times v$ is $mn\cos\theta$.

(41.) It is worth remarking that $mn\sin\theta$, the numerical magnitude of $u.v$, is the area of the parallelogram completed on u and v as sides.

(42.) *The Directrix of the Sum of two translations is the Sum of their Directrices.*—Let the two translations be $m\alpha.\beta$ and $m'\alpha'.\beta'$, and their directrices, of course, $m\gamma$ and $m'\gamma'$; let EB be the intersection of the two planes of these translations, and take AB$=m$ and BC$=m'$, AB and BC being drawn at right angles to BE, AB in the plane of $\alpha.\beta$, and BC in the plane of $\alpha'.\beta'$.

Fig. 22.



Now, since all units of lateral translation in the same plane are equivalent to each other, I may turn α and β, α' and β' about in their respective planes, until both α and α' coincide with BE; in which case β and β' will coincide with AB and BC respectively; then AB will become $m\beta$, and BC $m'\beta'$. Let the third side of the triangle ABC be $m''\beta''$ as shown in the figure. Then the sum of the two translations is

$$m\alpha.\beta+m'\alpha'.\beta',$$

which, since $\alpha'=\alpha$, becomes

$$\alpha.(m\beta+m'\beta');$$

and this, since $m\beta+m'\beta'=m''\beta''$, becomes

$$m''\alpha.\beta''.$$

Let $m''\gamma''$ be the directrix of this: then it is clear that γ, γ' and γ'', being each at right angles to α, lie in the plane of the triangle ABC, and consequently $m\gamma$, $m'\gamma'$ and $m''\gamma''$ are the three sides of the triangle ABC, supposing it to be turned round in its plane through 90°. It follows therefore that

$$m''\gamma''=m\gamma+m'\gamma',$$

that is, the directrix of the sum of the two translations is the sum of their directrices.

(43.) Generally, it is manifest from this result, that the *Directrix of the Sum* of any number of Translations is the *Sum of their Directrices.*

It is not necessary to point out the importance of this rule as regards the *summation* of translations, nor its identity with the well-known rule in the *Theory of Couples.*

(44.) For the sake of convenience it will be worth while to employ some abbreviated mode of specifying the directrix of any translation; for this purpose I shall adopt the following notation, which, it will be found, will answer all purposes, and at the same time very distinctly mark that property in which the peculiar relation between a translation and its directrix consists. The property I allude to is the theorem just proved in article 42.

I shall employ the letter D to stand as an abbreviation for the words "*directrix of*," and thus $D(u.v)$ will mean *the directrix of the translation* u.v. It will be borne in mind, then, that $D(u.v)$ denotes a line at right angles to the plane of $u.v$, and containing as many units of length as there are units of area in the parallelogram constructed on u and v.

(45.) *Distributiveness of the Operation thus represented.*—It is most important to notice the *distributive* nature of the symbol D. By art. 42, we have immediately

$$D(u.v)+D(u'.v')=D(u.v+u'.v');$$

whence, D denotes a distributive function.

(46.) *Consequences hence resulting.*—From the equation in art. 45 it follows, that, if m denote any numerical coefficient, positive or negative,

$$D(mu.v)=mD(u.v).$$

Again, since

$$D(u'.v')-D(u.v)=D(u'.v'-u.v),$$

we have, passing to limits,

$$d\big(D(u.v)\big)=D\big(d(u.v)\big);$$

whence also,

$$\int\big(D(u.v)\big)=D\int(u.v).$$

In short, in all operations in which differentiation and integration are concerned, D is to be regarded as if it were an ordinary constant coefficient.

Again, if we have an equation of the form

$$u.v+u'.v'+u''.v''+\&c.=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

there result from it

$$D(u.v)+D(u'.v')+D(u''.v'')+\&c.=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

And, conversely, (2.) gives (1.).

(47.) *Inverse of* D.—If w be the directrix of $u.v$, and therefore $w=D(u.v)$; I may of course, according to the true force of the index (-1), assert, that $u.v=D^{-1}w$. Thus $D^{-1}w$ comes to be an abbreviation for the words—"*the translation whose directrix is w.*"

(48.) It may be well to observe that the following relations result from what has been said, viz.

$$D(\alpha.\beta)=\gamma \qquad D(\beta.\gamma)=\alpha \qquad D(\gamma.\alpha)=\beta$$
$$D(\beta.\alpha)=-\gamma \qquad D(\gamma.\beta)=-\alpha \qquad D(\alpha.\gamma)=-\beta$$

whence

$$\alpha.\beta=D^{-1}\gamma \qquad \beta.\gamma=D^{-1}\alpha \qquad \gamma.\alpha=D^{-1}\beta$$
&c. &c. &c.

(49.) *Symbol for Units of Longitudinal Translation.*—It has been shown that these units are all equal to each other; a *unit of longitudinal translation* is therefore an *absolute constant.* No distinctive symbol is necessary, therefore, to represent these units, and it will be allowable to employ the common unit (1) for the purpose; just in the same way that we denote all units, whether they be linear, superficial, cubical, mechanical, by this common symbol. I shall therefore always represent a unit of longitudinal translation by 1; and thus put

$$\alpha\times\alpha=1, \qquad \beta\times\beta=1, \qquad \gamma\times\gamma=1;$$

and generally, if m and n denote the magnitudes of u and v, we have (art. 41)

$$u\times v=mn\cos\theta,$$

θ being the angle made by u and v.

(50.) Hence if α and α' be any two "*directions*," we have

$$\alpha\times\alpha'=\text{cosine of angle made by } \alpha \text{ and } \alpha'.$$

To this may be added

$$\text{magnitude of } \alpha.\alpha'=\text{sine of same angle.}$$

(51.) *Projections represented by the lateral and longitudinal translation-products.*— It is clear from the principles just established, that, if α, β, γ denote the directions of three coordinate axes, and v any line, the *projections* of v on the three axes are, numerically,
$$\alpha\times v, \qquad \beta\times v, \qquad \gamma\times v.$$

Again, since $u\times u$ is the square of the magnitude of the line u, the projection of v on u is, numerically,
$$\frac{u\times v}{\sqrt{u\times u}},$$
which, putting for u and v the values $x\alpha+y\beta+z\gamma$, $x'\alpha+y'\beta+z'\gamma$, becomes by *longitudinal multiplication*,
$$\frac{xx'+yy'+zz'}{\sqrt{x^2+y^2+z^2}}.$$

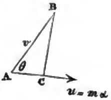
I may use the terms "*lateral* and *longitudinal multiplication*" to designate the operations denoted by $u.v$ and $u\times v$; for the word "multiplication" has quite lost its original and proper signification even in ordinary algebra.

(52.) If AB represent v, AC the projection of v on u, the magnitude and direction of u being m and α; then

$$CB=AB-AC=v-(\alpha\times v)\alpha.$$

I think CB might be advantageously called the *Complement of the Projection* of v on u, for CB added to AC makes up or *completes* v; and thus, employing the usual abbreviation, we may call CB the *coprojection* of v on u.

Fig. 23.



Hence the operation $(\alpha(\alpha\times)$ performed on v gives the *projection* of v on u, and the operation $(1-(\alpha\times\alpha)$ the *coprojection*; where α denotes the "*direction*" of u, *i. e.*

$$\frac{u}{m} \text{ or } \frac{u}{\sqrt{u\times u}}.$$

(53.) Since $\alpha.v=\alpha.(\text{AC}+\text{CB})=\alpha.\text{CB}$, it is clear that the symbol $\alpha.v$ denotes, in magnitude, the coprojection of v on α; and it also represents the plane of projection. But $\text{D}\alpha.v$ is a better symbol to use; for its magnitude is the same as that of $\alpha.v$, and, in direction, it denotes a line at right angles to the plane of projection.

(54.) *Repetition of the Operation* ($\text{D}\alpha.$). This operation is often to be performed twice in investigations, and, on this account, the following relation is important, viz.

$$(\text{D}\alpha.)^2\alpha'=(\alpha\times\alpha')\alpha-\alpha'. \quad (1.)$$

Of course $(\text{D}\alpha.)^2\alpha'$ means $\text{D}\alpha.\,(\text{D}\alpha.\alpha')$.

To prove this, let β be chosen so as to lie in the plane $(\alpha\alpha')$, and let θ denote the angle which α and α' make with each other; then (by art. 17)

$$\alpha'=\alpha\cos\theta+\beta\sin\theta$$

$$\text{D}\alpha.\alpha'=\gamma\sin\theta \text{ (art. 48)}$$

$$\text{D}\alpha.(\text{D}\alpha.\alpha')=-\beta\cos\theta$$

$$=\alpha\cos\theta-\alpha'$$

or

$$(\text{D}\alpha.)^2\alpha=(\alpha\times\alpha')\alpha-\alpha' \text{ (art. 50).}$$

If $u=m\alpha$ $v=n\alpha'$, we have

$$(\text{D}u.)^2v=m^2n(\text{D}\alpha.)^2\alpha'$$

$$=(m\alpha\times n\alpha')m\alpha-(m\alpha\times m\alpha)n\alpha',$$

or

$$(\text{D}u.)^2v=(u\times v)u-(u\times u)v; \quad (2.)$$

or, if $m=1$,

$$(\text{D}\alpha.)^2v=(\alpha\times v)\alpha-v. \quad (3.)$$

(55.) *Relation of the Operation* ($\text{D}\alpha.$) *to the operation* $\sqrt{-1}$ *or* $(-)^{\frac{1}{2}}$. The definition of the index $\frac{1}{2}$ in relation to operations is this. If Ω and $\Omega_{,}$ be symbols of operation, such that Ω performed twice on a quantity gives the same result as $\Omega_{,}$ once performed, then Ω is denoted by $\Omega_{,}^{\frac{1}{2}}$. Now, if we suppose that v is at right angles to α, and therefore $\alpha\times v=0$, the equation (3.) gives (what indeed is otherwise more easily shown from art. 48)

$$(\text{D}\alpha.)^2v=-v,$$

wherefore

$$\text{D}\alpha.=(-)^{\frac{1}{2}}.$$

In this case α is any unit line whatever at right angles to v, and therefore, in Solid Geometry, $(-)^{\frac{1}{2}}$ or $\sqrt{-1}$ has not two (as in Plane Geometry) but an infinite number of values.

(56). It is clear from the relation

$$(\text{D}\alpha.)^2v=-v,$$

that ($\text{D}\alpha.$) has all the properties of the sign $\sqrt{-1}$, *provided it be performed on lines at right angles to* α. But ($\text{D}\alpha.$) is a far better sign for actual use in Solid Geometry than

$\sqrt{-1}$, because the latter is indefinite, not distinguishing what particular root of $-$ is meant; but the former is perfectly definite, inasmuch as it indicates one particular root of $-$, namely, that root which denotes rotation through 90° about α as axis*.

* Sir W. Hamilton, in his System of Symbolic Geometry and Quaternions, which may be truly described as one of the most profound and beautiful theories in the whole range of abstract science, assumes the letters i, j, k to denote particular values of $\sqrt{-1}$. In a certain limited sense, the symbols $(D\alpha.)$, $(D\beta.)$, $(D\gamma.)$ are equivalent to these; for i, j, k denote rotation through 90° about the axes α, β, γ; and thus we have

$$\gamma=i\beta, \quad \alpha=j\gamma, \quad \beta=k\alpha;$$

and, therefore, since

$$\gamma=D\alpha.\beta, \quad \alpha=D\beta.\gamma, \quad \beta=D\gamma.\alpha,$$

it is clear that i, j, k, and $(D\alpha.)$, $(D\beta.)$, $(D\gamma.)$, so far, denote the same operations.

But the operations are different *in general*; for $i^2=-1$ *always*, but $(D\alpha.)^2$, as has been shown above, is *not* equivalent to -1, *except* when performed on lines at right angles to α.

There is one difficulty, I confess, I cannot get over in Sir W. Hamilton's Theory, no doubt from some misconception on my part, or from taking too narrow a view of the meaning of the sign $\sqrt{-1}$. The difficulty I allude to consists in this. Sir W. Hamilton assumes i, j, k not only to be particular values of $\sqrt{-1}$, but also *absolute directions* (i. e. *units of direction*): in short he uses i, j, k in the same sense as α, β, γ above, and in the same sense *also* as $(D\alpha.)$, $(D\beta.)$, $(D\gamma.)$. Now my difficulty arises from my not being able to see how particular values of $\sqrt{-1}$ can denote anything (geometrically) but *change* of direction, or to perceive, that they can be used with propriety as symbols of those rectangular units of direction.

However this may be, it is important to explain the fact, that, in the results to which I have been led by the conception of translation, there are no general relations corresponding to

$$i^2=-1, \quad j^2=-1, \quad k^2=-1.$$

In my method, α, β, γ are *simple units of direction* and *nothing more*; and instead of the relations just put down, I have been led, by the conception of translation, to the following, viz.—

$$\alpha.\alpha=0, \quad \beta.\beta=0, \quad \gamma.\gamma=0$$
$$\alpha\times\alpha=1, \quad \beta\times\beta=1, \quad \gamma\times\gamma=1;$$

though, as regards the last three relations, all that I have a right to assert as a matter of necessity is, that

$$\alpha\times\alpha=\beta\times\beta=\gamma\times\gamma.$$

Also, I find

$$(D\alpha.)^2=-1, \quad (D\beta.)^2=-1, \quad (D\gamma.)^2=-1,$$

but only when performed on lines at right angles to α, β, γ respectively.

I may observe that if uv denote the product of u and v according to Sir W. Hamilton's Theory, it may be thus expressed in terms of my *translation-products*; viz.—

$$uv=-u\times v+Du.v.$$

Hence, since

$$Du.v=-Dv.u, \text{ and } u\times v=v\times u,$$
$$vu=-u\times v-Du.v.$$

Wherefore

$$u\times v=-\frac{1}{2}(uv+vu)$$
$$Du.v=\frac{1}{2}(uv-vu).$$

I may also observe, that, according to my method, I might put

$$uv=u\times v+u.v,$$

supposing that uv denotes the complete product of the translation of u along v, *including both* the lateral and longitudinal effects. But I cannot make any nearer approximation to the equation

$$uv=-u\times v+Du.v;$$

nor can I see that the conception of translation furnishes any interpretation of the $-$ before $u\times v$.

If, in any way, I could show that -1 was the proper value for a unit of longitudinal translation, I should have

$$\alpha\alpha=\alpha\times\alpha+\alpha.\alpha=-1.$$

(57.) The lateral translation-product is therefore well adapted to denote *rotation*. Thus (always supposing that v is at right angles to α), the symbol of a line *making an angle θ with* v, *and in a plane at right angles to α*, is

$$\varepsilon^{\theta D\alpha.}v,$$

which in fact is the same thing as

$$\{\cos\theta+(D\alpha.)\sin\theta\}v,$$

for cos θ and sin θ represent

$$1-\frac{\theta^2}{1.2}+\frac{\theta^4}{1.2.3.4},$$

and

$$\frac{\theta}{1}-\frac{\theta^3}{1.2.3}+\frac{\theta^5}{1.2.3.4.5},$$

and $\varepsilon^{\theta D\alpha.}$ is the symbolic form for expressing

$$1+\frac{(\theta D\alpha.)}{1}+\frac{(\theta D\alpha.)^2}{1.2}+;$$

whence, since $(D\alpha.)^2=-\ (D\alpha.)^4=+$ &c., the symbolic equivalence of the forms

$$\varepsilon^{\theta D\alpha.} \text{ and } \cos\theta+(D\alpha.)\sin\theta$$

is manifest. But the quantity operated upon must necessarily be at right angles to α; for, otherwise, $(D\alpha.)^2$ is not equivalent to $-$, as appears from art. 54, equation (3.).

(58.) *Concluding Remarks.*—I have now said enough, I think, to explain the nature of the proposed symbolization, and the general rules which regulate the application of the two *translation-products*. I have based the whole theory *simply and exclusively* on the conception of translation, taking my clue from the three suggesting instances, the *parallelogram*, the *couple*, and *work*. As regards the lateral effect of translation, the theory is nothing but a general development of our *geometrical* notion of multiplication; for what is a rectangle, ABCD, *considered apart from arithmetical measure*, but the effect or *product* of the translation of AB along AD? and this we represent by writing AD before or after AB. But this method of representation is clearly incomplete when we put for AB and AD their numerical representations; and why? because the special superficial unit then is omitted. For, suppose AB=3, AD=4; then, if we say that the rectangle ABCD is the product of 3 and 4, or 12, we mean, 12 *superficial units*. Now, by omitting the superficial unit in our representation, we leave out all conception of *the plane* in which the rectangle lies. All that I have done above is *to restore the superficial unit*, and determine its proper representative symbol. As regards the *longitudinal-effect* (suggested by the conception of work), it appears that all units are absolutely equivalent, and therefore may be all confounded in the common symbol of unity.

Fig. 24.

I now proceed to give Applications of the Symbolic forms* $u.v$ and $u\times v$, &c.

* I may refer here to an imperfect attempt I made in a paper read before the Cambridge Philosophical Society (Nov. 1846) to base the symbolic form $D\alpha.v$ on the conception of perpendicularity in art. 19 above.

PART II.

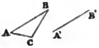
APPLICATIONS OF THE SYMBOLIC FORMS.

I. Geometrical Applications of the Symbolic Forms.

The geometrical applications which may be made of the principles and notation just explained are of great variety and importance; but, as I am anxious to dwell chiefly on physical applications, I shall only touch on this part of the subject.

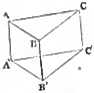
(59.) *Surface symbolically considered.*—In Symbolic Geometry the symbols of lines represent, not merely length but also direction. A line is supposed to be the geometrical effect of the motion of a tracing-point, and the equivalence of lines is considered altogether and exclusively with reference to the change which they represent in the position of the tracing-point. We consider that AB is equivalent to A′B′, when the two lines are *of equal length, parallel, and traced the same way*; for then they represent equivalent changes of position of their respective tracing-points. It is usual to employ the notation AB to denote the line AB *traced from* A *to* B, and BA to denote the same line traced from B to A; and thus $AB=A'B'$, but $BA=-A'B'$. It is clear then that equivalent lines must be, not only of equal length and parallel, but also must be traced *the same way*. Again, the equivalence being considered only with reference to the tracing-point's change of position, $AC+CB$ is equivalent to A′B′.

Fig. 25.



Now, following this analogy, and regarding surface as the effect of the motion of a *tracing-line*, just as a line is the effect of the motion of a *tracing-point*, we may employ symbols to denote surfaces, not merely as regards their numerical area, but also with reference to the manner in which they are generated by their tracing-lines; and we may also define the equivalence of surfaces altogether and exclusively with reference to the change which they represent in the position of the tracing-line. Thus, if ABA′B′, BCB′C′, ACA′C′ be three parallelograms, and if we conceive them to be generated, respectively, by the *translations* of A′A along A′B′, B′B along B′C′, and A′A along A′C′; it is clear that

Fig. 26.



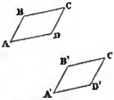
$$(ABA'B')+(BCB'C') \text{ is equivalent to } (ACA'C'),$$

just in the same sense that

$$(AB+BC) \text{ is equivalent to } (AC).$$

Again, if ABCD and A′B′C′D′ be two parallelograms, AB and AD being respectively parallel and equal to A′B′ and A′D′, we may consider these parallelograms as equivalent to each other, in the same sense exactly that the lines AB and A′B′ are regarded as equivalent. Only, just as it is necessary for the equivalence of AB and A′B′, that they should be traced *the same way* by their respective *tracing-points*, so it is necessary to the equivalence of ABCD and A′B′C′D′, that they should be traced *the same way* by their respective *tracing-lines*. Now here it is to be observed that a tracing-point is devoid of two important properties which

Fig. 27.

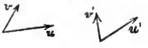


a tracing-line possesses; I mean, *direction* and *two-sidedness* (if I may so speak). An arrow lying on* a plane, not only points in a particular direction, but has *two distinct sides*, right and left (see art. 34). Hence when we speak of lines described *the same way* by their tracing-points, and parallelograms described *the same way* by their tracing-lines, the expression "*same way*" includes much more in the latter than in the former case. For example, the parallelograms generated by the translation of AB along AD, and of DC along DA, though coincident, are described *opposite ways*. Again, the parallelograms generated by the translation AB along AD, and of AD along AB, though coincident, are described opposite ways; for the former is described by a *right-side* motion, and the latter by a *left-side* (see art. 34).

(60.) *Definition of Surface by reference to Lateral Translation.*—It appears to me that the simplest way of including the considerations just alluded to in the general conception of surface requisite in Symbolical Geometry, is to define Surface by reference to the Lateral Effect of the Translation of one line along another. The *Fundamental Assumption* (article 29) is justified in this case, as appears from the remarks just made, and the peculiar relation $u.v=-v.u$ is naturally interpreted (see art. 34). I shall therefore define *Surface* in *Symbolic Geometry* to be *the Lateral Effect of the Translation of one line along another*. By "*Effect*" here I mean simple *Geometrical* effect, *i. e. change of position in space*. Also I only speak of *lateral effect*; because all notion of *longitudinal* effect is excluded by our ordinary conception of surface, and we may assume that no shifting which a line undergoes *in its own direction* can generate *surface*.

(61.) *Longitudinal Effect considered geometrically.*—But though the shifting of a line in its own direction generates no surface, it produces alteration of position; and hence it constitutes an important conception. The only difficulty, in considering the longitudinal effect *geometrically*, consists in this—How is it that all units of longitudinal translation are equivalent (see arts. 36, 37), while those of lateral translation are not, and on what principle can this difference be interpreted? The answer appears to be this: that $\alpha.\beta$ denotes the effect of translating the unit β along the perpendicular unit α; that this operation *conveys the conception of a particular plane*, and we must think of $\alpha.\beta$, $\beta.\gamma$ as different operations because they are performed in different planes. On the contrary, the translation of α along α, or $\alpha\times\alpha$, *conveys no conception of a particular plane*; in fact $\alpha\times\alpha$ and $\beta\times\beta$ may be regarded as performed in the same plane. Thus, that which before made the difference does not exist in this case.

Fig. 28.



Generally, $u.v=u'.v'$, when the two parallelograms generated are equal in magnitude, and lie in the parallel planes; but $u\times u$ and $u'\times u'$ may be always considered as lying in the same plane; and consequently difference of magnitude only remains to constitute a difference between $u\times v$ and $u'\times v'$.

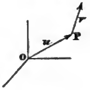
* Not *in* but *on* a plane, that is, on one particular side of the plane; *e. g.* on the *upper side* of this sheet of paper.

Again, as regards sign, there are *four* varieties of the form $u.v$, namely $u.v$, $(-u).(-v)$, both $+^{ve}$, and $(-u).v$, $u.(-v)$, both $-^{ve}$. But there are only *two* varieties of the form $u \times v$ when v becomes identical with u; inasmuch as v must be the same as u *in sign as well as in magnitude*, and consequently the notation does not admit of the variations $(-u)\times u$ and $u\times(-u)$. Now the other remaining variations are $u\times u$ and $(-u)\times(-u)$, and these have both the same sign.

Hence (supposing that u and v are units), since units of lateral translation may differ from each other in two particulars *only*, namely, *sign* and *plane of translation*, and since units of longitudinal translation are incapable of differing in these particulars, we may see the interpretation of the result in art. 37, and its perfect consistency with that in art. 36.

(70.) *Symbol of a line drawn from a given Point.*—If we assume that simple letters, such as u, v, w always denote lines of particular lengths and drawn in particular directions, *but all starting from the Origin of Coordinates* O; then the proper symbol for denoting a line v *drawn from the point* P, OP being u, will be

Fig. 29.



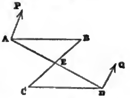
$$v+u.v+u\times v;$$

for v denotes the line v drawn from O, and $u.v+u\times v$ the lateral and longitudinal effects of translating it from O to P, which effects, as above stated, have reference only to the change of position of P. I shall reserve the consideration of this symbolization for a future occasion, as a striking instance of the same thing will be given in the next section.

II. Statical Application of the Symbolic Forms.

(71.) *Equivalence of Parallel and Equal Translations.*—As a necessary preliminary the *Fundamental Assumption* in art. 29 must be justified. To do this it is only necessary to bear in mind that all statical problems are reducible to the case of balancing forces acting *on the same rigid body*. Now let AB and CD be any two parallel and equal lines in the *same rigid body*; join A and D, C and B; the intersection E bisecting the two joining lines; and let P and Q be two equal parallel forces acting at A and D.

Fig. 30.



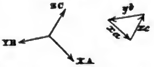
Then P and Q are equivalent to P+Q at E, and P+Q at E is equivalent to P at B and Q at C. We may therefore translate P from A to B, provided we, at the same time, translate Q from D to C. Whence it follows that the translation of Q from C to D must be equivalent to the translation of P from A to B. Thus the fundamental assumption is justified.

(72.) *Representation of Forces by Lines.*—It will be remembered that the general symbolic representation of forces by lines assumes the truth of the Parallelogram of Forces. As a matter of curiosity it may be asked, is it possible to apply this Symbolization of Translation to prove the Parallelogram of forces, *without assuming that*

forces are represented by lines? It may be done very simply, and I give the proof here as an example of the application of the method.

It may be seen, on referring to art. 21 and the following articles, that there is no assumption whatever of the possibility of representing forces by lines generally. The arrow representing the translated magnitude is used merely as a *conventional symbol*, just as a letter in algebra. I therefore may apply the notation $u.v$ to the present question, provided I consider v to be the symbol of a force, and not of a line representing that force. I must not however employ the reasoning in art. 32, for that distinctly assumes the point in question. The following then is the proof I shall give.

Fig. 31.



(73.) *Parallelogram of Forces.*—Let A, B, C denote *any* three units of force, and a, b, c units of length (directed units) parallel respectively to A, B, C; let X, Y, Z, x, y, z be pure numbers; suppose that the forces XA, YB, ZC balance each other, and that xa, yb, zc are the three sides of a triangle formed by lines drawn parallel to the forces. Then by *successive* addition we have

$$xa+yb+zc=0, \quad \ldots \ldots \quad (1.)$$

and by *simultaneous* addition

$$XA+YB+ZC=0. \quad \ldots \ldots \quad (2.)$$

Hence, from (1.) and (2.), we have

$$(-zc).(-ZC)=(xa+yb).(XA+YB);$$

or, observing that (since no lateral effect is produced by translating a magnitude in its own direction) $c.C$, $a.A$, $b.B$ are each zero, we have

$$0=xY(a.B)+yX(b.A). \quad \ldots \ldots \quad (3.)$$

Now, without altering the directions of the units A, B, a, b, let us put $X=Y$; in which case it is self-evident, that ZC must become equally inclined to XA and YB, and therefore zc must make equal angles with xa and yb, which gives $x=y$. Thus (3.) becomes

$$a.B+b.A=0, \text{ or } b.A=-a.B.$$

Wherefore, restoring the inequality of X and Y, we find from (3.),

$$(xY-yX)a.B=0, \text{ or } xY-yX=0.$$

And similarly, we may show that

$$yZ-zY=0$$
$$zX-xZ=0;$$

whence $$X:Y:Z::x:y:z.$$

And this is, virtually, the Parallelogram of Forces.

Thus it appears that the notation $u.v$ is capable of affording a simple proof of the great fundamental theorem of Statics; this application of the method is given, however, as I stated above, merely to show by example what can be done in this way. I may observe that the whole of the proof here given depends simply upon *two things*, the distributiveness of $u.v$, and the fact that numerical coefficients may be brought out

and incorporated by actual multiplication. I now proceed to give, generally, the mode of applying the notation to determine the conditions of equilibrium of a rigid body, or a system of particles.

(74.) *Remarkable symbolization of a Force acting at a Specified Point of a Rigid Body.* One of the most remarkable symbolizations which my proposed method leads to appears to me to be the following.

Let U denote a force (in magnitude and direction) *acting at any assumed origin* A, then

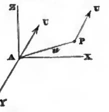
$$\mathrm{U}+u.\mathrm{U}$$

will *completely* denote *the same force supposed to act at the point u*, that is, at the point (P) whose distance from the origin is, symbolically, u. For if we apply the force at the origin A and then translate it to P, it will be the same thing as if we applied the force directly at P. Thus

effect of force at P=effect at A (or U)
+effect of translation from A to P.

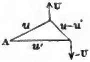
But, the lateral effect only of the translation need be considered, for the longitudinal effect is zero, inasmuch as we may suppose a force to act at any point of its line of direction on a rigid body. Hence the effect of the force acting at P is symbolically represented by the force U+ lateral effect of translation, or $\mathrm{U}+u.\mathrm{U}$.

Fig. 32.



(75.) *Symbol of a Couple.*—Let the couple consist of two forces, U at the point u, and $-\mathrm{U}$ at the point u': then these two forces are completely represented, as regards their effect on the rigid body, by the expression

Fig. 33.



$$(\mathrm{U}+u.\mathrm{U})+\big(-\mathrm{U}+u'.(-\mathrm{U})\big)$$

or

$$(u-u').\mathrm{U},$$

which is the general symbol for a couple; as indeed is clear beforehand from the fact, that the couple is that which translates the force U from the point (u') to the point (u), *i. e.* along the line ($u-u'$).

It will be generally simpler to employ the symbol in the form

$$u.\mathrm{U},$$

u here denoting, symbolically, the line drawn from the point of application of the force ($-$U) to that of the force (U). The *directrix* of this, *i. e.* the *axis of the couple*, is $\mathrm{D}(u.\mathrm{U})$.

(76.) *To combine a given Set of Couples.*—Let us put (see art. 15)

$$u=x\alpha+y\beta+z\gamma$$
$$\mathrm{U}=\mathrm{X}\alpha+\mathrm{Y}\beta+\mathrm{Z}\gamma.$$

Then we find, by *lateral multiplication*,

$$u.\mathrm{U}=(x\mathrm{Y}-y\mathrm{X})\alpha.\beta+(y\mathrm{Z}-z\mathrm{Y})\beta.\gamma+(z\mathrm{X}-x\mathrm{Z})\gamma.\alpha.$$

Hence, if we suppose the given couples to be $u.\mathrm{U}$, $u'.\mathrm{U}'$, $u''.\mathrm{U}''$, &c., and if we put, for brevity,

$$\Sigma(x\mathrm{Y}-y\mathrm{X})=\mathrm{N},\ \Sigma(y\mathrm{Z}-z\mathrm{Y})=\mathrm{L},\ \Sigma(z\mathrm{X}-x\mathrm{Z})=\mathrm{M},$$

the combined effect of the couples will be

$$\Sigma(u.\mathrm{U})=\mathrm{L}\beta.\gamma+\mathrm{M}\gamma.\alpha+\mathrm{N}\alpha.\beta$$
$$=\mathrm{D}^{-1}(\mathrm{L}\alpha+\mathrm{M}\beta+\mathrm{N}\gamma) \text{ (art. 48).}$$

Hence, if we assume G to denote the magnitude and γ' the direction of $\mathrm{L}\alpha+\mathrm{M}\beta+\mathrm{N}\gamma$, which gives (art. 17)

$$\mathrm{G}^2=\mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2 \;\ldots\ldots\; (1.), \qquad \gamma'=\frac{\mathrm{L}}{\mathrm{G}}\alpha+\frac{\mathrm{M}}{\mathrm{G}}\beta+\frac{\mathrm{N}}{\mathrm{G}}\gamma \;\ldots\ldots\; (2.);$$

we find

$$\Sigma u.\mathrm{U}=\mathrm{D}^{-1}(\mathrm{G}\gamma')$$
$$=\mathrm{G}\alpha'.\beta' \text{ (arts. 48 and 16).}$$

Here $\alpha'.\beta'$ denotes a unit-couple in a plane at right angles to γ'; and thus the resultant is a couple G in this plane; G and γ' being given by (1.) and (2.).

(77.) *To combine a given Set of Forces acting at given Points of a Rigid Body.*—Let the forces be U, U′, U,″ &c., and their points of application u, u', u'', &c.; then, by art. 74, their combined effect will be

$$\Sigma(\mathrm{U}+u.\mathrm{U}).$$

Hence, if we put

$$\Sigma\mathrm{X}=\mathrm{X}_{\prime}, \quad \Sigma\mathrm{Y}=\mathrm{Y}_{\prime}, \quad \Sigma\mathrm{Z}=\mathrm{Z}_{\prime},$$

and employ the notation of the preceding article, the combined effect becomes

$$\mathrm{X}_{\prime}\alpha+\mathrm{Y}_{\prime}\beta+\mathrm{Z}_{\prime}\gamma+\mathrm{G}\alpha'.\beta',$$

or

$$\mathrm{R}_{\prime}\alpha_{\prime}+\mathrm{G}\alpha'.\beta',$$

where

$$\mathrm{R}_{\prime}^2=\mathrm{X}_{\prime}^2+\mathrm{Y}_{\prime}^2+\mathrm{Z}_{\prime}^2, \;\ldots\ldots\ldots\ldots\; (3.)$$

and

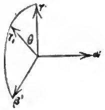
$$\gamma_{\prime}=\frac{\mathrm{X}_{\prime}}{\mathrm{R}_{\prime}}\alpha+\frac{\mathrm{Y}_{\prime}}{\mathrm{R}_{\prime}}\beta+\frac{\mathrm{Z}_{\prime}}{\mathrm{R}_{\prime}}\gamma. \;\ldots\ldots\ldots\ldots\; (4.)$$

Thus it appears that the set of given forces are combined into a single force $\mathrm{R}_{\prime}\alpha_{\prime}$ (given by (3.) and (4.)), and a single couple $\mathrm{G}\alpha'.\beta'$ (given by (1.) and (2.) previous article).

From the result just obtained the various well-known conditions and equations, relating to the effect of a set of forces on a rigid body, immediately follow. To exemplify the method I shall apply the formula $\mathrm{R}_{\prime}\alpha_{\prime}+\mathrm{G}\alpha'.\beta'$ to the following question.

(78.) *To combine the set of forces into Two Forces.*—Let us so choose α' and β' (which are arbitrary, except so far as they are perpendicular to γ'), that β' shall be in the same plane as $\gamma_{\prime}$ and γ'; and let θ be the angle which $\gamma_{\prime}$ makes with γ'. Then, by article 17,

Fig. 24.



$$\gamma_{\prime}=\gamma'\cos\theta+\beta'\sin\theta;$$
$$\therefore\; \mathrm{R}_{\prime}\gamma_{\prime}+\mathrm{G}\alpha'.\beta'=(\mathrm{R}_{\prime}\cos\theta)\gamma'+(\mathrm{R}_{\prime}\sin\theta)\beta'+\mathrm{G}\alpha'.\beta'$$
$$=(\mathrm{R}_{\prime}\cos\theta)\gamma'+(\mathrm{R}_{\prime}\sin\theta)\beta'+\left(\frac{\mathrm{G}}{\mathrm{R}_{\prime}\sin\theta}\alpha'\right)\cdot(\mathrm{R}_{\prime}\sin\theta)\beta'.$$

Now by article 74 this is the expression for the effect of two forces, viz.—

$\mathrm{R}_{\prime}\cos\theta$ in the direction γ',

and $\mathrm{R}_{\prime}\sin\theta$ in the direction β',

the former acting at the origin, and the latter at a point whose distance from the origin is

$$\frac{G}{R_{,}\sin\theta} \text{ drawn in the direction } \alpha'.$$

$R_{,}$, G, γ' and $\gamma_{,}$ are given by the equations (1), (2), (3), (4) above; α' is determined because it is at right angles to the plane $(\gamma'\gamma_{,})$, and β', because it is in the same plane, and at right angles to γ'. As regards θ, we have, taking the *longitudinal product* of (2.) and (4.),

$$\frac{LX_{,}+MY_{,}+NZ_{,}}{LR_{,}}=\gamma'\times\gamma_{,}=\cos\theta.$$

(79.) *Centre of Parallel Forces.*—Let the magnitudes of the parallel forces be R, R′, R″, &c., $\gamma_{,}$ their common direction, and u, u', u'', &c. their points of application. Then their combined effect is

$$\Sigma(R\gamma_{,})+\Sigma(u.R\gamma_{,})=(\Sigma R)\gamma_{,}+\frac{\Sigma Ru}{\Sigma R}\cdot(\Sigma R)\gamma_{,}.$$

Now, by art. 74, this is the symbol for a force ΣR, acting in the direction $\gamma_{,}$, and at a point whose distance from the origin is

$$\frac{\Sigma Ru}{\Sigma R},$$

or

$$\frac{\Sigma Rx}{\Sigma R}\alpha+\frac{\Sigma Ry}{\Sigma R}\beta+\frac{\Sigma Rz}{\Sigma R}\gamma,$$

which expresses the common formulæ for the centre of parallel forces.

III. Application of the Symbolic Forms to Dynamics.

(80.) *Effective Force, Vis-Viva, Work.*—If u be the distance of a moving particle m from the origin at any time t, it is clear that du represents, in magnitude and direction, the space described in the time dt; and thus the complete symbol of the velocity becomes

$$\frac{du}{dt}.$$

Also $d\left(\frac{du}{dt}\right)$ represents, in magnitude and direction, the alteration of velocity in the time dt; and thus the complete symbol of the effective force is

$$m\frac{d^2u}{dt^2}.$$

Again, if U denote any force acting on m, it is easy to see that the symbol of the work accumulated, while m is moving from the point u to the point u', is

$$\int_u^{u'} U\times du.$$

Lastly, if for U here we put the effective force, the *effective work* will be

$$\int_u^{u'} m\frac{d^2u}{dt^2}\times du$$

or*
$$\frac{m}{2}\left(\frac{du'}{dt}\times\frac{du'}{dt}-\frac{du}{dt}\times\frac{du}{dt}\right).$$

Here $\frac{du}{dt}\times\frac{du}{dt}$ denotes the square of the velocity (art. 49), and thus the expression denotes the ordinary *vis-viva*.

(81.) *Description of Areas.*—It is clear that $\frac{1}{2}u.du$ is the area described in the time dt by u; for it is the half parallelogram formed on u and du. But it is to be noticed that $\frac{1}{2}u.du$ represents this area, not only in magnitude, but also *in position.*

If we put $\frac{1}{2}u.du=\mathrm{A}dt$, we find
$$\frac{d\mathrm{A}}{dt}=\frac{1}{2}\left(u.\frac{d^2u}{dt^2}+\frac{du}{dt}.\frac{du}{dt}\right),\text{ and }\frac{du}{dt}.\frac{du}{dt}=0\text{ (art. 32)};$$
whence
$$\frac{d\mathrm{A}}{dt}=\frac{1}{2}u.\frac{d^2u}{dt^2}. \qquad (1.)$$

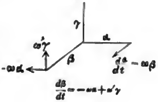
A, here, is a symbol which represents two important things:—1st, *in magnitude*, it is the ordinary *rate of description of area* $\left(r^2\frac{d\theta}{dt}\right)$; 2ndly, its plane is the plane containing the radius vector and the direction of motion, *i.e.* the *plane of the orbit* of m.

DA is the symbol of a line perpendicular to the plane of the orbit, and equal in magnitude to the rate of description of area.

If we put $u=x\alpha+y\beta+z\gamma$, and perform the operation indicated in equation (1.), we find,
$$2\frac{d(\mathrm{DA})}{dt}=\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)\gamma+\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)\alpha+\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right)\beta.$$

(82.) *Expression for Effective Force with reference to radius vector, angular velocity, and plane of orbit.*—Let r denote the magnitude and α the "*direction*" of u; let β denote a direction *in* the plane of the orbit of m (and, of course, at right angles to α); then γ will be a direction always perpendicular to the plane of the orbit: lastly, let ω denote the angular velocity of u, and ω' the angular velocity of the plane of the orbit about u, ω and ω' being numerical quantities. On referring to art. 19, the following relations are manifest,
$$u=r\alpha \quad \frac{d\alpha}{dt}=\omega\beta \quad \frac{d\beta}{dt}=-\omega\alpha+\omega'\gamma;$$

Fig. 35.



for $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ denote, in magnitude and direction, the velocities of the extremities of the directed units α and β, and the figure will show what their values are in terms of ω and ω'.

* $d\left(\frac{du}{dt}\times\frac{du}{dt}\right)=\frac{d^2u}{dt}\times\frac{du}{dt}+\frac{du}{dt}\times\frac{d^2u}{dt}$
$=2\frac{d^2u}{dt^2}\times du$ (art. 33).

Hence we immediately find the required expression for the effective force $\frac{d^2u}{dt^2}$ by simple differentiation, as follows,

$$\frac{du}{dt}=\frac{dr}{dt}\alpha+r\frac{d\alpha}{dt}=\frac{dr}{dt}\alpha+r\omega\beta$$

$$\frac{d^2u}{dt^2}=\frac{d^2r}{dt^2}\alpha+\frac{dr}{dt}\frac{d\alpha}{dt}+\frac{d(r\omega)}{dt}\beta+r\omega\frac{d\beta}{dt},$$

or effective force $\quad =\left(\frac{d^2u}{dt^2}-r\omega^2\right)\alpha+\left(\frac{dr}{dt}\omega+\frac{d(r\omega)}{dt}\right)\beta+r\omega\omega'\gamma.$

Hence it appears that the effective force is equivalent to the well-known expressions along and perpendicular to the radius vector, together with a third part, $r\omega\omega'$ perpendicular to the plane of the orbit.

(83.) *The use of the forms* u.v *and* u×v *exemplified in the case of motion about a centre of force varying as* r^{-2}.—This case appears to me to afford so good an illustration of the use of these forms, that I shall give it here briefly. Assuming α, β, γ, ω, r as in the preceding article, it is clear that the symbol of the central force is $-\frac{\mu}{r^2}\alpha$, and therefore we have

$$\frac{d^2u}{dt^2}=-\frac{\mu}{r^2}\alpha, \quad \ldots \quad (1.)$$

whence $\quad u.\frac{d^2u}{dt^2}=0$, since $u=r\alpha$, and $\alpha.\alpha=0$,

therefore (by art. 81), $\quad \frac{dA}{dt}=0. \quad \ldots \quad (2.)$

This indicates that there is no variation of A, and consequently (see art. 81) that the plane of the orbit, and the rate of description of area is constant.

If we put for A its value (art. 81) $\frac{1}{2}u.\frac{du}{dt}$ and for u and $\frac{du}{dt}$ their values $r\alpha$ and $\frac{d(r\alpha)}{dt}$, observing that $\frac{d\alpha}{dt}=\omega\beta$, we find

$$A=\frac{1}{2}r\alpha.\left(\frac{dr}{dt}\alpha+r\omega\beta\right)=\frac{1}{2}r^2\omega\alpha.\beta,$$

or, by (2.), $\quad A=\frac{1}{2}h\alpha.\beta \quad (h=r^2\omega$, as usual).

Now it is singular that (1.) admits of *immediate integration*, instead of requiring the well-known transformations: for, in it, put for α its value, $-\frac{1}{\omega}\frac{d\beta}{dt}$ (see art. 82), and we have

$$\frac{d^2u}{dt^2}=\frac{\mu}{r^2\omega}\frac{d\beta}{dt}=\frac{\mu}{h}\frac{d\beta}{dt};$$

wherefore, integrating, we find

$$\frac{du}{dt}=\frac{\mu}{h}\beta+\text{ constant.}$$

Here put for $\frac{du}{dt}$ its value, and there results

$$\frac{dr}{dt}\alpha+r\omega\beta=\frac{\mu}{h}\beta+\text{ constant }(c\beta'\text{ suppose*}).$$

Multiply this *longitudinally* by β, and, since $\beta\times\alpha=0$, $\beta\times\beta=1$, and $\beta\times\beta'=\alpha\times\alpha'$ $=\cos\theta$, we find

$$r\omega=\frac{\mu}{h}+c\cos\theta;$$

or, since $r\omega=\frac{h}{r}$, this gives

$$\frac{1}{r}=\frac{\mu}{h^2}+\frac{c}{h}\cos\theta,$$

the well-known equation, θ being the angle which α (the direction of the radius vector) makes with α' (an arbitrary constant direction).

The manner in which the symbolic forms have effected this integration appears to me to be worthy of notice.

(84.) *General Expression for the Momentum of a Rigid Body moving in any manner.*—The *Momentum* of a particle (m), moving with a velocity $\left(\frac{du}{dt}\right)$, is

$$m\frac{du}{dt}.$$

And this symbol represents the Momentum in direction as well as magnitude. Now momentum is really *a force*, estimated, however, somewhat differently from ordinary pressure, on the principle that the true *dynamical effect* of a force is proportional to absolute intensity and the time of its action conjointly†. Hence, regarding the momentum of m as a force, its complete symbol will be (by art. 74),

$$\frac{mdu}{dt}+u.\left(\frac{mdu}{dt}\right).$$

Thus the symbol of the *total momentum* of the rigid body will be

$$\Sigma m\left\{\frac{du}{dt}+u.\frac{du}{dt}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

(85.) If $\bar{u}$ denote the distance of the *centre of gravity* from the origin, and u' the distance of the point (u) from the centre of gravity, we have $u=\bar{u}+u'$, and thus, since $\Sigma mu'=0$, (1.) becomes (putting M for Σm)

$$\mathrm{M}\frac{d\bar{u}}{dt}+\bar{u}.\left(\mathrm{M}\frac{d\bar{u}}{dt}\right)+\Sigma m\left(u'.\frac{du'}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

* Of course the constant is the symbol for some constant line, as regards *direction* as well as magnitude, and therefore I put it in the form $c\beta'$, c being a number and β' a "*direction*."

† If the *pressure* P produces a velocity v in the time t, it produces the velocity $\frac{v}{t}$ per second, and therefore $\mathrm{P}=m\frac{v}{t}$; or $mv=\mathrm{P}t$. Pt then is the momentum, and this is proportional to P and t conjointly.

Now here, by (1.), M $\frac{d\bar{u}}{dt}+\bar{u}.\left(M\frac{d\bar{u}}{dt}\right)$ is the expression for the momentum of M concentrated into the centre of gravity, and $\Sigma m\left(u'.\frac{du'}{dt}\right)$ is the momentum of the body as regards its motion *relatively* to the centre of gravity (as is evident from (1.), for, if the origin have no momentum, $\Sigma m\,\frac{du}{dt}=0$).

In Section V. I shall show, that

$$\Sigma m\left(u'.\frac{du'}{dt}\right)=D^{-1}(A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma),$$

that is, *a couple whose axis* is $A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma$. Here A, B, C are the moments of inertia about the three principal axes, of which α, β, γ are supposed to be the directions; and ω_1, ω_2, ω_3 are the well-known component angular velocities.

(86.) *General Expression for the Energy of a Rigid Body moving in any manner.*—I venture to suggest the word "*energy*" as a proper designation for the *total effective force* by which the motion of a rigid body is produced, inasmuch as ἐνέργεια means force *actually exerted and effective.* The symbol, then, of *the energy* of a rigid body will be

$$\Sigma m\left(\frac{d^2u}{dt^2}+u.\frac{d^2u}{dt^2}\right).$$

Now, observing that $\frac{du}{dt}\cdot\frac{du}{dt}=0$, we have

$$u.\frac{d^2u}{dt^2}=d\left(u.\frac{du}{dt}\right).$$

Hence the expression for the energy is

$$\frac{d}{dt}\left\{\Sigma m\left(\frac{du}{dt}+u.\frac{du}{dt}\right)\right\}.$$

Comparing this with the expression for the *momentum* in art. 84, we have

$$energy=\frac{d\,(momentum)}{dt}\quad\ldots\ldots\ldots\ldots\quad(1.)$$

(87.) This, though a very simple result, is really one of importance; thus if the centre of gravity be fixed, we find (by art. 85.)

$$energy=D^{-1}\frac{d}{dt}(A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma).$$

Now it will be shown in Sect. V. that

$$\frac{d\alpha}{dt}=D\omega.\alpha,\quad\frac{d\beta}{dt}=D\omega.\beta,\quad\frac{d\gamma}{dt}=D\omega.\gamma,$$

where $\omega=\omega_1\alpha+\omega_2\beta+\omega_3\gamma$.

Hence, performing the differentiation, and observing that

$$D^{-1}\alpha=\beta.\gamma,\ D^{-1}\beta=\gamma.\alpha,\ D^{-1}\gamma=\alpha.\beta,\ \text{and}\ D^{-1}D=1,$$

we find $$energy = A\frac{d\omega_1}{dt}\beta.\gamma + B\frac{d\omega_2}{dt}\gamma.\alpha + C\frac{d\omega_3}{dt}\alpha.\beta$$
$$+\omega.(A\omega_1\alpha + B\omega_2\beta + C\omega_3\gamma).$$

Lastly, if we perform the *lateral multiplication* denoted by ω. here, we find

$$energy = \left\{A\frac{d\omega_1}{dt} + (C-B)\omega_2\omega_3\right\}\beta.\gamma$$
$$+\left\{B\frac{d\omega_2}{dt} + (A-C)\omega_3\omega_1\right)\gamma.\alpha$$
$$+\left\{C\frac{d\omega_3}{dt} + (B-A)\omega_1\omega_2\right)\alpha.\beta.$$

I need not point out the meaning of this formula in relation to EULER's equations for a rigid body.

(88.) *General Expression for the "Vis-Mortua" or "Dead Pull" on a Rigid Body.*—I may use the almost obsolete word "*Vis-Mortua*" (which has been so well translated by the familiar expression "*Dead Pull*") in the sense in which it was originally employed to denote simple pressure or *impressed force.* I shall therefore designate the total mechanical effect of the impressed forces acting on a rigid body as the *Vis-Mortua* or *Dead Pull* on that body. If U denote the force acting at the point (u), we have, therefore, by art. 74,

$$Vis\text{-}Mortua = \Sigma(U + u.U):$$

this of course is the same expression as that in art. 77.

If we put, as before, $u = \bar{u} + u'$, this expression becomes

$$\Sigma U + \bar{u}.\Sigma U + \Sigma u'.U.$$

Here $\Sigma U + \bar{u}.\Sigma U$ is the symbol of the force ΣU acting at the point ($\bar{u}$), and $\Sigma u'.U$ is (art. 75) the symbol of a couple.

(89.) As an example, the result of which I shall require in Sect. V., I shall calculate the Vis-Mortua of a rigid body acted on by the attraction of a distant particle m', taking the centre of gravity of the rigid body as origin, and α, β, γ as the directions of the three principal axes; assuming also u' to denote the distance of m', r' the magnitude of u', and r that of u.

Here U is a force acting along the line joining the two points (u) and (u'), and inversely proportional to the square of the magnitude of that line. The line alluded to is $u' - u$, its magnitude is $\sqrt{(u'-u)\times(u'-u)}$, and its direction therefore is

$$\frac{u'-u}{\sqrt{(u'-u)\times(u'-u)}}.$$

Hence $$U = mm'\frac{(u'-u)}{\{(u'-u)\times(u'-u)\}^{\frac{3}{2}}}.$$

Also $$\{(u'-u)\times(u'-u)\}^{-\frac{3}{2}} = (u'\times u' - 2u'\times u + u\times u)^{-\frac{3}{2}}$$
$$= (r'^2 - 2u'\times u + r^2)^{-\frac{3}{2}}$$
$$= \frac{1}{r'^3}\left(1 + \frac{3u'\times u}{r'^2}\right) \text{ nearly.}$$

Hence, observing that $u.u=0$, and $\Sigma mu=0$, we have

$$\Sigma u.\mathrm{U}=\frac{3m'}{r^5}\Sigma m(u'\times u)(u.u')$$

$$=-\frac{3m'}{r^5}u'.\{\Sigma m(u'\times u)u\}.$$

Now $\Sigma m(u'\times u)u=\Sigma m(xx'+yy'+zz')(x\alpha+y\beta+z\gamma)$

$=(\Sigma mx^2)x'\alpha+(\Sigma my^2)y'\beta+(\Sigma mz^2)z'\gamma$, by properties of principal axes,

$=(\Sigma mr^2-\mathrm{A})x'\alpha+(\Sigma mr^2-\mathrm{B})y'\beta+(\Sigma mr^2-\mathrm{C})z'\gamma$

$=(\Sigma mr^2)u'-(\mathrm{A}x'\alpha+\mathrm{B}y'\beta+\mathrm{C}z'\gamma)$.

Wherefore, observing that $u'.u'=0$, we find

$$\Sigma u.\mathrm{U}=\frac{3m'}{r^5}u'.(\mathrm{A}x'\alpha+\mathrm{B}y'\beta+\mathrm{C}z'\gamma).$$

Also $\qquad \Sigma\mathrm{U}=\Sigma mm'\frac{u'-u}{r^3}$ (neglecting $\frac{3u'\times u}{r^2}$ for obvious reasons)

$$=\mathrm{M}m'\frac{u'}{r^3} \qquad \begin{cases}\mathrm{M}=\Sigma m\\ \Sigma mu=0.\end{cases}$$

Hence we have

$$\textit{Vis Mortua}=\mathrm{M}m'\frac{u'}{r^3}+\frac{3m'}{r^5}u'.(\mathrm{A}x'\alpha+\mathrm{B}y'\beta+\mathrm{C}z'\gamma),$$

which expresses a force $\mathrm{M}m'\frac{u'}{r^3}$ acting at the origin, and a couple,

$$\frac{3m'}{r^5}u'.(\mathrm{A}x'\alpha+\mathrm{B}y'\beta+\mathrm{C}z'\gamma).$$

If we perform the *lateral multiplication* indicated by $(u'.)$, this couple becomes

$$\frac{3m'}{r^5}\Big\{(\mathrm{C}-\mathrm{B})y'z'\beta.\gamma+(\mathrm{A}-\mathrm{C})z'x'\gamma.\alpha+(\mathrm{B}-\mathrm{A})x'y'\alpha.\beta\Big\},$$

which gives the three well-known couples in the theory of Precession and Nutation.

IV. Application of the Symbolic Forms to determine the Correction for the Earth's Rotation in Problems relating to motion on or near the Earth's Surface.

(90.) The best way of defining that which is commonly called the *Centrifugal Force* appears to be the following, viz. that it is an imaginary force which may be introduced as a correction for the error of not taking into account the rotation of the radius vector r. Suppose P to denote the accelerating force acting along r, and let us for a moment forget that r has an angular velocity $\left(\frac{d\theta}{dt}\right)$, then we put

$$\frac{d^2r}{dt^2}=\mathrm{P};$$

but this is erroneous, and we must correct it for the rotation by adding to P the term $r\left(\frac{d\theta}{dt}\right)^2$, as is well known. Hence we may regard the centrifugal force as a *correction for neglected rotation.*

But it only corrects the error so far as the motion along r is concerned; another correction (supposing still the rotation is neglected or forgotten) is necessary to be applied at right angles to r, namely, the imaginary force $-\frac{1}{r}\frac{d}{dt}\left(\frac{r^2d\theta}{dt}\right)$. Thus the true and complete correcting force is the resultant of the two forces

$$r^2\frac{d\theta}{dt}, \text{ and } -\frac{1}{r}\frac{d}{dt}\left(\frac{r^2d\theta}{dt}\right).$$

It appears to me that the idea here suggested might be applied with great advantage to cases of motion on or near the earth's surface. The beautiful pendulum experiment which made so much noise last year, and the various investigations respecting it, give great interest to such cases of motion. I propose therefore to investigate here, by the aid of the Symbolic Forms, the proper symbol of the imaginary force which corrects completely for the earth's rotation supposed to be neglected. By the aid of this symbol, it will be found that the greatest possible simplicity is introduced into investigations such as those relating to the pendulum experiment. It will enable the investigator to forget altogether the earth's rotation in framing his equations of motion, and at the same time to *correct his error by the introduction of a simple term.*

(91.) Let ω denote a line pointing in the direction of the earth's polar axis (north suppose), and representing, by its length, the earth's angular velocity. In other words, let ω be the *directrix of the earth's rotation*, then (as will be shown in Sect. V.) it is easy to see, that, if u denote (symbolically) the distance of any point from the earth's centre, the velocity communicated to it by the earth's rotation (if it be fixed to the earth) is represented by the symbol*

$$\mathrm{D}\omega.u.$$

Now let d denote differentiation (of u) on the erroneous supposition that the earth is fixed, and δ the true and complete differentiation; then the true velocity of the point u is $\frac{\delta u}{dt}$, and this must be the resultant of the erroneous velocity $\left(\frac{du}{dt}\right)$ and the velocity ($\mathrm{D}\omega.u$) due to the rotation. Hence we have

$$\frac{\delta u}{dt}=\frac{du}{dt}+\mathrm{D}\omega.u.$$

The effective accelerating force will be obtained by the true and complete differentiation of the correct velocity, that is,

$$\frac{\delta}{dt}\left(\frac{du}{dt}+\mathrm{D}\omega.u\right),$$

or $$\frac{d}{dt}\left(\frac{\delta u}{dt}\right)+\mathrm{D}\omega.\frac{\delta u}{dt} \quad \text{(observing that } \omega \text{ is a constant),}$$

* For $\mathrm{D}\omega.u$ denotes a line at right angles to both ω and u, and its magnitude is $nr\sin\theta$; where n is the magnitude of ω, r that of u, and θ the angle which u makes with ω. Therefore $\mathrm{D}\omega.u$ is manifestly the velocity caused in the point u by the rotation ω.

which, putting for $\frac{\delta u}{dt}$ the above value, becomes

$$\frac{d^2u}{dt^2}+2\mathrm{D}\omega.\frac{du}{dt}+\mathrm{D}\omega.(\mathrm{D}\omega.u).$$

Hence if U denote (symbolically) the resultant of the accelerating forces, whatever they may be, which act on the point (u), we find

$$\frac{d^2u}{dt^2}+2\mathrm{D}\omega.\frac{du}{dt}+(\mathrm{D}\omega.)^2u=\mathrm{U}. \quad \text{(1.)}$$

I may observe, in passing, that, since $d+(\mathrm{D}\omega.)$ represents the *complete* differentiation of u, we might have written down the equation of motion *immediately*, in the form

$$\frac{\{d+(\mathrm{D}\omega.)\}^2}{dt^2}u=\mathrm{U},$$

which, expanded, is identical with (1.).

Now, if we had forgotten the earth's rotation, we should have put, instead of (1.),

$$\frac{d^2u}{dt^2}=\mathrm{U}.$$

Hence it appears that, if we neglect the rotation in forming the equation of motion, we may *correct the error*, by supposing that there is the imaginary force

$$-\left\{2\mathrm{D}\omega.\frac{du}{dt}+(\mathrm{D}\omega.)^2u\right\} \quad \text{(2.)}$$

acting in addition to the real force represented by U; for on this supposition we find

$$\frac{d^2u}{dt^2}=\mathrm{U}-2\mathrm{D}\omega.\frac{du}{dt}-(\mathrm{D}\omega.)^2u, \quad \text{(3.)}$$

which is equivalent to (1.).

As regards terrestrial problems, however, the expression (2.) admits of an important simplification; for the accelerating force of gravity (g), which of course is included in U, is supposed to be the resultant of the earth's attraction, and the *common centrifugal force*. Now this common centrifugal force is that which is conceived to be in action upon a particle rigidly connected with the revolving earth; in other words, it is what (2.) becomes when $\frac{du}{dt}=0$. Wherefore the expression for the *common centrifugal force* is

$$-(\mathrm{D}\omega.)^2u. \quad \text{(4.)}$$

As this therefore is included among the forces which U represents, it ought to be omitted in the equation (3.). Thus we find that

$$-2\mathrm{D}\omega.\frac{du}{dt} \quad \text{(5.)}$$

is the force which must be supposed to act on the point (u) *as a correction for the neglected rotation.*

We may, therefore, in all problems of motion relative to the earth, forget altoge-

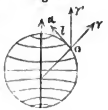
ther the earth's rotation, provided we introduce the force (5.), in addition to the accelerating forces, whatever they may be, which are really in action upon the point (u); it being understood that the common centrifugal force is allowed for in g.

(92.) In (5.) $\frac{du}{dt}$ is the apparent velocity of the point u (apparent, that is, to an observer unconscious of the earth's motion); and if n denote the earth's angular velocity, and γ' the direction of the polar axis, $\omega=n\gamma'$. Thus (5.) becomes

$$-2n\mathrm{D}\gamma'.\frac{du}{dt}. \quad \ldots \quad (6.)$$

Now this represents a force at right angles to γ' and $\frac{du}{dt}$, *i. e.* to the *polar plane* in which the apparent velocity is taking place at the instant t. Also the magnitude of this force is $2n$ times the apparent velocity $\left(\frac{du}{dt}\right)$ multiplied by the sine of the angle it $\left(\frac{du}{dt}\right)$ makes with the polar axis (γ').

Fig. 36.



(93.) This force may be expressed with reference to horizontal and vertical coordinates at any place, as follows:

Let O be the place, γ the vertical at O, and then the plane ($\alpha\beta$) will be horizontal: also let α be chosen so as to lie in the meridian plane; and let l denote the latitude of O (*i. e.* the angle γ' makes with α). Then

$$\gamma'=\alpha\cos l+\gamma\sin l. \quad \ldots \quad (7.)$$

Also if we take u to denote the distance of the moving point from O at any time t, and therefore put

$$u=x\alpha+y\beta+z\gamma, \quad \ldots \quad (8.)$$

$\frac{du}{dt}$ will be the same as the $\frac{du}{dt}$* in (6.), for all that we have to express by $\frac{du}{dt}$ is the apparent velocity of the point u.

Hence, differentiating (8.), and performing the operations indicated by $2n\mathrm{D}\gamma'.$, (6.) becomes

$$-2n\mathrm{D}(\alpha\cos l+\gamma\sin l).\left(\frac{dx}{dt}\alpha+\frac{dy}{dt}\beta+\frac{dz}{dt}\gamma\right),$$

or

$$2n\left\{\left(\frac{dy}{dt}\sin l\right)\alpha+\left(-\frac{dx}{dt}\sin l+\frac{dz}{dt}\cos l\right)\beta-\left(\frac{dy}{dt}\cos l\right)\gamma\right\}. \quad \ldots \quad (9.)$$

Hence the ordinary equations of motion will be

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\mathrm{X}+2n\frac{dy}{dt}\sin l\\ \frac{d^2y}{dt^2}&=\mathrm{Y}-2n\frac{dx}{dt}\sin l+2n\frac{dz}{dt}\cos l\\ \frac{d^2z}{dt^2}&-\mathrm{Z}-2n\frac{dy}{dt}\cos l\end{aligned}\right\} \quad \ldots \quad (10.)$$

* Originally u was measured from the earth's centre.

where X, Y, Z represent the accelerating forces, whatever they may be, that are in action on the point u.

(94.) The formation of these equations, here given, affords a good example of the use of the symbolic form $u.v$: but, to illustrate the method more clearly, it will be worth while to employ, in some particular problem, the general equation,

$$\frac{d^2u}{dt^2}=\mathrm{U}-2n\mathrm{D}\gamma'.\frac{du}{dt}, \qquad (11.)$$

which includes the three equations (10.), U denoting $\mathrm{X}\alpha+\mathrm{Y}\beta+\mathrm{Z}\gamma$. The problem I shall choose will be that of the *Pendulum Experiment*.

Fig. 37.

Let QP represent the string at any time t, QO its vertical position, c its length; then

$$\mathrm{OQ}=c\gamma, \quad \mathrm{OP}=u, \quad \mathrm{PQ}=c\gamma-u,$$

also *direction* of $\mathrm{PQ}=\frac{c\gamma-u}{c}$.

Let T denote (in magnitude) the tension of PQ; and then, since the direction of T is that of PQ, the symbol of the tension is $\mathrm{T}\,\frac{c\gamma-u}{c}$; to which if we add $-g\gamma$, the symbol of the force of gravity, we find

$$\mathrm{U}=\mathrm{T}\frac{c\gamma-u}{c}-g\gamma=(\mathrm{T}-g)\gamma-\frac{\mathrm{T}}{c}u.$$

Hence (11.) becomes

$$\frac{d^2u}{dt^2}=(\mathrm{T}-g)\gamma-\frac{\mathrm{T}}{c}u-2n\mathrm{D}\gamma'.\frac{du}{dt}. \qquad (12.)$$

Now, for greater simplicity, I shall suppose that u represents a small excursion, and c a long string. On this supposition we may regard u as always horizontal. Also, if we put for γ' its value (7.), the equation (12.) becomes

$$\left\{\frac{d^2u}{dt^2}+2n\sin l\mathrm{D}\gamma.\frac{du}{dt}-\frac{\mathrm{T}}{c}u\right\}+\left\{2n\cos l\mathrm{D}\alpha.\frac{du}{dt}-(\mathrm{T}-g)\gamma\right\}=0.$$

Hence, since $\frac{du}{dt}$ is horizontal, $\mathrm{D}\alpha.\frac{du}{dt}$ is vertical, and $\mathrm{D}\gamma.\frac{du}{dt}$ is horizontal. The equation (12.), therefore, is separated into two parts, horizontal and vertical, which, being equated to zero, we obtain

$$\frac{d^2u}{dt^2}+2n\sin l\mathrm{D}\gamma.\frac{du}{dt}-\frac{\mathrm{T}}{c}u=0 \qquad (13.)$$

$$(\mathrm{T}-g)\gamma=2n\cos l\mathrm{D}\alpha.\frac{du}{dt}. \qquad (14.)$$

We have to substitute for T in (13.) its value derived from (14.), which on account of the smallness of n, and the fact that T is multiplied by $\frac{u}{c}$ in (13.), gives $\mathrm{T}=g$ for a first approximation. Hence (13.) becomes

$$\frac{d^2u}{dt^2}+2n\sin l\mathrm{D}\gamma.\frac{du}{dt}-\frac{g}{c}u=0. \qquad (15.)$$

Now, here, the operation (Dγ.) is performed only on lines at right angles to γ; we may therefore put $\sqrt{-1}$ for (Dγ.) (see art. 55); and thus (15.) becomes

$$\frac{d^2u}{dt^2}+2n\sin l\sqrt{-1}\,\frac{du}{dt}-\frac{g}{c}u=0. \qquad (16.)$$

The roots of the equation

$$\left(\frac{d}{dt}\right)^2+2n\sin l\sqrt{-1}\left(\frac{d}{dt}\right)-\frac{g}{c}=0$$

are

$$-n\sin l\sqrt{-1}\pm\sqrt{-n^2\sin^2 l-\frac{g}{c}},$$

or

$$(-n\sin l\pm m)\sqrt{-1},$$

if, for brevity, we put $n^2\sin^2 l+\frac{g}{c}=m^2$. Hence the solution of (16.) is

$$u=\varepsilon^{-nt\sin l\sqrt{-1}}(\mathrm{A}\varepsilon^{mt\sqrt{-1}}+\mathrm{B}\varepsilon^{-mt\sqrt{-1}}). \qquad (17.)$$

The constants A and B here denote two arbitrary lines in the horizontal plane.

If the earth were fixed, the form of the solution would have been

$$u=\mathrm{A}\varepsilon^{mt\sqrt{-1}}+\mathrm{B}\varepsilon^{-mt\sqrt{-1}} \qquad \left\{\text{for then } m^2=\frac{g}{c}\right\}.$$

Now $\varepsilon^{-nt\sin l\sqrt{-1}}$ * indicates a uniform *backward* rotation of $\mathrm{A}\varepsilon^{mt\sqrt{-1}}+\mathrm{B}\varepsilon^{-mt\sqrt{-1}}$ with an angular velocity $n\sin l$. Thus it appears that the apparent curve of motion of the pendulum will be the same form as if the earth were fixed, only there will be a slow angular regression of the whole about the vertical as axis at the rate $n\sin l$ per second.

I may observe, in passing, that the simplest interpretation of (17.) is this; that the motion of the point (u) results from the superposition of two motions,

$$\mathrm{A}\varepsilon^{(m-n\sin l)t\sqrt{-1}} \text{ and } \mathrm{B}\varepsilon^{-(m+n\sin l)t\sqrt{-1}};$$

and these are two uniform circular* motions, the former that of the line A forward with an angular velocity $(m-n\sin l)$; the latter that of B backward with an angular velocity $m-n\sin l$.

(95.) As my object is simply to exemplify the application of my notation, I shall not proceed to a second approximation; which however is very easily effected by substituting for T in (13.) its complete value given by (14.), after having put for u in (14.) the value (17.) just obtained. The result is important, especially as regards motion near the equator.

* $\mathrm{A}\varepsilon^{\theta\sqrt{-1}}$ denotes A turned out of its position (round γ) through an angle θ, and therefore $u=\mathrm{A}\varepsilon^{nt\sqrt{-1}}$ is an equation indicating that the motion of the point (u) results from rotation round the origin at the distance A, the angle nt being described in the time t.

V. Application of the Symbolic Forms to determine the Motion of a Rigid Body about its Centre of Gravity.

(96.) The symbolic forms $u.v$ and $u\times v$ are singularly useful, as it appears to me, in all cases of the Motion of a Rigid Body in space, especially as regards Rotation. Considerable simplification is also gained by employing $d\alpha$, $d\beta$, $d\gamma$ to denote the angular motions of the three axes α, β, γ. I shall now proceed to consider this case.

I shall take α, β, γ to denote three rectangular directions fixed in the Rigid Body, and x, y, z the coordinates of any particle (m) of the body. On this supposition x, y, z are constants as regards t, while α, β, γ are variables. The origin is the fixed point (the centre of gravity, namely,) about which the body moves. u denotes the distance of m from the origin, and therefore

$$u=x\alpha+y\beta+z\gamma. \quad \ldots \quad (1.)$$

(97.) Now the rigidity of the body requires that the velocity $\left(\frac{du}{dt}\right)$ of m shall be at right angles to u always; this may be expressed (see art. 44) by putting

$$\frac{du}{dt}=\mathrm{D}\omega.u, \quad \ldots \quad (2.)$$

where ω denotes some unknown line. It may be shown, as follows, that ω is a function of t *only*, or, in other words, that ω is the same for all points of the body, *i. e.* for all values of u.

Let (u') be *any point in space*, and let us assume, as we may, that this point moves always with a velocity $\mathrm{D}\omega.u'$; then

$$\frac{du'}{dt}=\mathrm{D}\omega.u'; \quad \ldots \quad (3.)$$

and hence, by (2),

$$\frac{d(u'-u)}{dt}=\mathrm{D}\omega.(u'-u). \quad \ldots \quad (4.)$$

Now $u'-u$ is the line joining the two points (u) and (u'), and the square of its length is

$$(u'-u)\times(u'-u),$$

and

$$\frac{d}{dt}\{(u'-u)\times(u'-u)\}=2\frac{d(u'-u)}{dt}\times(u'-u)=0;$$

for (4.) shows that $\frac{d(u'-u)}{dt}$ and $u'-u$ are at right angles. Consequently the length of the line $u'-u$ is invariable.

In precisely the same way (3.) shows (what indeed is otherwise obvious) that the line u' is of invariable length.

Hence the point (u') is rigidly connected with the origin and with the point (u); and consequently (u') *is a point of the rigid body*. Therefore, comparing (2.) and (3.), it appears that ω does not vary when we pass from one point to another of the rigid body.

This result is of great importance, and furnishes, in the simplest possible manner, every formula necessary for determining the motion of the rigid body.

(98.) *The symbol ω represents, in direction*, the *instantaneous axis of rotation*. For (2.) shows that, when u coincides in direction with ω, $\frac{du}{dt}=0$; consequently all the points of the body which lie in the direction of ω are quiescent at the instant t.

(99.) *The magnitude of ω is the instantaneous angular velocity*. For, let α' denote any unit line fixed in the body at right angles to ω; then (see art. 19) $\frac{d\alpha'}{dt}$ is the angular velocity, at least in magnitude. But, by (2.), $\frac{d\alpha'}{dt}=\mathrm{D}\omega.\alpha'$; and, since ω is at right angles to α', $\mathrm{D}\omega.\alpha'$ has the same magnitude as ω (see art. 40.). Wherefore ω has the same magnitude as $\frac{d\alpha'}{dt}$, and therefore represents the angular velocity in magnitude.

(100.) Hence the result above obtained, namely,

$$\frac{du}{dt}=\mathrm{D}\omega.u, \quad \ldots \ldots \ldots \quad (4.)$$

may be thus enunciated:—the rigid body is, at the time t, moving about a certain instantaneous axis, with a certain angular velocity; and if we assume ω to denote that axis, *in direction*, and the angular velocity *in magnitude*, then the velocity $\left(\frac{du}{dt}\right)$ of any point (u) of the rigid body is obtained by performing the operation ($\mathrm{D}\omega$.) upon u.

(101.) If we put

$$\omega=\omega_1\alpha+\omega_2\beta+\omega_3\gamma, \quad \ldots \ldots \ldots \quad (5.)$$

where ω_1, ω_2, ω_3 denote numerically the projections of the line ω on the three coordinate directions α, β, γ, we find by (4.),

$$\frac{du}{dt}=\omega_1\mathrm{D}\alpha.u+\omega_2\mathrm{D}\beta.u+\omega_3\mathrm{D}\gamma.u. \quad \ldots \ldots \quad (6.)$$

Hence it appears that $\frac{du}{dt}$ results from the superposition of three angular velocities ω_1, ω_2, ω_3 about the axes α, β, γ respectively: for, by (4.), $\mathrm{D}(\omega_1\alpha).u$ (or $\omega_1\mathrm{D}\alpha.u$) denotes a velocity of the point (u) resulting from an angular velocity ω_1 about the axis α, $\omega_2\mathrm{D}\beta.u$ that resulting from ω_2 about β, and $\omega_3\mathrm{D}\gamma.u$ that resulting from ω_3 about γ: and (6.) shows that the actual velocity of u is the resultant of these three velocities.

If we put for u its value $x\alpha+y\beta+z\gamma$, (6.) becomes immediately

$$\frac{du}{dt}=(\omega_1y-\omega_2x)\gamma+(\omega_2z-\omega_3y)\alpha+(\omega_3x-\omega_1z)\beta.$$

Whence it follows that the velocity of the point (xyz) is equivalent to the three component velocities $\omega_2z-\omega_3y$ parallel to x, $\omega_3x-\omega_1z$ parallel to y, and $\omega_1y-\omega_2x$ parallel to z.

(102.) The theory of the composition and resolution of angular velocities is com-

pletely expressed by (4.); for, if it be required to find the effect of the superposition of two angular velocities represented by ω and ω', we find by (4.) that the velocity produced by ω in any point (u) is $D\omega.u$, and that produced by ω' is $D\omega'.u$. The actual velocity of (u) will be the resultant of these, that is,

$$D\omega.u+D\omega'.u, \text{ or } D(\omega+\omega').u.$$

Now by (4.) $D(\omega+\omega').u$ is the effect of an angular velocity represented by $\omega+\omega'$. Hence it follows that the two angular velocities ω and ω' superposed produce the same effect as the angular velocity $\omega+\omega'$; and $\omega+\omega'$ is the third side of the triangle formed of the two lines ω and ω'.

(103.) The equation of motion of a rigid body acted on by any forces (about its centre of gravity) is easily obtained as follows.

Let U denote the accelerating force in action on m, *i. e.* at the point (u); then, by Sect. III., we have

$$\Sigma mu.\frac{d^2u}{dt^2}=\Sigma mu.U,$$

which is the same thing (see art. 86) as

$$\frac{d}{dt}\left(\Sigma mu.\frac{du}{dt}\right)=\Sigma mu.U. \qquad (7.)$$

Now, putting for u its value (1.), and supposing that α, β, γ are the *principal axes* of the body, we find

$$\Sigma mu.\frac{du}{dt}=\left(\alpha.\frac{d\alpha}{dt}\right)\Sigma mx^2+\left(\beta.\frac{d\beta}{dt}\right)\Sigma my^2+\left(D\gamma.\frac{d\gamma}{dt}\right)\Sigma mz^2.$$

But, by (4.) and (5.),

$$\frac{d\alpha}{dt}=D\omega.\alpha=-\omega_2\gamma+\omega_3\beta;$$

$$\therefore \quad D\alpha.\frac{d\alpha}{dt}=\omega_2\beta+\omega_3\gamma^*.$$

Similarly, $$D\beta.\frac{d\beta}{dt}=\omega_3\gamma+\omega_1\alpha,$$

and $$D\gamma.\frac{d\gamma}{dt}=\omega_1\alpha+\omega_2\beta;$$

whence, introducing D, we find

$$\Sigma mDu.\frac{du}{dt}=(\omega_2\beta+\omega_3\gamma)\Sigma mx^2+(\omega_3\gamma+\omega_1\alpha)\Sigma my^2+(\omega_1\alpha+\omega_2\beta)\Sigma mz^2=A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma,$$

where A, B and C denote respectively $\Sigma m(y^2+z^2)$, $\Sigma m(z^2+x^2)$, $\Sigma m(x^2+y^2)$.

Thus (7.) becomes

$$\frac{d}{dt}(A\omega_1\alpha+B\omega_2\beta+C\omega_3\gamma)=\Sigma mDu.U. \qquad (8.)$$

* Or thus: $D\alpha.\frac{d\alpha}{dt}=D\alpha.(D\omega.\alpha)=-(D\alpha.)^2\omega=\omega-(\omega\times\alpha)\alpha=\omega_1\alpha+\omega_2\beta.$

This is the general equation of motion of a rigid body about a fixed point. It gives the three well-known equations immediately by equating the coefficients of α, β, γ*. But the equation (8.) as it stands is more available for the solution of problems, and furnishes results much more simply, than the three equations alluded to. Along with (8.) we must employ the equation

$$\frac{du}{dt}=\mathrm{D}\omega . u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

And these two are completely equivalent to the six equations commonly employed.

(104.) The line represented by the symbol $\mathrm{A}\omega_1\alpha+\mathrm{B}\omega_2\beta+\mathrm{C}\omega_3\gamma$ has a remarkable relation to the instantaneous axis $\omega_1\alpha+\omega_2\beta+\omega_3\gamma$, which may be thus interpreted. Suppose the rigid body to undergo a *distortion* or *unequal expansion* of such a nature, that all lines *in it* parallel to α become A times longer than before, all lines parallel to β, B times longer, and all lines parallel to γ, C times longer. The effect of this will be to convert the unit α into $\mathrm{A}\alpha$, β into $\mathrm{B}\beta$, γ into $\mathrm{C}\gamma$; and thus the line $\omega_1\alpha+\omega_2\beta+\omega_3\gamma$ will be converted into $\mathrm{A}\omega_1\alpha+\mathrm{B}\omega_2\beta+\mathrm{C}\omega_3\gamma$. This latter line, therefore, I may call the *Distorted Instantaneous Axis.*

(105.) The *distortion* here alluded to is one of great importance to be noted, because it indicates an operation which has immediate connection with many remarkable physical phenomena as well as with various theories in Solid Geometry. As regards its geometrical meaning, if we conceive the rigid body to be a solid composed of spherical shells having a common centre at the origin, each shell will be converted into an ellipsoid by the distortion. The sphere whose radius is unity will be changed into an ellipsoid whose axes are $\mathrm{A}\alpha$, $\mathrm{B}\beta$, $\mathrm{C}\gamma$; and the axes of the other ellipsoids will be parallel and proportional to these.

The line represented by the symbol

$$\mathrm{A}\alpha+\mathrm{B}\beta+\mathrm{C}\gamma$$

is an important determining element. If we assume ω' to denote what ω becomes in consequence of the distortion, it may be easily seen that ω' is a *distributive function* of ω and $\mathrm{A}\alpha+\mathrm{B}\beta+\mathrm{C}\gamma$; and from this fact a number of curious symbolical relations may be deduced. But I must not dwell upon this subject of distortion now further than my immediate purpose requires.

(106.) Using ω' for brevity to denote the *distorted instantaneous axis,*

$$\mathrm{A}\omega_1\alpha+\mathrm{B}\omega_2\beta+\mathrm{C}\omega_3\gamma,$$

I may observe that the equation (8.), that is,

$$\frac{d\omega'}{dt}=\Sigma m\mathrm{D}u.\mathrm{U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

* Observing that $\frac{d\alpha}{dt}=\mathrm{D}\omega.\alpha$, $\frac{d\beta}{dt}=\mathrm{D}\omega.\beta$, $\frac{d\gamma}{dt}=\mathrm{D}\omega.\gamma$, we find the coefficient of γ in the first member to be

$$\mathrm{C}\frac{d\omega_3}{dt}+(\mathrm{B}-\mathrm{A})\omega_1\omega_2;$$

and putting $\mathrm{U}=\mathrm{X}\alpha+\mathrm{Y}\beta+\mathrm{Z}\gamma$, we find, in the second member,

$$\Sigma m(x\mathrm{Y}-y\mathrm{X}).$$

gives the velocity of the point (ω') *in space*, the differential letter, d, denoting *absolute* change of position. But it is often important to determine the motion of this point *relatively to the rigid body*; and this may be done as follows:—

Let us assume δ to denote *relative* change of position with reference to the rigid body; then it is evident that

$$\frac{d\omega'}{dt}=\frac{\delta\omega'}{dt}+\mathrm{D}\omega.\omega'; \quad \ldots \quad (11.)$$

for the velocity of the point (ω') is the resultant of two velocities, namely, that relative to the body, and that arising from the motion of the body; the former is $\frac{\delta\omega'}{dt}$, and the latter, by (9.), is $\mathrm{D}\omega.\omega'$.

Hence, and by (10.), we find

$$\frac{\delta\omega'}{dt}=\Sigma m\mathrm{D}u.\mathrm{U}-\mathrm{D}\omega.\omega'. \quad \ldots \quad (12.)$$

This equation gives the velocity of the point (ω') *relatively to the rigid body*.

Now it is clear that if we can solve (10.) and (12.) the motion of the rigid body is determined; for we shall then know (by (12.)) how the line ω' moves *in the rigid body*, and by (10.) how ω' moves *in space*; and thus, by the intervention of ω', we shall obtain the motion of the rigid body in space.

(107.) As an example of this I shall take the case of the earth attracted by the sun, and point out briefly how (10.) and (12.) determine the motion of the polar axis. In this case $\mathrm{A}=\mathrm{B}$, and $\mathrm{C}=(1+\lambda)\mathrm{A}$, when λ is a small number: also the instantaneous axis ω very nearly coincides in direction with the polar axis γ. Hence, and by art. 89, we have

$$\omega'=\mathrm{A}(\omega+\lambda\omega_{3}\gamma)$$

$$\Sigma m\mathrm{D}u.\mathrm{U}=\mathrm{A}\frac{3m'}{r'^5}\mathrm{D}u'.(u'+\lambda z'\gamma).$$

Here u' denotes (symbolically) the sun's distance, r' is the magnitude of u', m' the sun's mass.

Thus, observing that $\mathrm{D}u'.u'$ and $\mathrm{D}\omega.\omega$ are zero, (10.) and (12.) become

$$\frac{d\omega}{dt}+\lambda\frac{d(\omega_3\gamma)}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma$$

$$\frac{\delta\omega}{dt}+\lambda\frac{\delta(\omega_3\gamma)}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma-\lambda\omega_{3}\mathrm{D}\omega'.\gamma.$$

In the terms multiplied by λ we may approximate on the supposition that γ is fixed and $\omega=n\gamma$, where n is the earth's angular velocity about its polar axis. This reduces the two equations to

$$\frac{d\omega}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma. \quad \ldots \quad (13.)$$

$$\frac{\delta\omega}{dt}=\lambda\frac{3m'}{r'^5}z'\mathrm{D}u'.\gamma. \quad \ldots \quad (14.)$$

Now let us take new directions $(\alpha', \beta', \gamma')$ so, that γ' shall coincide for a moment with the polar axis γ, while the plane $(\alpha'\gamma)$ contains the sun's distance u'; then

$$u'=x'\alpha'+z'\gamma$$
$$Du'.\gamma=-x'\beta';$$

wherefore (14.) becomes

$$\frac{\delta\omega}{dt}=\left(-\lambda\frac{3m'}{r'^5}x'z'\right)\beta'.$$

The coefficient of β' here, being multiplied by λ, may be regarded as invariable during one day, inasmuch as x' and z' take a year to go through their values: also, since δ implies that the earth is considered as fixed, the sun, and therefore the direction β', must be supposed to revolve about γ from east to west, through 360° in the day. The velocity $\frac{\delta\omega}{dt}$ therefore is constant in magnitude but changes its direction (which is always perpendicular to γ) uniformly through 360° in the day. It appears therefore that the point (ω) describes a daily circle, and therefore the line ω describes a daily cone about γ. From this it follows (observing that ω and γ make a very small angle with each other), that the mean daily angular motion in space of γ and that of the direction of ω (manifestly $\frac{\omega}{n}$ very nearly) are identical. Wherefore

$$\frac{d\gamma}{dt}=\frac{1}{n}\frac{d\omega}{dt}=\frac{\lambda}{n}\frac{3m'}{r'^5}z'Du'.\gamma, \text{ by (13.)};$$

or, since

$$z'=u'\times\gamma,$$
$$\frac{d\gamma}{dt}=\frac{\lambda}{n}\frac{3m'}{r'^5}(u'\times\gamma)Du'.\gamma. \quad \text{(15.)}$$

If now we assume γ' to be at right angles to the plane of the ecliptic, and α' to point towards the first point of Aries, we have

$$u'=r'(\alpha'\cos n't+\beta'\sin n't),$$

where $n't$ is the sun's longitude, n'^2 being $\frac{m'}{r'^3}$.

Wherefore, observing that γ is at right angles to α', we find

$$\frac{d\gamma}{dt}=\frac{3n'^2\lambda}{n}(\gamma\times\beta')\sin n't\,(\cos n'tD\alpha'.\gamma+\sin n'tD\beta'.\gamma).$$

If we integrate this between the limits 0 and $\frac{2\pi}{n'}$, we find the *annual variation of* γ, which therefore is

$$\frac{3n'\lambda\pi}{n}(\gamma\times\beta')D\beta'.\gamma.$$

This represents in magnitude and direction the actual space described by the point (γ) in one year, *i. e.* the angular motion of the pole, or the precession. If ϖ denote the obliquity of the ecliptic,

$$\gamma=\gamma'\cos\varpi+\beta'\sin\varpi,$$

and
$$\therefore \gamma\times\beta' = \sin\varpi$$
$$D\beta'.\gamma = \alpha'\cos\varpi.$$

Wherefore the annual precession of the pole is

$$\left(\frac{3n'\lambda\pi}{n}\cos\varpi\sin\varpi\right)\alpha',$$

α' indicating that its direction is perpendicular to the solstitial colure, and retrograde as regards the sun's motion.

I have gone through this example because in every step it shows the use of the symbolic forms.

VI. Application of the Symbolic Forms to Physical Optics.

(108.) The use of the symbolic forms $u.v$ and $u\times v$ in Physical Optics is very remarkable; but as this paper is already so long, I can only just allude to the subject.

In the Transactions of the Cambridge Philosophical Society I have shown (in a paper read March 17, 1847), that if v denote the displacement at the point (u) of an uncrystallized medium, where

$$u = x\alpha + y\beta + z\gamma$$
$$v = \xi\alpha + \eta\beta + \zeta\gamma,$$

and if A and B be two constants (namely, the coefficients of direct and transverse elasticity); then

$$\frac{d^2v}{dt^2} = B\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)v + (A-B)\left(\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz}\right)\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right).$$

And this result I obtained by merely considering the *disarrangement* of the medium about the point u, without any assumption respecting the constitution of the medium, except that it possessed direct and transverse elasticity.

Now if we employ the letter Ω to denote the operation

$$\alpha\frac{d}{dx}+\beta\frac{d}{dy}+\gamma\frac{d}{dz},$$

the above equation, by the aid of the symbolic forms, immediately assumes the simple form

$$\frac{d^2v}{dt^2} = A\Omega(\Omega\times v) - B(D\Omega.)^2v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

The symbol Ω has a very important signification when written before any quantity, U, which is a function of x, y and z: for the *direction* of the line ΩU is that direction *perpendicular to which there is no variation of* U; while the magnitude of ΩU is the *rate of variation* of U *in that direction, i. e.* as we pass from point to point of the medium in the direction of ΩU.

Again, $\Omega\times v$ denotes the *rarefaction* of the medium, at the point (u), resulting from

the displacements represented by v; while $\Omega.v$ denotes, *in magnitude and plane*, the *lateral disarrangement* of the medium.

It is clear, therefore, that the symbolic forms $u.v$ and $u\times v$ must be of great use in Physical Optics; indeed the facility they give of following out investigations respecting undulatory movements is so great, that the whole subject of reflexion and refraction, in crystallized as well as in uncrystallized media, and the mathematical explanations of the phenomena connected with polarization, double refraction, &c., may be reduced to a state of simplicity which could hardly be expected in such a difficult subject.

(109.) In the paper above alluded to, I obtained also the equation of vibratory motion generally, for any crystallized medium, without any of those assumptions which mathematicians have found it necessary to make in order to render the investigation manageable; especially, without assuming the vibrations of a plane polarized ray to be *in* the plane of polarization, which appears to me to be a highly objectionable assumption. By the aid of the symbolic forms, the general equation of vibratory motion, where the transmission of transverse vibrations is possible, is thus expressed:

$$\frac{d^2v}{dt^2}=\left(A_1\alpha\frac{d}{dx}+A_2\beta\frac{d}{dy}+A_3\gamma\frac{d}{dz}\right)(\Omega\times v)$$
$$+D\Omega.\left\{\left(B_2\frac{d\eta}{dz}-B'_3\frac{d\zeta}{dy}\right)\alpha+\left(B_3\frac{d\zeta}{dx}-B'_1\frac{d\xi}{dz}\right)\beta+\left(B_1\frac{d\xi}{dy}-B'_2\frac{d\eta}{dx}\right)\gamma\right\}. \quad . \quad . \qquad (2.)$$

Here A_1, A_2, A_3 are coefficients of *direct elasticity*, corresponding to A in equation (1.); and B_1, B_2, B_3, &c. are six coefficients of *transverse elasticity*, corresponding to B in (1).

Fresnel's hypothesis, that the vibrations of a plane polarized ray are *perpendicular* to the plane, makes

$$B_1=B'_1, \qquad B_2=B'_2, \qquad B_3=B'_3,$$

while the hypothesis, that the vibrations are *in* the plane of polarization, makes

$$B_1=B_2, \qquad B_2=B'_3, \qquad B_3=B_1.$$

On the former hypothesis the equation becomes

$$\frac{d^2v}{dt^2}=\left(A_1\alpha\frac{d}{dx}+A_2\beta\frac{d}{dy}+A_3\gamma\frac{d}{dz}\right)(\Omega\times v)-(D\Omega.)^2(B_1\xi\alpha+B_2\eta\beta+B_3\zeta\gamma); \quad . \quad . \quad . \qquad (3.)$$

while, on the latter, it becomes

$$\frac{d^2v}{dt^2}=\left(A_1\alpha\frac{d}{dx}+A_2\beta\frac{d}{dy}+A_3\gamma\frac{d}{dz}\right)(\Omega\times v)$$
$$-(D\Omega.)\left\{B_1\left(\frac{d\eta}{dz}-\frac{d\zeta}{dy}\right)\alpha+B_2\left(\frac{d\zeta}{dx}-\frac{d\xi}{dz}\right)\beta+B_3\left(\frac{d\xi}{dy}-\frac{d\eta}{dx}\right)\gamma\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4.)$$

For transverse vibrations the rarefaction $(\Omega\times v)$ (see above) is zero, which further simplifies (3.) and (4.). By equating the coefficients of α, β, γ in (4.), thus simplified, we obtain MacCullagh's three equations. The equation (3.) coincides in every way with Fresnel's theory.

The conception of *distortion* alluded to in Sect. V. applies to (3.) in a very remarkable manner; for if we put

$$A_1\alpha\frac{d}{dx}+B_1\beta\frac{d}{dy}+C_1\gamma\frac{d}{dz}=\Omega',$$

$$B_1\xi\alpha+B_2\eta\beta+B_3\gamma\zeta=v',$$

then Ω' is Ω *distorted* by the line $A_1\alpha+A_2\beta+A_3\gamma$; and v' is v *distorted by* the line $B_1\alpha+B_2\beta+B_3\gamma$. In using the expression "distorted by," I anticipate a signification which I hope to explain in a future paper; but the fact is, Ω' is a *distributive function* of Ω and $A_1\alpha+A_2\beta+A_3\gamma$ (see art. 105), and therefore, symbolically, Ω' is Ω *multiplied by* $A_1\alpha+A_2\beta+A_3\gamma$.

Now the equation (3.) becomes

$$\frac{d^2v}{dt^2}=\Omega'(\Omega.v)-(D\Omega.)^2v',$$

or, for transverse vibrations, simply

$$\frac{d^2v}{dt^2}=-(D\Omega.)^2v'.$$

But I must reserve the consideration of this remarkable equation, merely remarking here that it shows very clearly how the force brought into play by the disarrangement of a medium resulting from transverse vibrations is altered by the crystallization. In the uncrystallized medium the force is, by (1.),

$$-(D\Omega.)^2(B\xi\alpha+B\eta\beta+B\zeta\gamma),$$

and in the crystallized medium it is

$$-(D\Omega.)^2(B_1\xi\alpha+B_2\eta\beta+B_3\zeta\gamma).$$

In one case it is found by performing the operation $-(D\Omega.)^2$ on v, in the other by performing the same operation on v *distorted* as by the threefold expansion of the medium, consisting of a drawing out in the direction α in the proportion of B_1 to B, in the direction β, of B_2 to B, and in the direction γ, of B_3 to B.

Upper Norwood, Surrey,
April 19, 1852.

PHILOSOPHICAL TRANSACTIONS.

XIII. *On the Anatomy of* Doris. *By* ALBANY HANCOCK *and* DENNIS EMBLETON, *M.D., F.R.C.S.E., Lecturer on Anatomy and Physiology in the Newcastle-upon-Tyne College of Medicine in connection with the University of Durham. Communicated by Prof.* EDWARD FORBES, *F.R.S.*

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WHILST engaged in arranging the materials for their Monograph on the Nudibranchiate Mollusks now in course of publication by the Ray Society, it became evident to Messrs. ALDER and HANCOCK, that not only the external characters, but also the internal structure, of this interesting group of animals, should be fully investigated. Since then the writers of the following paper have taken up the anatomical part of the subject, and have already been enabled to publish in the 'Annals of Natural History' an account of the anatomy of one genus, *Eolis*; another genus, *Doris*, forms the subject of the present communication.

On entering on the present investigation, we were naturally anxious to avail ourselves of the stock of knowledge already collected by our predecessors in this walk of science.

CUVIER, the great master of Comparative Anatomy, had dissected and studied the genus *Doris*, and had left a considerable amount of information for those who might follow in his steps: indeed his memoir contains a more complete account of the anatomy of *Doris* than any yet published. Within the last few years Herr HEINRICH MECKEL has described the generative organs in MÜLLER'S Archiv, 1844, and more recently Messrs. MILNE-EDWARDS and BLANCHARD have given accounts of the circulatory and nervous systems of *Doris*. We will not here dilate upon the works of these authors, but shall refer to them in the course of our description of the various organs.

Since the reading of our paper on the subject before us at the Edinburgh Meeting of the British Association in 1850, we have gone more fully into the details of the anatomy of *Doris* than it had been in our power up to that time to do, and having verified our observations in many particulars, and thrown additional light on our

results by comparative dissections, we feel more confidence in placing on record an extended account of our researches.

Before we commence the anatomical description, we may premise that for examination, we have selected *D. tuberculata*, not only as the species most highly typical of the genus, but as that which, on our coast, is the most abundant and the largest, and therefore in all respects most convenient for our purpose. We have not however confined ourselves to this single species, but have examined several others, with a view of obtaining a more accurate knowledge of the anatomy of the genus. Of these latter we may mention *D. pilosa* and *D. bilamellata*, as representing very fairly, with *D. tuberculata*, the three great divisions which have been characterized in the Monograph on the British Nudibranchiata, now in course of publication by the Ray Society.

D. tuberculata is 2 or 3 inches long, pretty regularly oval, rather depressed, arched above, with the mantle extending over the whole back, concealing the head and foot, and covered with numerous obtuse, spiculose tubercles of various sizes; the dorsal tentacles are retractile within cavities, of a clavate form, with the upper portion laminated: the mouth projects a little in front between the foot and mantle; on each side is a small depressed, obtuse, angular tentacle: there are nine branchial plumes united at the base, and retractile within a cavity; they form a circle on the median line of the back near the posterior extremity, enclosing the anal and renal orifices: the foot is broad, truncated in front, and rounded behind: the genital orifice is situated towards the anterior extremity, on the right side, between the mantle and foot.

D. pilosa differs from *D. tuberculata* chiefly in having the back more elevated, and in being clothed with soft conical papillæ; the oral tentacles too are fused into a veil, which is at either side expanded into a leaf-like process, and the branchial plumes are non-retractile, being merely contractile. This species is about 1 inch long.

D. bilamellata is not more than three-quarters of an inch in length, and is considerably depressed, having the mantle covered with large, spiculose tubercles: branchiæ of about twenty, simply pinnate, non-retractile plumes, set separately in an open circle: head furnished with a semicircular veil. In other respects this does not differ materially from the above species.

Digestive System.—On removing the skin of the back, the viscera, including the buccal apparatus, are seen bound down and completely covered by a transparent fibrous membrane, which is fixed on each side to the inner surface of the skin just above the track of its great lateral venous canals, and to the circumference of the pericardium. For want of a better name we call this membrane the peritonæum. It has been removed to show the viscera*. The mouth opens on the inferior surface of the body between the cloak and the foot; it is surrounded by a protuberant, fleshy, outer lip divided below on the median line. The channel to the buccal cavity is very

* Plate XI. fig. 4, and Plate XII. fig. 1.

short, and provided with a thin projecting inner lip. Immediately behind this is the buccal mass, which is strong and muscular, of considerable size, somewhat ovate, with the small end forwards; it is continuous with the channel of the mouth, and the œsophagus leads away from its upper and posterior part.

There are two sets of retractor muscles for the buccal apparatus*; one composed of radiating bands, three or four on each side, inserted into the channel to the mouth in front, and attached to the skin at the sides of the body behind; the other consists of only two large strong straps, one on each side, having their insertions at the sides of the posterior part of the buccal mass by means of a band which there encircles that body, and running a good way backwards to be fixed to the fleshy foot. These two sets of muscles acting together will pull backwards the buccal mass, and mouth channel, with the external orifice; if they act separately, of course they will retract the parts into which they are respectively inserted.

A thin layer of numerons delicate parallel muscular bands coats the lateral parts of the buccal mass. These bands are attached in front to the channel of the month, mingling with other fibres of that part, and behind to the circular band that receives the attachment of the buccal retractors. When the channel of the mouth is fixed, these layers of mnscle will advance the buccal mass towards the external orifice.

The remaining muscles, which are proper to and form nearly the whole of the buccal mass, are for the movements of the tongue and the buccal lip, and enclose, immediately, the cavity of the mouth.

On opening the buccal mass† from above, by cutting through the buccal lip and a strong layer of transverse bundles forming the posterior part of the roof of the buccal cavity, and therefore assisting in prehension and deglutition, we see projecting upwards and forwards from the floor of the cavity a broad conical mass surmounted by rows of teeth, and placed directly in front of the opening of the œsophagus; this mass is that of the lingual muscles supporting and moving the spiny tongue. In advance of this mass is a powerful sphincter or circular belt of muscle commanding the entrance to the mouth; this we propose to call the third or buccal lip, for reasons to be afterwards given.

The tongue consists of a tubular, dentigerous membrane, the upper or anterior portion of which is partly enclosed within, and partly expanded npon, the conical mass of lingual muscles; the lower or posterior portion is continued downwards and backwards, through the centre of the mass, into a delicate pouch which projects beyond the posterior and under part of the buccal organ. This spiny tongue‡ is easily detached from the muscular mass to which it is connected, and can be drawn out of the tube in which it is lodged; and it can then be observed that the part of the muscular mass§ which has been covered by the tongue is composed of numerous parallel transverse plates or flakes corresponding apparently in number and direction to the rows of teeth. These flakes are bound together by a thin layer of fibres which crosses

* Plate XIII. fig. 1. † Plate XIII. fig. 2. ‡ Plate XIII. fig. 3. § Plate XIII. figs. 4 and 5.

them at right angles. The transversely placed muscular plates above mentioned are most highly developed around the posterior lateral parts of the central cavity, and towards the front of the cavity show as a layer on each side, gradually decreasing in thickness, and in front are continuous with each other, completing at that part the cavity containing the dentigerous membrane. This cavity is a laterally compressed tube, somewhat elongated from above downwards, and has projecting from the whole length of its posterior wall, almost as far as the front of the cavity, a fleshy lamina, in transverse section, wedge-shaped above, fusiform below. This, which we shall call the cuneiform lamina, is not only connected with the posterior wall of the cavity, but also with the bottom of it,—the pouch at the lower and back part of the buccal mass; its anterior border and upper end are both free. It divides the cavity into two nearly equal lateral parts. A further notice of it will find place in the description of the teeth and their development.

If the upper portion of the above muscular conical mass be cut away, we find exposed on each side, a flattened dense substance* placed vertically, and being made up for the most part of indistinct transversely fibrous texture, and partially enclosed by the transverse muscular flakes before named. In front, the inner opposed surfaces only of these bodies or nuclei are clothed by the muscular flakes, which there form a thin layer, and their fibres are short, extending from the lower to the upper border of the nuclei; further back the muscular coating is thicker and the fibres longer; these being continued gradually more and more over the upper border of the nuclei, and then by degrees down the external surface, till at length they enclose the posterior end of the nucleus altogether, which is then free as it were in a muscular cavity, and the muscular flakes being then attached to the lower border only of the nucleus, but by one end to the inner and the other to the outer edge, play round the upper border somewhat like a belt over a pulley.

By the action of this apparatus the tongue is not only partially everted from its tube, bringing numerous rows of spiny teeth to bear upon the food, but is retracted again into its previous position.

Something similar to this perfect arrangement we have elsewhere described in *Eolis*, only there the lingual mass is single, whereas in *Doris* it is divided into two lateral halves.

On the earliest approach of decomposition the dentigerous membrane† can be easily removed; it is then seen as an incomplete tube, being quite open at the bottom, behind, and at the part corresponding to the cuneiform lamina, but retaining its original form. On looking into this tube, a fine membrane, continuous with the dentigerous, is seen stretched transversely across; and if the tube be replaced in its cavity, that membrane is found to fit upon, and clothe the top of, the cuneiform lamina. Thus we find that the tube of the tongue is divided into two unequal parts by this membranous septum, which occurs about the end of the upper or anterior third of the tube. The lower two-thirds of the tube thus constitute a large follicle, in which

* Plate XIII. fig. 6. † Plate XIII. fig. 3.

are formed and developed the spiny teeth in regular rank and file; the upper third contains the mature teeth, which can all more or less readily be used in prehension.

Development of Teeth.—If we make a transverse section of that lower part of the tube* which projects obliquely behind from the buccal mass, and examine it, we find in the centre, the cuneiform lamina having a somewhat pyriform shape, the base forwards, the apex backwards and attached to the muscular envelope of the pouch. Except at this point the lamina is free and lies in a cavity bounded by two parallel membranes, between which are developed the rows of teeth; the outer of these is the dentigerous layer, which rests upon the mucous membrane lining the muscular envelope of the tube. If we make another section† lower down, and obliquely through the end of the pouch, we find the cuneiform lamina fixed at both ends, the free edge has ceased, and the lamina is found attached along the median line of the fundus of the tube. Along each side of this attachment is found an enlargement of the mucous membrane, which must be of glandular nature, as from it seem to originate the two parallel membranes mentioned in noticing the former section just above.

Between these membranes, as already stated, the teeth are formed. These organs can be observed at a very early period of their growth. They may be discerned upon the newly-secreted edge of the outer or dentigerous layer as delicate rows of elongated and attenuated, soft and colourless cells. At the distance of a row or two further on they assume the exact form and size of the mature teeth, still, however, retaining their soft cell-like character. The next rows seem orange-coloured, and have put on some of the hardness and of the other characters of perfect teeth, which are however somewhat paler. The teeth examined just below the transverse membrane that closes in the teeth follicle above, are found to have attained all that extreme hardness and temper which characterize them when in use, and possess the most perfect pellucidity and polish.

The space between the two parallel membranes is packed with a soft matter which is derived from the inner layer, and which, when placed in spirit, becomes flocculent; in this matter the teeth are imbedded. These two membranes may not inaptly be compared to the pulp and enamel membranes of the teeth of the Vertebrata, though the denticles themselves, as far as we have been able to observe, are quite homogeneous in structure. We are inclined to think the denticles are siliceous in composition, as they resist the action of the stronger mineral acids, as do those of *Eolis*, which, although they possess the same hardness, transparency and polish, are in some measure affected by hydrofluoric acid.

Those teeth which are placed upon the top of the lingual mass, and are actually engaged in the act of prehension, become gradually blunted, broken or wholly detached, and that the more rapidly the more they lie away from the median line: hence the whole apparatus is worn to a point in front and on the median line. This continual wear and tear of the dental rows necessitates a fresh supply of them, and this is constantly being provided by the secretion of dentigerous membrane at the

* Plate XIV. figs. 1 and 3.

† Plate XIV. fig. 2.

bottom of the follicle above described. The advance upwards of the teeth is assisted, in all probability, by the secretion of some matter supplementary to the dentigerous membrane by the surface of the tube in which it is contained. This surface is formed by a fine membrane, which in the follicle below is visible enough, but in the more muscular part of the tube above is not so easily demonstrated. This membrane, which is continuously spread over the whole of the buccal cavity, is analogous to the mucous membrane of the mouth of higher animals; the tube formed by it, and the pouch which terminates it, compose a follicle, the secretion of which is the dentigerous membrane beset with ranks of denticles, the analogues of epithelium and teeth. The inner of the two parallel membranes, and the flocculent matter in which the teeth are at first imbedded, gradually become detached and fall off, leaving the teeth bare as they reach their field of action.

The mode of growth of the spiny tongue of *Doris* is thus evidently quite analogous to the growth and advance of the teeth of Rays, Sharks, &c., or of the hoofs or nails of Mammalia.

In *D. tuberculata* the number of rows of denticles is forty-three or forty-four, eighteen of which are situated above the transverse septum or top of the follicle, and each contains 140 denticles; in this species therefore there are 6160 little teeth. These are not all of the same size, but diminish from near the ends of each row towards the centre, where the row is slightly interrupted, there being no central tooth*. The five or six external denticles also rapidly decrease in size towards the end of the row. Each denticle† is expanded broadly at the base, and is well arched backwards and sharp pointed, somewhat resembling the spines of the dog-rose.

In *D. verrucosa* and *D. tuberculata*, Verany, the buccal organ and tongue are constructed after the above model.

In *D. Johnstoni* the buccal and lingual apparatus follow the same arrangement, the rows of denticles being twenty-four in number, of which eleven are used in prehension. There are about fifty denticles to each row; the five outermost at the ends of the rows are much attenuated, being nearly linear‡.

In *D. coccinea* the above parts are after the same type; the external attenuated teeth of each row being much more numerous than in the last-named species.

In *D. repanda* we have the buccal mass and the tongue resembling those of *D. tuberculata*. There are sixty-eight rows of toothlets, and thirty-six toothlets in each row, but they are of more elaborate character, having their edges serrated: in each row there is a centre toothlet bearing four cusps§. Three or four of the little teeth on each side of the centre one are short and robust, those further out are much more produced.

In addition to the above lingual organ, we find superadded in this species a collar of crowded minute bifid spines arranged on the anterior surface of the buccal lip; to this curious organ we shall again shortly refer.

In *D. bilamellata* we have a very interesting modification of the buccal apparatus.

* Plate XII. fig. 8. † Plate XII. fig. 9. ‡ Plate XII. fig. 10. § Plate XII. figs. 11, 12 and 13.

It is small in proportion, and in general form has resemblance to that of *D. tuberculata*; but in addition there is the singular appendage of a lentil-shaped organ, having all the characters of a gizzard*. This is attached by a short tubular pedicle to the upper part of the buccal mass, a little in front of the œsophagus. This buccal gizzard is not much inferior in size to the buccal mass itself: its walls are thick, its external surface on each side presents numerous muscular bundles radiating from a centre, and its peripheral margin is bounded by a strong belt of muscle. Its inner surface is smooth, but tough, and its cavity communicates with the mouth through the pedicle, which is itself muscular. There is nothing to discredit the idea that we have here an organ added to the buccal mass for the more perfect trituration of the food, and which may probably at the same time act in promoting deglutition, for there is along the roof of the mouth apparently a groove leading from the pedicle in the direction of the œsophagus.

In this species there is a modification of the tongue: it is here long and narrow; there are twenty-six or twenty-eight rows of toothlets, two only in each row; one toothlet is placed on each side of the median line. About eleven rows are engaged in prehension. The toothlets are long, stout, arched spines. There is likewise in this species a prehensile collar, which, instead of being denticulated, is roughened with transverse irregular rows of short minute papillæ. Moreover, there exists a minute pair of rudimentary jaws; but as both these parts exist also in *D. pilosa*, the description given under that head may suffice for *D. bilamellata* also.

D. aspera† and *D. depressa* have likewise the narrow tongue and buccal gizzard.

In *D. pilosa* the buccal mass‡ differs a good deal from the above forms. It is large, and the channel of the mouth opens into its under surface instead of in front, as in the other species. The difference of form is owing mainly to the apparent incorporation of a gizzard-like organ with the anterior part of the buccal mass. That organ has the radiating muscular fibres, and the peripheral belt observable in the gizzard of *D. bilamellata*. It is placed immediately in front of the œsophagus, and its connexion with deglutition is more evident. The tongue§ is of the same type as that of *D. bilamellata*, but the teeth have their margins denticulated.

The prehensile collar‖ is present, and is different in form from that of *D. repanda*. It is divided into two lateral halves; its spines, which are directed inwards, are very densely crowded, being for the most part like bifid stumps. The two parts of the collar are broad below, where they are somewhat separated, and taper to fine points above. We have here, in addition to the prehensile collar, a pair of minute rudimentary triangular horny plates or jaws¶, which are situated between and immediately behind the lower separated ends of the collar. They are united in front for more than two-thirds of their length, and are imbedded in the lower part of the buccal lip, having only their free points exposed. Attached to the posterior border of each of these free parts of the plates is a thin transparent tough membrane, which

* Plate XII. fig. 5. † Plate XII. fig. 14. ‡ Plate XII. fig. 4.
§ Plate XII. figs. 6 and 7. ‖ Plate XIII. figs. 7, 8, 9, 10 and 11. ¶ Plate XIII. fig. 12.

runs up the inner surface of the buccal lip, tapering away to nothing near the top. One edge of this membrane is attached to the buccal lip along the inner margin of the collar, the other is free, projecting into the aperture of the mouth. These rudimentary jaws and the prehensile collar are doubtless formed from the mucous membrane of the mouth, but we have not seen evidence to prove that they are, together or separately, the homologous parts or part to the horny jaws of *Eolis*, but they seem to be engaged in the same function, that of prehension.

The œsophagus in our typical species* comes from the upper posterior part of the buccal mass, is somewhat dilated at first, and then pretty uniform in diameter, exceeds the stomach in length, is rather delicate in texture, and its interior is longitudinally plicated. It runs straight backwards, and opens freely into the posterior end of the stomach at the under part, and rather at the right side directly in advance of the great hepatic duct.

The Salivary Glands are a pair of long delicate tubes tapering backwards, where they lie against the liver, and open into the mouth on each side at the coming off of the œsophagus.

The Stomach is a large ovate membranous bag lying on the left side of the body, with its posterior end resting in a funnel-shaped depression in the anterior face of the liver. The great duct from the liver† opens into this posterior or cardiac end of the stomach so widely, that it is difficult to determine the line of demarcation between them. This duct, at its union with the stomach, lies above the cardia or entrance of the œsophagus, which is rather to the right; it lies also above the opening of a small duct from the pancreas, which is situated somewhat towards the left side. At the pyloric end, which is anterior in position, the stomach becomes rapidly constricted and converted into the intestine, which appears to come off from the under part, bends upwards to the right side, and then backwards, running along the right side of the stomach between it and the genitalia, inclining gradually towards the right; it then passes under the heart and right side of the pericardium, winds round and under the right side of the posterior end of the liver, and lastly, ascends behind the end of the liver to terminate at a projecting tubular anus in the centre of the branchial circle. The internal surface of the stomach is more or less closely and finely corrugated; the rugæ are most strongly marked at the upper posterior part of the organ in front of the hepatic duct.

The Intestine, which is of smaller diameter than the œsophagus, but pretty uniform in calibre throughout, has its internal surface longitudinally laminated.

The Pancreas‡ is a somewhat elliptical pouch lying below and at the left side of the stomach. Its interior is divided by numerous projecting transverse folds of the lining membrane, strongly resembling valvulæ conniventes, and usually contains a dark brown matter. It opens very freely into the stomach. We have not examined microscopically the muscular coat of the alimentary canal, but on the external surface of the stomach and pancreas, after the aorta has been injected, can readily be

* Plate XII. fig. 1. † Plate XIII. fig. 13. ‡ Plate XVII. fig. 5.

seen a fine network or delicate arborescence of very minute vessels enveloping the organs. The whole alimentary canal, the pancreas and the great hepatic duct are likewise enveloped in the irregular meshes of the sympathetic system of nerves.

Varieties.—In *D. tuberculata*, VERANY, and *D. verrucosa*, CUV., we have the alimentary apparatus formed on the whole after the same type as that in the British *D. tuberculata*, only in *D. verrucosa** we have at the commencement of the œsophagus an extensive membranous pouch or crop projecting forwards, and the stomach is much reduced in size. The salivary glands are folliculated.

In *D. Johnstoni*†, we have extending from the buccal mass to the cleft of the liver, where it receives the hepatic duct, a simple tube dilated at its upper part; this dilatation may be looked upon as a sort of crop or anterior stomach. The intestine comes off upwardly immediately after the junction of the hepatic ducts, and though there is here no well-marked dilatation, we believe from analogy that the anterior part of what is here called intestine may perform the part of stomach; the salivary glands are very slender, almost linear, plain tubes.

In *D. pilosa*‡, at the commencement of the œsophagus, is an extensive membranous pouch; to this succeeds a much-contracted tube, which suddenly dilates into a fusiform sac like the crop or anterior stomach of *D. Johnstoni*. A tube from the opposite end of the sac plunges into the middle of the anterior surface of the liver, which is not cleft as in the previous species. The tube becomes dilated within the liver, and penetrates to about one-third the length of that organ, after which it curves upwards and slightly forwards, and emerges from its upper surface. After this the tube, still dilated, runs a little forward, and is then bent upon itself, at the same time becoming contracted in calibre, and having appended to it a small pancreas, assumes the usual course to the anus. That part of the tube enclosed within the substance of the liver receives three or four large hepatic ducts at the posterior and under part of the curve, and together with its continuation as far as the pancreas, must be regarded as the true stomach. The present species has its salivary glands curved and folliculated at the buccal end.

The alimentary tube of *D. bilamellata*§ is formed on the same plan as the last described; only the crop or anterior stomach differs somewhat in form, and has its walls folliculated. The salivary glands are here reduced to a small granulated body surrounding the commencement of the œsophagus.

In *D. repanda*, *D. coccinea*, *D. aspera*, and *D. depressa*, the alimentary canal appears to be cast in the same mould as the last-mentioned species. The inner surface of the stomach is plicated in all, and in all the intestine is rather wide, and plicated longitudinally throughout.

The liver‖ is the largest organ in the body, and usually fills up more than the posterior half of the visceral cavity. It is of rather a conical form; the base, which is somewhat cleft above, and deeply hollowed out into the shape of a funnel, is

* Plate XII. fig. 3. † Plate XII. fig. 2. ‡ Plate XII. fig. 4. § Plate XII. fig. 5. ‖ Plate XIII. fig. 13.

placed forwards, and has the great ducts* emerging from the apex of the funnel, and immediately joining the posterior end of the stomach which is lodged in the wider part of the concavity. From the cleft in the upper border of the base runs backward along the upper surface and median line of the organ, a considerable groove which lodges the trunk and the ramifications of the renal organ, and contains at its posterior part the trunk of the hepatic vein. The thin stratum of the ovarium lies spread out upon, and closely adherent to, the upper part and sides of the mass of the liver. The external surface of the liver itself is distinctly granular, and of a darkish brown colour, and together with the ovarium is overspread by the ramifications of the aorta which form a network upon these organs. Four or five principal ducts join together in the short wide common channel to the stomach, and if these ducts are traced into the substance of the liver, they are found to divide and subdivide very rapidly and minutely; and we believe, though we have not microscopically examined this, that the extreme branchlets end in the granules seen on the external surface of the organ. Owing to the number, size, and frequent division of the hepatic ducts, the interior of the liver has quite a spongy aspect; and there can be no doubt that, from the great size of the principal ducts, the food during digestion can easily enter them, as was remarked formerly by CUVIER, and as is the case in the Eolididæ.

Varieties.—The liver in *D. verrucosa*† is rounded in front, and less evidently cleft for the reception of the stomach, but is otherwise as in *D. tuberculata.*

In *D. Johnstoni*‡ the organ is deeply cleft, but the right lobe or side of the cleft is much reduced in size, whilst the left is larger in proportion than the corresponding part in *D. tuberculata.*

In *D. bilamellata*§ these modifications are carried to a much higher degree, so that the left lobe appears as a mere rudiment; and in *D. pilosa*‖ that lobe is no longer distinguishable, the liver appearing as if truncated on that side, which is perforated by the œsophagus, as before mentioned.

The texture of the organ in all appears to be the same, though the colour varies somewhat, being greenish or yellowish, or even inclined to orange, and sometimes of a deep purple brown.

Generative organs.—The organs of this system are remarkable for their large size, high development, and complicated arrangement, being very analogous in their complex hermaphrodite character to those which we have elsewhere described in *Eolis*¶. They lie on the right side of the body (with the exception of the ovary, which is spread upon the liver), in front of the hepatic organ, behind the buccal apparatus, and on the right side of the œsophagus and stomach.

There is one common external aperture placed on the right side of the body, about a third of the way down from the head between the mouth and the foot. Imme-

* Plate XII. fig. 1. † Plate XII. fig. 3. ‡ Plate XII. fig. 2.
§ Plate XII. fig. 5. ‖ Plate XII. fig. 4. ¶ Plate XI. figs. 1, 2, 3, 4; and Plate XII. fig. 1.

diately within this aperture is a very short common vestibule, on the inner wall of which are three orifices; an anterior, which admits of the exsertion of the penis; a posterior, the vulva, leading to the female channel; and an upper, the vagina, leading to the androgynous apparatus, and receiving during coition the penis of the accoupled individual.

The organs are male, female, and androgynous.

Male organs.—These consist of, first, an intromittent organ, capable of being protruded from and retracted within the body; and secondly, a testis.

In *D. tuberculata*, in which these organs* are most like those of *Eolis*, the penis lies in front of all the rest. When it is fully retracted within the body, we find a membranous pouch of a conical form attached by its base to the inner surface of the margin of the external opening for the passage of the penis; the pouch receives at its apex the external end of the testis. When laid open, there is found in it, continued from and through the apex, a small tube continuous with the testis; this tube runs down to be attached to the side of the apex of a smaller conical bag within, and about half the length of the other. The interspace between the cones is filled with a filamentous woolly-looking tissue, which fixes the inner cone and its attached tube in their position; the bases of the two cones are continuous with each other at the inner margin of the external orifice. When exsertion takes place the inner cone is everted like the tentacle of a snail, forming the apparent external penis; whilst this is taking place the apex of the outer is drawn after that of the inner cone, by means of the tube by which they are connected; and the filamentous tissue connecting the two cones with a considerable portion of the outer cone, which becomes gradually everted as exsertion proceeds, come at length to be contained within the body and base of the fully exserted penis. The process of exsertion seems to be brought about by the contraction of the walls of the outer cone, in the first instance pressing upon the inner cone through the medium of the filamentous tissue between them, into which tissue some fluid, most likely the blood, may be rapidly introduced during venereal excitement.

The testis is a long, simple, pale flesh-coloured, convoluted tube, the coils loosely bound together with filamentous tissue; the bundle thus formed lies partly on the penis, and partly on the oviduct and anterior margin of the mucus-gland. The walls of the tube are firm, thick and muscular; and the interior, which is of very small calibre, lined with a glandular membrane, the inner surface of which is beset with minute cells. This tube is connected at the outer end with the apex of the cone described as the penis, and at the other it opens into the oviduct at a sudden turn, which that tube makes before entering into the channel of the mucus-gland, not far from the vulva, an arrangement identical with that in *Eolis*.

Female organs.—These are ovarium, oviduct, mucus-gland, and channel leading to vulva.

* Plate XIV. fig. 7.

When fully developed, the *ovarium* is spread over the liver as a thin layer of tubes and follicles. The tubes are ramified from the general oviduct, and end partly in anastomoses, and partly in free extremities, the ramifications being all studded with innumerable sessile follicles. This stratum in the breeding season is very conspicuous, but is scarcely discernible at other times, and extends over the whole liver, with the exception of a small space on the under surface along the median line. The main branches of the oviduct all tend towards the anterior and upper edge of the liver, along which the common trunk courses from left to right. The oviduct before leaving the liver dips somewhat downwards; it then quits that organ as a fine tube, which is at first free, and then becoming attached to the left side of the mucus-gland, is suddenly dilated; it then passes sinuously on as far as the front of the mucus-gland, where it is found somewhat diminished in size; it is then suddenly bent backwards, and again forwards, and at the second bend receives the inner end of the testis as above stated. It then sinks into a fissure in the opake portion of the mucus-gland, and being joined by the duct from the androgynous apparatus, debouches into the female channel.

The mucus-gland is an irregular rounded compressed mass; the left side is flatter than the right, and shows most easily the connexions of the different portions of the generative organs; and here also it can be seen that the gland in question is composed of two parts, one semipellucid, formed of the convolutions of a large tube, the other opake, brownish red, imbedded in the former and made up of the closely compacted folds of a minute tube. There is a large channel in the interior of the gland communicating with the vulva externally,—the female channel; this receives the termination of the convoluted tube of the gland, and also of the oviduct, after it has been joined by the duct of the spermatheca.

The androgynous organs are two spermathecæ and connecting channels; one channel, the vagina, is large, longitudinally plicated within, leads from the common external vestibule, and opens freely into the principal spermatheca; this is a globular sac of a purple brown colour, owing to its contents, and lies between the male and female parts. Just where the vagina opens into the spermatheca a small duct leaves it, which is soon joined by one, still smaller, from the accessory spermatheca, and then opens into the oviduct, where it dips into the opake part of the mucus-gland. The accessory is a much smaller sac than the principal spermatheca; it lies against the vagina, is of lighter colour than the other, and its duct is very short.

Of the reproductive system of *Doris* there are two varieties, the most remarkable of which we shall now proceed to notice. In *D. bilamellata** the testis is considerably shorter and thicker than in *D. tuberculata*; it is suddenly much constricted at its junction with the oviduct, which is also suddenly and strikingly constricted a little way before it reaches the testis, and the whole of the twice bent part and its continuation into the opake portion of the mucus-gland is elongated, and of the

* Plate XIV. fig. 8.

same minute size. The androgynous apparatus is modified in this species in the following way: the vagina and the channel, from the principal spermatheca to the oviduct, are united and communicate with that bag as a single tube; the channel to the oviduct is of considerable calibre, tortuous, and is continued into a blind dilated pouch, from the side of which comes off a short duct leading direct to the oviduct. The cæcum here mentioned may be regarded perhaps as an accessory spermatheca; it, as well as a portion of the dilated tube leading into it, is imbedded in the mucus-gland, which is not the case with the corresponding parts of *D. tuberculata*.

In *D. pilosa* the testis is shorter than in the last species, and somewhat stouter, but otherwise as in *D. tuberculata*. The vagina is very long and wide, and before reaching the spermatheca becomes constricted into a very fine tube, which a little before its entrance into the principal spermatheca, has appended to it a small accessory pouch with a delicate duct; between the junction of this duct and the principal spermatheca, and quite close to the latter, a very fine channel passes off nearly at right angles to join the oviduct at the usual point.

*D. repanda** has the testis reduced to a very short but wide tube, abruptly constricted at each end; at the outer end succeeds a narrow duct of nearly the same length as the testis itself, leading to the penis; this duct may be compared to the vas deferens. At the inner end, the constriction is in close contiguity to the junction of the testis with the bent part of the oviduct. The dilated portion of the oviduct is longer and wider than usual, and is suddenly constricted in front as in *D. bilamellata*; but in *D. repanda*, in which all these parts are remarkably open to observation, this constricted part of the oviduct is only once bent upon itself, where it is joined by the testis; a little further on it receives at another simple bend the channel from the androgynous apparatus, and as a small duct may be seen to enter the mucus-gland. The vagina is short, and soon receives a duct from the principal spermatheca; the tube then bends upon itself, becomes somewhat convoluted, and runs to terminate in the accessory spermatheca; the channel to the oviduct being given off as a very fine duct from the convoluted part, a little way in front of this last receptacle. The spermathecæ are more nearly of a size than in previous examples, and are both globular. The mucus-gland is here formed of the convolutions of a much larger tube than in any of the before mentioned species.

Hitherto we have noticed those species which possess a single loosely convoluted and comparatively short tube, which alone we have considered testis. We have now to call attention to another set of species, which are provided over and above with a dense fusiform mass bent upon itself, composed entirely of a very fine convoluted tube, the coils of which are compactly cemented together, forming a concentrated gland enclosed within a fine capsule.

Of this second set, *D. tuberculata*†, Verany, is here first noticed, as it will lead us naturally to the rest; it has both the loosely convoluted tube and the fusiform gland

* Plate XV. fig. 5.

† Plate XV. fig. 4.

largely developed. These two, at first sight dissimilar parts, we look upon as portions of the same gland, the testis. The looser part of the tube is much larger in calibre than that which is compacted in the fusiform mass, the latter tube being almost microscopic; the former is external, and at one end is in connexion with the penis by means of a much-attenuated and long duct, analogous to that which in *D. repanda* we have named vas deferens; at the other end it merges into the tube forming the spindle-shaped mass; the internal end of this latter tube opens into the oviduct, at the usual part at which the testis is received in the former category of species. The dilated part of the oviduct in this species is of unusual length, and the spermatheca of large size, as if in correspondence with the great elaboration of the testis. A small glandular sac, moreover, provided with a small duct, opens a little within the external orifice of the vagina, and this tube, which is short, communicates directly and separately with the spermatheca. The duct from that reservoir is short and delicate, comes off close to the junction of the vaginal tube, and shortly before it is united to the oviduct receives the duct of the accessory spermatheca.

In *D. coccinea** the testis is represented by the fusiform mass alone, which communicates by a very short simple duct with the penis at one end, and at the other, as usual, with the oviduct, which, as in some previously described examples, is here much constricted. The androgynous apparatus is arranged as in the last species, except that the vaginal gland is absent.

D. verrucosa has all the organs as last described, except the androgynous parts, which are in their connexions with each other slightly modified.

D. Johnstoni† has the fusiform gland composed of a very fine tube densely packed and connected at both ends, precisely as in *D. tuberculata*, VERANY, and *D. coccinea*, and the other organs are likewise disposed on the same plan as in those species.

In this species, however, and in *D. tomentosa*, VERANY, a closely allied species, we have the following addition to the already complicated male organs. There is an elongated hollow pouch, longer than the penis, placed in front of that organ, and opening alongside of it into the common vestibule. This pouch, into which projects, when the parts are quite retracted into the body, the finely-pointed end of a dart or stiletto, which in *D. Johnstoni* is straight, in *D. tomentosa* curved, is capable of being everted like the tentacle of a snail. To one side, and near the apex or internal end of this pouch, is attached by a short pedicle, a small ovate sac‡, from the side of which again a small twisted tube leads to a large, long, irregular, curved membranous bag, lying on the outer side of the mucus-gland. The stiletto mentioned above has its thicker end lodged in, and growing from the inner extremity of the ovate sac, whilst its shaft projects along the narrow pedicle of the sac into the hollow pouch beyond. This stiletto is evidently the production of the lining membrane of the ovate sac; that membrane forms also a sheath for a considerable portion of the stiletto, and projects with it into the pouch; here however we find it succeeded by a

* Plate XV. fig. 3. † Plate XIV. figs. 9 and 10. ‡ Plate XV. fig. 1.

horny layer, also of its own secreting, which sheathes the dart to the end *, guarding its extremely fine point, but being open. The elongated pouch lying on the other genitalia is, we believe, of a glandular nature, and transmits along its tube or duct to the ovate sac its secretion. This fluid is necessarily therefore poured into the sheath of the stiletto. Now the pouch into which the stiletto projects, we have stated, is capable of undergoing eversion, and when this takes place, the ovate sac with the dart becomes thrust into the pouch, and is thus brought to the external orifice. This being effected, it is not difficult to conceive that, by successive longitudinal contractions and relaxations of the ovate sac, the dart may be projected and withdrawn by turns, and that the fluid which has been poured into the sheath of the dart will of necessity be driven out with the dart, and will therefore inevitably be shed into any punctured wound that the dart may have made in the soft body of a conjoined *Doris*.

The above apparatus is in all probability destined for the inoculation of another individual, previous to or during the act of congress, with a fluid of a stimulating or aphrodisiac character, the stiletto being analogous to the dart of the common Snail.

It is perhaps worthy of remark that the penis of *Limapontia* is pointed with a crystalline spur-like appendage; and we may be allowed to make the suggestion, founded on this circumstance, and on the above description of the genitalia of *D. Johnstoni* and *D. tomentosa*, that the two penis-like organs noticed in *Onchidium* by Cuvier in his 'Mémoires pour servir,' &c. as of doubtful character, are, one a penis with a hard spur-like end, the other a stiletto, such as has just been described.

In the latter set of species of *Doris* there is the same division of the mucus-gland into two parts as in the former, but these parts vary somewhat in proportionate size, and in the diameter of the tube, which is convoluted in them.

It will be found, from the foregoing description, that the reproductive organs do not differ essentially in *Doris* from those in *Eolis*, as given by us in the Annals and Magazine of Natural History for February 1848, and hence we infer that the two sets of observations and the results drawn from them support each other. At the above date we were not aware that Herr Heinrich Meckel had been engaged upon the sexual organs of the hermaphrodite Mollusks, as we have since learnt from the Report on Zoology for 1844, published by the Ray Society, and we are sorry that it has not been in our power to make ourselves acquainted with the full nature of the original memoir. From the Report, we find that Meckel takes the same view as Siebold, with regard to the organ seated upon the liver, namely, that it is androgynous, and has the vas deferens included in the oviduct. Since we became acquainted with the above views, we have examined the generative organs of *Helix*, *Limax*, *Onchidium* and *Aplysia*, and it would indeed appear that in these genera the part we call testis is insufficient for the function attributed to it; and in all of them we have detected spermatozoa in the ovary, as will shortly be seen we have found to

* Plate XV. fig. 2.

be the case in *Doris*. We have failed to find the double sacculi and the inclusion of the vas deferens in the oviduct, at the same time we acknowledge that it is matter of great difficulty to determine the exact functions of the several parts of the apparatus.

But however the matter may be in the above Mollusks, as regards *Doris* we are still inclined to adhere to our own opinion founded on observation, namely, that the disputed organ attached to the liver is simply an ovarium; that there is no duct included in the oviduct, but that it is a simple tube, the function of the dilated portion of which is not merely to give passage to the ova, but to serve as a sort of reservoir for the seminal fluid, which is allowed to pass from the spermatheca along it, even as far as the ova in the ovarium. The testis and vas deferens, it will be seen, we recognize in that tube, of whatever length and however convoluted, which extends from the oviduct to the penis.

The position and arrangement of this testicular tube and its connection would seem of themselves to warrant this recognition; but when we find an almost microscopically convoluted tubular gland, almost the type of the mammalian testis, superadded, as in *D. tuberculata*, VERANY, and occupying part of the position of the larger or loosely convoluted tube of the true *D. tuberculata*, and that in some species, as in *D. repanda*, *D. verrucosa* and *D. coccinea*, there is a lengthened duct that may very aptly be compared to a vas deferens, leading as it does from the fusiform gland to the penis, then we must look upon our conclusions as in some degree warranted. It is true that this convoluted tubular testis joins the oviduct; but in this circumstance, instead of seeing anything that militates against our theory, we find a certain degree of confirmation of our views of the condition of hermaphroditism in which these creatures are placed. The connection with the oviduct we suppose is to provide the means of self-impregnation, when the extremely solitary habits of some of these mollusks render such necessary; and lastly, the evidence to be elicited from the contents of the testis and other genitalia would seem further to support what we have here advanced.

In *D. tuberculata* of our coasts, the testis, examined immediately after coitus, contains granular tenacious mucus, having imbedded in it numerous large granular vesicles, each having one or more clear cells developed in its interior. These vesicles* appear to be spermatophora in an early stage of development. In the fusiform mass, making part of the testis in *D. tuberculata*, VERANY, the same kind of bodies has been observed. The same specimen of *D. tuberculata* of our coasts, the contents of whose testis have been just noticed, was examined as to its spermatheca; this was found to contain multitudes of minute elliptical cells, apparently enclosed in delicate fusiform membranous sacs, together with a few scattered spermatozoa. These sacs are the developed spermatophora†, and enclose cells,—spermatozoa in an incipient state. The spermatheca of other specimens taken during the breeding season, we found to be filled with large fusiform sacs, containing either simple cells,

* Plate XV. fig. 6.

† Plate XV. fig. 7.

or these intermingled in various proportions with apparently fully developed spermatozoa. The accessory spermatheca, examined in season, is always found quite filled with perfectly developed spermatozoa arranged in parallel order, and in masses as if still surrounded by the membranes of the spermatophora, the simple cells having entirely disappeared. The spermatozoa*, when fully developed, are elongated slender waved filaments, having at one end a small curved fusiform enlargement obliquely attached.

The dilated portion of the oviduct affords vast numbers of spermatozoa lying about without any obvious arrangement, though they have been seen on one occasion in bundles as if contained in spermatophora. Spermatozoa are also found abundantly, and also perfectly formed, in the ovarium itself; but we have never seen them having that exact relationship to the ova which is believed in by H. Meckel, neither has anything like spermatophora been observed in the ovarium. If the observation and theory of this anatomist be correct with regard to *Doris*, then the development of the spermatozoa ought to proceed from the ovary in exactly the reverse order to that just described. In the ovarium young spermatophora ought to be found; these ought to burst either in the vas deferens or in the tube leading to the penis, where one ought naturally to expect to find the perfect zoosperms. These however we have never found therein; and indeed it would not appear correct to suppose that a thoroughly elaborated secretion, like the normal semen, should have to pass through such a minute and extensively convoluted tube as we find coiled up in the more solid part of the testicular apparatus of *D. tuberculata*, Verany, *D. Johnstoni* and others; for this, from its composition, size and arrangement, is clearly itself an originator of secretion, and not merely the duct of a gland. Further, the spermatheca ought, for the same reason, to be filled with perfect spermatozoa always ready for fecundation; but we have always seen them here, with the exception of a few scattered accidentally about, still included in the spermatophora and in process of development. Indeed, the evidence we have already adduced, it will be perceived, strongly corroborates our view, namely, that the tube we call testis is really the secreting organ of a tenacious mucus-like semen, having imbedded in it numerous incipient spermatophora at the time of its emission; that this fluid is poured during coitus into the spermatheca of the conjoined individual; that there the spermatophora are matured and the spermatozoa within them; as these last are developed they are passed on in their spermatophora into the accessory spermatheca, thence to be shed into the oviduct as occasion may require. Along this tube they are gradually conveyed to the ovarium, where fecundation is in all probability effected. It may appear somewhat anomalous to state that the seminal fluid of one individual should have to be matured in the body of another, and yet from the contents of the different parts of the generative apparatus, and particularly of the spermatheca, we can adopt no other conclusion in the present state of our knowledge. We are aware, however, that, multiplied as our

* Plate XV. fig. 8.

observations have been, the matter requires still more extended investigation before any very positive conclusion can be arrived at.

Organs of Circulation and Respiration.—These consist of central organs of propulsion, arteries, veins and sinuses, and of a plumose branchial organ arranged in a more or less complete circle.

First, we have the systemic heart*, consisting of auricle and ventricle, inclosed within a membranous pericardium, of an oval form, best seen when distended, which is attached all round to the general peritoneal investment of the viscera, except at the entrance of the two lateral venous trunks from the skin and the vein from the branchiæ: there it is continued upon the veins themselves. On removing the heart, there may be seen, on each side on the floor, a number of small oblique perforations or pores.

The heart lies upon the upper surface of the posterior part of the liver-mass in front of the branchial circle, and the upper surface of the pericardium is in contact with the skin of the back. The heart, when distended†, almost fills the pericardium, the auricle being somewhat larger than the ventricle. The latter is very muscular, subtriangular in form, provided with numerous and strong carneæ columnæ, and having the auriculo-ventricular opening guarded by a double valve, the edges of which come together horizontally and project into the ventricular cavity. There is also a valve at the aortic opening. The walls of the auricle are much thinner and more delicate than those of the ventricle; the interior is lined with an irregular open network of much-attenuated fleshy columns, or rather threads.

The circulation appears pretty rapid in *Doris*; in *D. pilosa* there are seventy-two beats of the heart in a minute; in *D. tuberculata* upwards of fifty.

Secondly, we have lying under the pericardium and opening into it another propelling organ, which has not hitherto been noticed as such, of considerable interest, which shall be more fully described after we have traced the general course of the circulation. The general systemic artery‡ comes from the front or apex of the pyriform ventricle, and is almost at once divided into three principal trunks; of these, two are lateral and opposite; and each results in three branches, which ramify upon the upper surface of the sides and posterior part of the common mass of the liver and ovarium, dividing and subdividing into ramuscules of extreme tenuity, forming a network around the lobules of the ovarium and then plunging into the liver. The third or anterior trunk or aorta passes forwards on the right lobe of the liver, to which it gives several branches; it then gives off on the left side a strong branch, the gastric artery; after this another large tube passes off on the same side, supplying copiously a spongy glandular-looking organ, analogous perhaps to some of the vascular ductless glands of the Vertebrata, overlying the buccal mass. The same artery distributes branches to supply the supra-œsophageal ganglia of the nervous system. The aorta, soon after, gives off the genital artery from the right side; this supplies

* Plate XI. figs. 1, 2 and 4. † Plate XVI. fig. 2. ‡ Plate XII. fig. 1.

all the genitalia except the ovary, and a network of its fine ramifications, when injected, can be seen spread over the surface of the mucus-gland. The aorta next passes forward and gains the median line beneath the buccal mass, and is resolved into two branches, a buccal and a pedial, the former going to the buccal mass and channel of the mouth; the latter, dividing as it goes, runs backward on the floor of the visceral cavity, supplying the foot from end to end.

In the viscera, except in the liver and in the skin, after repeated injections and examinations, we have failed to discover veins, and must therefore conclude that the blood, after having passed through the network of arterial ramifications, falls into spaces or sinuses, among the tissues, between the viscera and in the general peritoneal cavity. We are unable as yet to say whether the spaces into which the blood thus escapes are or are not provided with a delicate proper lining membrane, as has been advanced by M. le Dr. ROBIN in his admirable 'Rapport à la Société de Biologie sur le Phlébentérisme, 1851.' The general peritoneal cavity communicates freely by many apertures with a network of canals or sinuses in the skin, but we have as yet found no veins leading from the general cavity of the body directly to the respiratory organ. The spongy network of sinuses in the skin opens freely and widely into a great trunk sinus, running backward on each side of the body at the angle of union of the mantle with the foot. These trunks, opposite the posterior angles of the auricle, turn suddenly inwards, perforate the inner surface of the skin, gaining the general cavity of the body; they then, in the form of distinct systemic veins, penetrate the pericardium and empty themselves into the auricular cavity. Thus the blood current which we have traced from the heart along the arterial system through the sinuses of the visceral cavity and the skin, is brought by venous canals through the skin back at once to the heart without having previously passed through the branchiæ.

How then, it will be asked, does the blood of *Doris* find its way to the branchiæ? It will be remembered we have accounted for the return to the heart of all the blood sent to the skin, and to the viscera, with the exception of the liver-mass; now it is that portion of the blood which has circulated through the liver-mass, and that only, which is made to traverse the specialized respiratory apparatus before reaching the heart. We before mentioned that the arteries distributed to the liver-mass can be seen to form on its exterior a delicate and minute network; we find by injection that the veins of this organ are similarly arranged, and that the principal branches of the arteries and veins run commonly side by side; moreover, we have succeeded in filling numerous arterial plexuses by injecting coloured fluid by the veins: here these sinuses or lacunæ must be reduced to their minimum, if indeed they exist at all. The principal venous branches of the liver converge and unite in a common hepatic trunk situated on the median line, at first concealed in the substance of the organ, and then in its backward course emerging at a groove at the posterior end of the liver. This trunk then turns upward and opens widely into the anterior limbus

of an internal or venous branchial circle* which closely surrounds the anus. The blood from this circle passes on to traverse the branchial leaflets or plumes, by running up the inner side to the apex and then down the outer side to the base of the division, and then falls into a second, or external, or arterial circular canal at the base of the branchial crown. From the anterior limb of this outer circle a short wide trunk, the true branchio-cardiac or efferent vein of the branchiæ, opens forwards on the median line into the posterior border of the auricle of the heart. This efferent vein lies immediately over the hepatico-branchial or afferent vessel of the branchiæ. Thus it is that in the auricle of the heart, the blood from the liver-mass, having been aërated in the special respiratory organ, becomes mixed up with that from the other viscera, which has been returned through the general, though imperfect, respiratory surface of the skin, by the two great lateral systemic venous trunks before described.

Having now gone over the general course of the circulation, there remains to be noticed the additional blood-propelling organ situated, as before mentioned, beneath the pericardium. This is the organ which Cuvier calls a vesicle acting as a reservoir to a canal, which coming backward from the liver opens at the external orifice placed close to the anus. This canal and the vesicle act, according to the same high authority, as the agents for the production of an excrementitious fluid, which is prepared either by the liver itself in addition to the bile, or by some other gland, the lobes of which are so intimately interlaced with the lobes of the liver that the eye cannot distinguish one set from the other. M. Milne-Edwards conjectures, we see, in his 'Observations sur la Circulation,' Article premier, in the Annales des Sciences Naturelles, 1845, "that the pore which exists by the side of the anus in *Doris* may be for the purpose of admitting water into the interior of the organism, there to be directly mixed with the blood."

We began the study of this organ impressed with a high idea of the labour which Cuvier had bestowed upon it, and with every disposition to believe in the accuracy of his results; but after repeated dissections, and injections, and careful observations, we find ourselves obliged, however reluctantly, to differ from the views that he and M. Milne-Edwards have taken of these parts, and we therefore submit the following description, which we confidently believe will be found correct by those who will take the pains to examine into the matter with the minuteness it demands.

The vesicle, or heart† as we term it, is a hollow pyriform organ, lying somewhat transversely under the right side of the pericardium, with its base opening into that cavity just in advance of the posterior angle of the right side. The narrow end tapers to a tube, which after perforating the wall of the large sinus, to be presently mentioned, turns suddenly forwards along the median line of the liver, where it overlies the great hepatico-branchial vein, partially concealing it. In this course the tube

* Plate XVI. figs. 2 and 6.

† Plate XI. figs. 1, 2, 3 and 4; Plate XII. figs. 1 and 5; Plate XVI. fig. 1.

gives off several offsets on both sides, and ends at the anterior border of the liver by dividing into several branches. The inner surface of the pyriform organ is, as CUVIER has pointed out, strongly plicated*; the plicæ are arranged on an intricate plan, and so that they can act as a valvular apparatus to prevent the return of blood that has once passed through. The orifice is capable of being contracted like the mouth of a purse. This is the condition of the parts, as we have observed them, in *D. tuberculata*; but we have not yet seen how the branches of this tube terminate. In *D. repanda*, however, in which, as in the other species we have examined, the same organ, and branched tube proceeding from it, exist, and in which the tube does not extend so far forward as the anterior border of the liver, the terminal branches are lost among a minute network of twigs from the left side of the aorta, which are found to dip through the ovarium into the liver. In *D. bilamellata* also, we find that fresh light is thrown upon this curious apparatus†. The tube attached to the pyriform organ or heart, after passing forwards for about half the length of the liver, giving off numerous twigs, resolves itself into a multitude of other small branches. All these offsets go to form a very complete and close network lying over the hepatic vein in a superficial depression, and extending from one end to the other of the liver-mass; the sides of this plexus are united to a similar arrangement of numerous hepatic branches from both sides of the aorta. These branches, in *D. repanda*‡, are extremely numerous, and have a regular and very beautiful parallel arrangement; and the two lateral posterior branches in this species, as well as in *D. pilosa* and *D. bilamellata*, give off from their outer sides numerous much-ramified twigs. In these two species of *Doris*, therefore, the branches of the tube leading from the pyriform organ inosculate and form a network with those of the aorta; they therefore convey blood. This blood, coming as it does into the pericardium through the minute pores, already mentioned as existing on the floor of that organ and flowing through the vesicle or heart attached to it from the visceral cavity and intervisceral sinuses, is evidently venous; and our belief is that the vesicle of CUVIER is a ventricle, the office of which is to propel venous blood along its tube and branches, which are arterial in character, into the network formed by these branches and those of the aorta. This new apparatus then has a decidedly portal character.

The blood which has gone through the above network is conveyed through the liver to the hepatic vein, and we have satisfied ourselves that there is no channel of communication between this vesicle or heart and the external orifice near the anus; indeed, its office necessarily precludes the idea of such connection. The orifice near the anus is small, and leads into an extensive, more or less ramified cavity or sinus§, the trunk of which extends forwards along the upper surface and median line of the liver; its principal offsets follow the course of the chief arterial trunks, and appear to terminate on the surface of the liver.

* Plate XVI. figs. 3, 4 and 5.

† Plate XII. fig. 5.

‡ Plate XI. fig. 3.

§ Plate XI. figs. 1, 2 and 3; Plate XII. fig. 1.

This cavity is circumscribed by a delicate membrane, which is found beneath the pericardium, and has the aorta running along its roof, and the great hepatic vein in the liver beneath it. It is intimately adherent to and undistinguishable from the capsule of the liver-mass. In the wall of the cavity lies the network from the two sources already named. The inner surface of the whole of this membrane is covered with a fine spongy-looking tissue*, which is most abundant over the tracks of the blood-vessels; examined under the microscope, this tissue presents, particularly in *D. repanda*†, a remarkable honeycombed appearance, produced by slightly elevated lines of membrane, enclosing irregular five- or six-sided spaces, each holding a single, large, clear, globular vesicle, containing a few smaller cells of different sizes, together with some granules. In *D. tuberculata* a similar spongy tissue exists, but not so strongly marked. Considering that we have here a branched tube with a fine network of arterial and venous twigs on its wall, and the lining membrane evidently of a glandular nature, though of unusual form, there seems little room for doubt that this is an apparatus for the elaboration of some fluid from the blood; further, we find that this tube opens externally, and that the position of the orifice is close to the anus: hence we infer that this organ is one for excretion, and we have little hesitation in pointing it out as the renal organ of *Doris*. To return for a moment to the vesicle of Cuvier, it is now evident that it may with propriety be termed an accessory renal or hepatic heart; for its function is to propel venous blood, first to the renal and then to the hepatic organ; but if we may reason from what has been positively ascertained in the Vertebrata, we are inclined to consider it rather as belonging to the latter than to the former organ, particularly as we find the renal organ in *Doris* has not acquired that complete speciality and independence of other organs which it has attained in higher animals.

In recapitulation of what has been said, we will now endeavour to follow the course of the circulation in *Doris*. The principal or systemic heart propels a mixed stream of blood, which has come partly from the skin and partly from the branchial circle, and which therefore is not completely aërated, to all the viscera and the foot. This blood, with the exception of a small portion sent to the pericardium, passes from the viscera (except the liver, ovarium and kidney) and from the foot through the intervisceral sinuses, the common visceral cavity and the network of canals in the skin, by two lateral veins into the auricle. The liver, ovary and kidney are supplied with a current of blood brought to them, partly by the branches of the aorta, and partly by those of the portal heart, to which the pericardium acts as an auricle. This current, which on its passage to the liver acquires an additional venous character by traversing the walls of the renal sinus, penetrates the liver either through sinuses or capillary vessels, and is conveyed along a system of hepatic veins into the great hepatico-branchial trunk or afferent branchial vein; by this it reaches the branchiæ, whence, after having undergone the influence of the surrounding medium, it is returned to

* Plate XIV. fig. 5. † Plate XIV. fig. 6.

the auricle by a single trunk; there it is mixed with the imperfectly aërated blood from the skin, and propelled again by the ventricle along the arteries as before.

Thus then we find here a systemic circulation divided into two portions, one general, the other partial; the latter is combined with a portal circulation. It is from this hepatic course, in which the blood is most completely deteriorated, that it is sent to the branchiæ, and being returned thence to the heart, joins the current of the general portion of the circulation in the auricle. This part of the circulation seems not to have been hitherto noticed in the Nudibranchiata, and in 1845 we first pointed out the fact of the blood being returned to the heart, both from the skin and the branchiæ in *Doris*.

From what has been here said, it will be observed that we have assumed the skin in *Doris* to be to some extent an agent in respiration. That it is so, seems to be a fair inference from these facts following; that a large quantity of blood traverses it on its way to the heart continually; that although in most species the skin is stiffened by spicula, it is nevertheless sufficiently delicate to admit of the necessary changes taking place through it; and that in some, as in *D. pilosa*, this membrane is very soft, and clothed with numerous soft and delicate papillæ, which, whilst they materially increase the extent of surface, are well adapted to the above purpose; that in this last, as in the majority of the species, even the most spiculose, the whole of the mantle, and even the foot itself is covered with vibratile cilia. Moreover, if the skin be not a respiratory organ, the whole of the blood which supplies the viscera, with the exception of that sent to the liver-mass, must be returned to the heart unaërated, which is not likely.

In *Eolis* the skin in part performs the function of respiration, and in *Limapontia* entirely. Again, in Terebratula Professor Owen has proved that the mantle alone is a respiratory organ; and in Lingula, he has shown that there is the first appearance of a portion of that membrane becoming specialized as a gill; and as we ascend in the scale of organization in the Mollusca, it is evident that this relationship between the gill and the mantle is always maintained, or in other words, the gill is a development of the mantle, and not of any of the internal membranes.

Thus, therefore, it is quite in accordance with what might have been expected, to find the blood partly aërated in the skin, and partly in the branchiæ; and that such a state of things is no anomaly we learn from M. Milne-Edwards, who states that in the Great Triton of the Mediterranean, in *Haliotis, Patella* and *Pinna*, the blood is returned to the heart in a mixed condition, part of it coming from the mantle, and part from the gill; and from Mr. Garner, who several years ago pointed out the same condition of parts and functions in the *Lamellibranchiata*. And when the whole subject shall have been fully investigated, it will probably be found that the above condition of the circulatory and respiratory organs predominates in the Molluscan type. If this should be so, then the conditions of the circulation in the Mollusca and the higher crustacea will be found to approximate more closely than has generally been thought; inasmuch as, on the authority of John Hunter, and

more lately of Professor Owen, it can be shown that a mixed stream of aërated and unaërated blood flows into the auricle of the heart.

Lastly, there remains to be described the special branchial organ*. This is situated at the posterior part of the dorsal surface of the body, surrounding the anus and the renal orifice. It is composed of a variable number of plumes more or less divided or pectinated, and arranged in a more or less complete circle. In *D. tuberculata* they are three or four times pinnate. The plumes in some species, as in the last named and others, are retractile within a common cavity, the individual plumes being themselves contractile. In *D. pilosa*, *D. bilamellata*, &c., the plumes are merely contractile. Around the base of the plumes run two concentric canals† within the skin. The inner contains venous blood poured into it from the front by the hepatico-branchial or afferent vein, and communicates with as many channels as there are plumes. These channels run up the inner side of the stems of the plumes, and divide to apply themselves to their branches. At the ends of the branches these channels communicate with others corresponding to them on the outer side; the outer channels converge as they pass down to the stems of the plumes and debouch into the outer circle, which therefore receives aërated blood, and then transmits it to the branchio-cardiac or afferent vein, which opens into the posterior border of the auricle. In passing through these plumes, the blood follows the same course in all their subdivisions, running up the inner and down the outer surface; the main trunks of the plumes and all their offsets are clothed with vibratile cilia.

There is a peculiarity of structure in the branchial plumes of *D. pilosa*, the presence of which is indicated by their well-known white star-like centre. This appearance is owing to the existence of a double row in each plume stem of irregularly globular, hollow bodies‡, with elastic walls separating the inner and outer channels of the stem from each other. The function of this apparatus is somewhat doubtful; but we are inclined to believe that it is for the purpose of giving resilience to the breathing organ, and thus enabling the species in which it exists to continue its respiration for a time even out of water; and it is worthy of remark, that of all the species with which we are acquainted, the last-named enjoys the widest range of sea depth, being found from the coralline zone, to more than half-way between low and high-water marks. *D. tuberculata*, which is another species in which this apparatus is found, but in a less developed form, is to be taken both below and above low-water mark, and is frequently left dry among the crevices of the rocks.

Nervous System.—Of this there are two divisions; the first is made up of two series of ganglia, supra- and infra-œsophageal, interconnected by three commissures or collars. The former series is asymmetrical. These ganglia give off about twenty pairs of nerves, and four single nerves.

The second division consists of a complete and extensive network of minute ganglia and intercommunicating nerves spread out upon the viscera.

* Plate XI. fig. 4. † Plate XVI. figs. 2 and 6. ‡ Plate XVI. figs. 6 and 7.

The first division corresponds to the excito-motor, or indeed to the cerebro-spinal system, the second to the sympathetic system, of the Vertebrata. These two systems in *Doris* are at several points distinctly in connection with each other, and all the ganglia of the two systems are in *D. tuberculata* at least, from which the principal description of the nervous system is taken, of a bright orange colour.

The supra-œsophageal ganglia* of the first system are five pairs and a single ganglion, the pairs being symmetrically placed with regard to each other, and to the median line; and the single ganglion, not hitherto described, and which we shall call the visceral, lies on the right side. Of these, three pairs predominate in size, the anterior of which, or sensorial, the cerebroid of M. Blanchard, lie next the median line, across which they are connected, and are of a somewhat conical shape, the base forwards. Springing from about the middle of the anterior border, and from the upper surface of these, is a pair of ganglia corresponding to the pedunculate pair, which in *Eolis* we have called olfactory. In *Doris* they are almost sessile, and though in size much inferior to the cerebroid, are nevertheless of notable dimensions. They give off each a large nerve to supply the dorsal tentacles, the first pair. The next three pairs of nerves come off from the under surface of the anterior border of the cerebroid ganglia, external to the attachment of the first pair. They run forward and supply the muscles and the skin at the side of the channel of the mouth and the lips: the second of these pass to the oral tentacles. The fifth pair comes off close after the fourth, passes forwards and downwards, and soon divides into two branches; one goes on to the under part of the channel of the mouth and lips, the other courses round under the buccal mass, and unites with the corresponding nerve of the opposite side, to form the first or anterior collar (*k*), which is slender and wide. External and close to, and of about the same size as the fifth pair, comes off a nervous trunk, which curving round the top of the œsophagus, joins the principal infra-œsophageal ganglion. This trunk, instead of being an ordinary nerve, constitutes with its fellow of the opposite side the second or middle collar (*l*) or commissure between the anterior supra-œsophageal and the larger infra-œsophageal ganglia, and for this reason we omit it in the enumeration of the nerves. The sixth pair are very short, and come off from two minute elliptical ganglia, almost sessile upon the external border of the anterior or cerebroid ganglia near their posterior end: these are the optic nerves and ganglia. The seventh pair of nerves appear to have no trunk, for the auditory capsule is sessile on the ganglia directly behind the optic nerve.

The posterior or branchial ganglia are broadly ovate, and connected in front with both the cerebroid and the other pair. The eighth, ninth and tenth pairs arise from these ganglia, and are distributed to the whole of the mantle; the two first of these pairs go to the anterior, and the last to the posterior portion, sending a branch of communication (10′) to the branchial ganglia of the sympathetic system. The

* Plate XVI. fig. 8.

eleventh and twelfth nerves arise also from these centres, and are distributed to the sides of the body between the mantle and the foot.

The lateral or pedial ganglia lie on a plane rather beneath the others, with both of which they are connected, and in shape and size are like the branchial. Three large nerves, the thirteenth, fourteenth and fifteenth pairs, are given off from these centres, and supply the whole foot.

The single supra-œsophageal or visceral ganglion* is round, and about the size of the olfactory; it is sessile on the under surface of the anterior border of the right branchial ganglion, where this is in contact with the pedial. The nerves *g*, *h*, *i*, *j* issue from this ganglion; (*g*) the first in origin passes down by the side of the aorta, to which it gives branches; one of these has a small ganglion (*n*) in its course; the other, the larger, passes further on, and ends in two or three ganglia (*p*) placed at the root of the aorta, which send branches to the pericardium and heart. The trunk of the nerve passes down towards the branchiæ, and in so doing, gives off next another branch to the systemic heart, and then a large offset to the portal heart; after this the nerve comes into connection with the branchial ganglia and plexus (*q*) of the sympathetic system, and communicates with the renal plexus (*v*), apparently a dependence of the former or branchial. The next in order (*h*) runs backwards to the right side towards the base of the bag of the penis, where it merges into the principal sympathetic ganglion of the generative organs. The third nerve (*i*), smaller than the last, runs beneath the organs of reproduction and the stomach, and in its course distributes two or three branches to the mucus-gland, joining an open plexus (*s*), with a few small ganglia situated on that organ; the nerve then inclines towards the left side, presenting a small ganglionic swelling in its course; and shortly afterwards passes into the largest ganglion (*t*) of the gastro-hepatic plexus, that which receives the right par vagum nerve. The last nerve (*j*) runs straight backwards to the vicinity of the branchiæ, and after giving off two or three twigs apparently to the intestinal plexus (*w*), as it passes under the intestine, terminates in a small ganglion closely connected with the larger one, into which nerve (*g*) runs, both of these ganglia belonging to the branchial plexus (*q*).

We now come to the third, or posterior, or great œsophageal collar (*m*); it is stouter, more closely invests the œsophagus than the other two, and is composed of three parallel cords, two of which are attached to the under surface of the pedial ganglia; the third has one end in connection with the left branchial, and the other with the visceral ganglion.

The infra-œsophageal ganglia† are two pairs symmetrically disposed on the buccal mass; the larger or buccal, as we have seen, are connected by means of a collar with the cerebroid. They are elliptical, and are united by a short commissure. They give off laterally, and in union with the collar, two pairs of nerves, the sixteenth and seventeenth, which are distributed to the buccal mass. The eighteenth is a small

* Plate XVII. fig. 1. † Plate XVI. fig. 8.

pair which passes off backwards to the base of the tongue, and is joined by buccal filaments of the sympathetic. To the front of the buccal are attached the gastro-œsophageal ganglia, which are very small and give off three pairs of nerves. The smallest of these, the nineteenth pair, is given to the salivary gland. The twentieth supplies the top of the œsophagus, round which it curves on each side, communicating with the œsophageal sympathetic plexus. Lastly, the two nerves constituting the twenty-first pair, by far the largest of the three, are continued down on the under surface of the œsophagus, on each side of the median line, nearly parallel with each other, communicating by slender filaments with a fine open network of nerves and ganglia upon that tube, and unite with two of the largest ganglia of the sympathetic system of the stomach. This pair is the counterpart of the gastric portion of the par vagum of the higher animals, and is analogous to the nerves which in insects have been named stomato-gastric.

The principal varieties that have been observed in the cerebro-spinal nervous centres are as follows: in *D. pilosa** and *D. repanda*†, the first three pairs of ganglia are very distinct, and only slightly altered in relative position from those in *D. tuberculata*, the pedial becoming quite lateral as regards the œsophagus, and indeed almost meeting under that tube, the third collar being consequently of extreme brevity. In *D. Johnstoni*‡, *D. verrucosa*§, *D. coccinea*, *D. bilamellata*‖, and *D. aspera*, the cerebroid and branchial are more or less fused into one mass, which in some of the species is elongated in the antero-posterior direction, and in others obliquely. In *D. Johnstoni*¶ the visceral ganglion is pedunculate; no material variation in the origin and distribution of the nerves has been observed. The visceral ganglion is present in all.

The two pairs of infra-œsophageal ganglia just described, have been noticed in the Annales des Sciences Naturelles for 1848, by M. EMILE BLANCHARD, in an article on the Opisthobranchiate Mollusks; the larger pair he names œsophageal, and gives all their branches to the alimentary apparatus; the smaller he calls *angeïens* or *aortiques*; these latter he announces as a new discovery, and states that they are placed on each side of the aorta to which they give filaments. The corresponding ganglia in *Eolis* were discovered by us in 1846, and are to be found described and figured in the third part of the Monograph on the Nudibranchiate Mollusca, published by the Ray Society in the following year. These branches have now been so often verified by us, both in *Eolis* and *Doris*, that we are quite satisfied of the truth of the account we have given of them. We cannot therefore see the propriety of the names that M. BLANCHARD has imposed upon these ganglia. Further, M. BLANCHARD looks upon the infra-œsophageal ganglia and nerves as the representatives of the splanchnic or sympathetic nervous system of the higher animals; and anatomists in general seem to have a confused idea that the œsophageal ganglia of the Mollusca,

* Plate XVII. fig. 8. † Plate XVII. fig. 9. ‡ Plate XVII. fig. 2.
§ Plate XVII. fig. 4. ‖ Plate XVII. fig. 6. ¶ Plate XVII. fig. 3.

taken all together, represent not only the cerebro-spinal system of the Vertebrata, but their sympathetic system as well. We are now, however, we hope, prepared to show the error of such views, and to supply a desideratum in Comparative Anatomy, by pointing out the true sympathetic nervous system of the Mollusca. This system, it will be found, bears a very remarkable resemblance to, and correspondence with, its counterpart in the Vertebrata; and whilst its demonstration will, with the other new points which we have recorded in this paper, prove the very high degree of organization enjoyed by the Nudibranchs, it will also throw a clearer light upon the physiology of the whole of the ganglia surrounding the œsophagus than previously existed.

*The Sympathetic System**.—This exists, and is more or less demonstrable, in the skin, the buccal mass, and in all the internal organs. It consists of a vast number of minute distinct ganglia, varying in size and form, the larger quite visible to the naked eye, of a bright orange colour like the ganglia round the œsophagus, and interconnected by numerous delicate white nervous filaments, arranged in more or less open plexuses or networks. This beautiful system is in several points, as already indicated, connected with both sets of œsophageal ganglia.

In the skin the sympathetic system is not very easily detected, nor indeed have we much sought for it there; but we have seen enough to prove its presence in the fact of the existence of two or three ganglia in connection with filaments of the anterior branches of the branchial ganglion.

In the buccal mass this system is only a little more evident, and is very difficult of investigation; we have, however, succeeded in making out a few ganglia and nerves in the neighbourhood of the œsophagus. First, there is a nerve on the posterior surface of the buccal mass passing forward on each side of the infra-œsophageal ganglia, and having itself a small ganglion with other nerves at each of its ends imbedded in the buccal mass. This nerve runs within the second collar, communicating with it, and giving a twig to the salivary gland; it also furnishes a twig which joins the lingual nerve. This nerve is probably the rudimentary representative of that large nerve in *Eolis*, which passing from the same part of the buccal mass runs also within the corresponding collar with which it is in connection, and is distributed to the glands of the papillæ.

Around the œsophagus there is a fine open network † of very minute nerves and a few ganglia, frequently connected with both of the gastro-œsophageal nerves from the infra-œsophageal ganglia. This network at the top sends some offsets to the salivary glands, and others to the nerves and ganglia of the buccal mass. The œsophageal sympathetic plexus, with the gastro-œsophageal nerves, is continued down as far as the cardiac orifice of the stomach; at this part the gastro-œsophageal or vagi nerves pass into two comparatively large ganglia, situate under the cardia; and the œsophageal plexus, besides giving twigs to these ganglia, is continued into the gastro-hepatic

* Plate XVII. fig. 1. † Plate XVIII. fig. 3.

sympathetic plexus. Of these two great ganglia, the larger (*t*), that of the right side, receives likewise a distinct branch from nerve (*i*) of the visceral supra-œsophageal ganglion, and thus these gastro-hepatic centres are brought into relation with the cerebro-spinal centres. These two cardiac centres are the chief links of a complete chain or collar of ganglia* and commissures that surround the posterior or cardiac end of the stomach, just behind the œsophagus, and at the entrance of the great duct of the liver. There are ten or twelve, or more centres in this collar, and from it, branches are given off on both sides, and chiefly from the centres; these pass forwards on the stomach, and backwards on the hepatic duct to the liver; the latter are fine, and form with small ganglia an intricate plexus on the duct; these have not been followed into the liver on account of the extreme delicacy of the parts; the former are larger, and four of them, which may be regarded as principal trunks, pass forwards upon the different aspects of the stomach, forming, with microscopic filaments and ganglia, a complete interlacement all over that organ, particularly on its under surface. The pancreas also is crowded with almost microscopic ganglia and filaments. We propose to denominate the whole of this very extensive and complicated network, of which the circular collar of ganglia may be considered as the centre, the gastro-hepatic plexus of the sympathetic system. Altogether it forms, with its almost countless orange-tinted ganglionic nodules of various forms and sizes, a novel anatomical spectacle of extraordinary beauty and interest. These nerves and ganglia are visible without much difficulty in a favourable specimen, and lie for the most part slightly imbedded on the external surface of the organs. Towards the pyloric end of the stomach the ganglia are thickly strewn, presenting the appearance of another collar or circle around that part: this may be called the pyloric plexus (*u*)†. From this there is a continuous minute and singularly beautiful plexus of nerves and ganglia, down the whole length of the intestinal canal to the anus. These ganglionic nodules are usually smaller on the whole than the gastric, varying less in size and in form, being mostly globular, and having the same orange colour. They are most numerous at the two extremities of the intestine; taken as a whole, they may very properly be called intestinal plexus (*w*, *w*)‡.

The branchial or branchio-cardiac portion, as far as we have observed, consists of a somewhat irregular chain of large orange ganglia and nerves lying across, and in front of the base of, the branchial crown, and over the hepatic vein. Twigs pass backwards from this to the branchial plumes, and forwards towards the heart, and two branches of communication (*g* and *j*) are received from the visceral supra-œsophageal ganglion. A twig reaching these ganglia laterally from the mantle nerve of each side keeps up a connection between them and the branchial supra-œsophageal centres.

An open network (*v*) with two ganglia has been observed on the floor of the renal organ, and over the hepatic vein; this is all we have found to represent a renal plexus,

* Plate XVIII. fig. 1. † Plate XVIII. fig. 2. ‡ Plate XVIII. fig. 4.

or it may more properly belong to the vascular system; the nerve (*g*) from the visceral ganglion also gives a twig to this plexus: this part of the sympathetic system will, however, require more close investigation than we have yet been able to bestow upon it.

The last portion of this system which now remains, is that belonging to the reproductive organs. Here at the under part of the base of the penis exists an intricate network of numerous nerves and ganglia of varying size; the largest of these centres (*r*) receives nerve (*h*) from the visceral ganglion of the cerebro-spinal system. The network is continued in a more open form over the whole sac of the penis, and prolonged also over the ovarian channel, and can also be traced upon the anterior part of the mucus-gland, where there are several ganglia; one or two ganglionic nodules have also been seen upon the oviduct. A slight network of nerves has been met with upon the posterior part of the mucus-gland, in connection with two or three twigs of nerve (*i*) from the visceral ganglion above mentioned.

The ganglia of the sympathetic system* contain seldom more than a few nerve-globules; these are granular, nucleated and coloured, like those of the cerebro-spinal centres, but generally have two instead of one caudate prolongation. On account of the simplicity of the arrangement of the globules and nerve-tubes, the connection of these two elements may here be studied with great advantage. Various ganglia from the stomach have been submitted to the microscope, and what has usually presented itself to the eye is, that at the place where the ganglionic swelling occurs one or more nerve-tubes or fibres, according to the size and complexity of the ganglion, and the number of the globules contained, appear to dilate gradually or suddenly, and each to enclose a globule, the nerve-tube contracting again at the opposite pole of the globule, and resuming its former dimensions, the neurilemma also being bulged out to an extent correspondent with the size of the enclosed globule or globules.

The globules are generally more or less elliptical or fusiform, though some are pyriform: these last appear to be prolonged into a nerve-tube only at their apex. It is possible, however, that there may be, or may have been, another prolongation; but if so, it has certainly escaped us; at all events, as they have been observed, they remind us forcibly of the pyriform nucleated nerve-vesicles, formerly described by us as existing in the supra-œsophageal ganglia of *Eolis papillosa*. The globules vary greatly in size, are all provided with a large rounded or oval nucleus, having a distinct nucleolus; the whole globule is of a fine yellow colour, approaching orange, the nucleus being of a bright full orange; the nucleus and the whole cell are alike granular. The nerves of this system closely resemble those of the cerebro-spinal system, being semitransparent, pearly, and in showing parallel nerve-tubes enclosed in a common neurilemma. In spirit they become of an opake white, and are then best seen and dissected. What we have seen in *Doris* touching the connection of

* Plate XVIII. figs. 5, 6 and 7.

the nerve-tubes and nerve-vesicles, goes to confirm what we have elsewhere said respecting the corresponding parts in *Eolis*, and support the now almost universally received ideas on the subject of the relation of these parts to each other in the higher animals.

Extensive traces of the sympathetic system have been detected in several other species of *Doris* as well as *D. tuberculata*; and it is interesting to remark, that on the œsophagus, stomach and genitalia of *Eolis papillosa* ganglia and nerves of the same system have been observed. The same system has been seen in *Arion ater*, and there can be little doubt that it will be found in all the higher Mollusks.

It is worthy of notice, that in these Mollusks we have found no special relation between the nerves of the sympathetic system and the blood-vessels, such as are well known to exist in the Vertebrata, the only exception being that the nerve (g) from the visceral cerebro-spinal ganglion that goes to the branchial plexus, and to the two hearts, gives branches to the aorta and one of its divisions; on each of these branches appears a small ganglion. It is possible that on further and more minute scrutiny plexuses on the great vessels at least may be discovered.

After this detailed account of such a complicated system of visceral nerves, few perhaps will be disposed to doubt that we have here a true sympathetic nervous system. The fact, however, of the existence of ganglia and intercommunicating nerves forming plexuses, the mode of disposition of these networks upon the organs, their connections with the principal nervous centres grouped around the œsophagus, and lastly, the microscopic structure of both nerves and ganglia, all combine to prove the correctness of what we have advanced.

Assuming it, then, as proved that the system before us is a true sympathetic or splanchnic nervous system, let us now see what light this new fact is capable of throwing upon the physiology of the ganglia about the œsophagus. Although hitherto physiologists have seemed to concur in the belief that somehow or other the œsophageal centres preside at the same time over the functions of both animal and organic life, still there have been differences of opinion as to the mode of the assignment to the individual ganglia of functions apparently so different in nature as those of the cerebro-spinal and sympathetic systems. The principal theories to be met with are three; first, that all the ganglia, both those above and those below the œsophagus of the Mollusca, perform in some way or other the entire functions of cerebro-spinal and sympathetic systems; second, that the supra-œsophageal ganglia represent the cerebro-spinal, and the infra-œsophageal the sympathetic system of the Vertebrata; third, that whilst the two series of ganglia in the Mollusca are the counterparts of nearly all the cerebro-spinal in the Vertebrata, the nerves lying along the œsophagus and going to the stomach, represent the "par vagum" or the "sympathetic" or the "visceral nerves."

The discovery of the true sympathetic nervous system not only proves that the third theory here noticed is that which most nearly approximates to the truth of

nature, but also supplies a point of analogy between the nervous system of the Mollusca and that of the Vertebrata, which was previously wanting. We are now free to compare the nervous centres and their offsets grouped round the œsophagus in the Mollusca, with those which constitute the cerebro-spinal system of the Vertebrata. This we shall now proceed to do; and it is highly interesting to find how remarkably close the analogy is, even to the details, between the sets of organs in the two Sub-kingdoms. But first let us take a brief review of the ganglia and their relation to each other in *Doris*, and then of the relation of the ganglia in *Doris* to those in other Mollusca. It has been believed by CUVIER and others that these ganglia are to a greater or less degree fused into one mass; but if in *D. tuberculata* the contents of the capsule* or dura mater be pressed out completely, the true manner of the connection becomes evident. It is then seen that the communication between the ganglia is kept up through narrow apertures by means of small oval commissures, and that the masses are closely pressed as it were together, the opposed surfaces being flat. In such a preparation it can be observed that the cerebroid ganglia are divided by a strong septum, which is perforated by an oval opening; that the branchial are connected to both the cerebroid and the pedial ganglia by commissures somewhat similar; that the pedial are similarly united to the cerebroid; and lastly, that the visceral ganglion is joined to the branchial through a small aperture. The third or principal collar round the œsophagus is in such a preparation distinctly seen to be divided into three strands, two of which enter the pedial ganglia, the other passes from the left branchial to the visceral, which again, as above stated, is connected with the right branchial. From these intercommunications between the ganglia, and from the distributions of the nerves arising from them, a strict analogy may be drawn between the nervous centres of *Doris*, in which they have attained a high degree of concentration, and those of other Mollusks in which they are comparatively disjoined. We may take *Aplysia hybrida* as a fair specimen of the latter class. Here the cerebroid ganglia are slightly apart, being united by a short commissure. These, as in *Doris*, give their branches to the dorsal tentacles, the eyes, the channel of the mouth and lips. At a little distance behind these centres are the branchial, which are on the same plane, considerable in volume, rounded, and connected with the former by a stout commissure. The branchial ganglia supply the mantle, and are doubtless the homologues of the parts of the same name in *Doris*. Backwards from them pass two stout trunks, which go to two ganglia in the vicinity of the branchiæ; these ganglia are little inferior in size to the branchial, are united to each other by a distinct commissure, and send their nerves to the generative organs, heart and intestine; they are in some species fused into one, as we see in CUVIER's memoirs, and this he believed to have the office of a sympathetic system. These ganglia, or this ganglion, from its connections with the branchial, and from the distribution of its nerves, is undoubtedly the homologue of our single visceral ganglion in *Doris*. The

* Plate XVI. fig. 9.

remaining supra-œsophageal ganglia are the pedial. These are placed on a plane below the branchial, on the side of the œsophagus, and are connected with the cerebroid and branchial by stout commissures of considerable length. Three large nerves are given to the foot from these ganglia. Thus we find that the homology of the parts is complete; and were the ganglia concentrated, we should find them connected in precisely the same manner as in *Doris*. There is moreover an exact correspondence as to the infra-œsophageal ganglia and their nerves in *Doris* and *Aplysia*.

The examination of the nervous centres of *Arion* and *Onchidium* strongly support the view we take of the functions of the different nervous ganglia in *Doris* and *Aplysia*. In both these instances we have the cerebroid ganglia giving their nerves to the organs of the senses, and to the channel of the mouth and lips, the branchial giving off their nerves to the mantle, and the pedial supplying the foot. *Arion* and *Onchidium*, however, differ from each other as regards the visceral ganglion; but this difference itself only corroborates our views of the functions of that organ. In *Arion* it is still formed, as in *Aplysia*, of two large centres; but these, instead of being placed at a distance and connected by long commissures with the branchial, are placed close between and united to them, and overlie the pedial. In *Onchidium* there is fusion of the pair into a single ganglion, which lies in the same position with regard to both branchial and pedial, and has the same connection with the branchial as in *Arion*. The nerves arising from this centre in *Onchidium*, and from this pair of centres in *Arion*, we have traced, though not quite perfectly, to all the viscera as in *Doris*. In *Onchidium* a little alteration in the relative position of the ganglia is all that is necessary to make the resemblance complete between its nervous centres and those of *Doris*.

Taking this view of things, we find ourselves at issue to some extent with M. E. Blanchard; for instance, the large posterior nerve from the branchial ganglion, the cervico-cardiac of this author, sends a branch of communication, as we have seen, to unite with the branchial sympathetic ganglia; these ganglia he takes to be the same as those in the posterior part of the body of *Aplysia*, and which we believe to be the homologues of our visceral ganglion; hence it is that he was led to consider the connectives between these ganglia and the supra-œsophageal centres to be the homologues of the great posterior nerve from the branchial ganglion; but in *Aplysia* we have the true representative of this nerve, which arises as in *Doris* from the branchial ganglion, and supplies the posterior part of the mantle. Moreover, we have discovered near the roots of the respiratory organ in *Aplysia* a minute ganglion and nerves, the real counterparts of the sympathetic branchial ganglion of *Doris*. We have therefore little or no doubt of the accuracy of our views on this point.

If we compare the nervous system of the higher Mollusks with that of the *Lamellibranchiata*, we shall find that the principal centres are sufficiently well represented in the lower forms. In *Mya truncata* we have two large ganglia placed anteriorly at

the sides of the œsophagus, connected by a longish commissure; these, which are usually called the labial, answer to the cerebroid of *Doris*, and give branches to the anterior margins of the mantle, the seats of sensation; and the corresponding ganglia in some species are said to have the auditory capsules attached to them. From these centres run backward two commissures, one of which runs into a ganglion placed in the muscles of the foot, the pedial, the other passing further back enters a large ganglion situated on the posterior adductor muscle. This mass sends large nerves to the branchiæ, the siphonal tubes and the posterior portion of the mantle. This is the branchial ganglion of authors, and as the ganglia which we have so called in *Doris*, supply nerves to the mantle, and are brought into connection with the branchial ganglia of the sympathetic system; we believe them to be the homologues of the last-named ganglion of the *Lamellibranchiata*. It is worthy of note that we have discovered in *Mya truncata* two small elliptical ganglia attached to the anterior and underpart of the branchial, and united together by a commissure. These send filaments to the ovary, and ventricle of the heart, and therefore probably represent the visceral ganglia of *Doris*. Thus we have the representation of the nervous centres of *Doris* complete in *Mya*.

We shall now, then, return to the comparison of the nervous œsophageal ganglia in *Doris* with the ganglia making up the cerebro-spinal system of the Vertebrata. First, then, we have a small anterior pair of ganglia, which we have called olfactory, for reasons given in our anatomy of *Eolis*, and which it is perhaps unnecessary to repeat here. This is in position, and probably in function, the counterpart of the same organ in the Vertebrata, the rhinencephalon of Professor Owen. Secondly, we have the large anterior supra-œsophageal or sensorial or cerebroid ganglia supplying the lips and channel of the mouth with both motor and sensitive nerves, and not only supplying the olfactory, but also the optic ganglia and the auditory capsules. This second pair is clearly comparable to either the hemispherical ganglia or the centres of sensation and volition, or to both these sets together, with a portion of the anterior extremity of the spinal cord, and probably also the cerebellum. They are analogous therefore to the prosencephalon, a portion of the mesencephalon and of the anterior end of the myelon, and possibly also of the epencephalon of the same high authority. Thirdly, we have the optic ganglia; these come the next in order in the Mollusca as in the Vertebrata, and in the former are to all appearance dependences of the sensorial or cerebroid. The optic ganglia from which the optic nerves arise, are analogous to the corpora quadrigemina, or a portion of the mesencephalon. Fourthly, come the branchial centres; these are next the median line, on the same plane as the cerebroid and posterior to them, and in several species, as we have seen, more or less joined with them into one mass. The nerves from these ganglia go to the respiratory organs, that is the skin or mantle, and to the ganglionic nervous centres of the branchiæ. The branchial ganglia correspond in function to the pneumonic portion of the pneumogastric apparatus of the medulla oblongata or macromyelon of the

higher animals. Fifthly, the visceral ganglion attached to the right branchial is the only evidence we have of the heterogangliate type in the œosophageal nervous centres of *Doris*. Its nerves go to the stomach, to the respiratory organs, to the circulatory and reproductive apparatus, being at these several viscera connected with their sympathetic ganglia. This single ganglion we take to be the representative in *Doris* of that part of the spinal cord in the Vertebrata which gives off a series of nervous branches, and communicates with either the chain of ganglia of the trunk of the great sympathetic or the plexuses more immediately attached to the several viscera. Sixthly, next in order come the pedial pair of ganglia, placed on a plane rather below the other two pairs and on the side of the sensorial. They subserve the locomotive organ, and hence represent those parts of the spinal cord of the higher animals, which also in them supply the organs of locomotion. Seventhly, the first infra-œsophageal or buccal ganglia supplying the buccal or lingual apparatus with sensation and motor power, and, as we suppose, with taste as well, answer to that part of the medulla oblongata from which the lingual and gustatory nerves arise. Eighthly, the second infra-œsophageal or gastro-œsophageal correspond to the gastric division of the pneumogastric ganglion of the medulla oblongata of vertebrate animals.

In *Doris*, then, we see the pneumogastric apparatus resolved into two parts; the pneumonic appears in the branchial ganglia, the cardiac is wanting, the gastric is seen in the gastro-œsophageal. These give off nerves, passing to the œsophagus and stomach, which are the counterparts of the pharyngeal branches of the glossopharyngeal, and of the pharyngeal, œsophageal and gastric portions of the par vagum of Vertebrata.

The sum of these comparisons is, that the whole of the ganglia arranged in *Doris* around the top of the œsophagus are analogous to the encephalon and a portion of the enrachidion of the Vertebrata.

There seems great probability that the cerebral hemispheres and the cerebellum, with the seats of consciousness and volition, and also of emotion of the higher animals, are but very faintly shadowed forth in the cerebroid ganglia of the Mollusca.

Organs of the Senses. The organs of Hearing.—These are two very delicate, microscopic, ovoid capsules, sessile on the outer margin of the cerebroid ganglia immediately behind the eyes. When magnified considerably they show an inner and an outer capsule, enclosing fluid in which exists an agglomeration of minute otolithes. When these are more highly magnified their form is seen to be pretty accurately oval, presenting a central darkish spot or nucleus. When extracted with their capsule they present a continual vibratile motion.

In *Doris aspera* the number of otolithes was found to be upwards of forty. On account of the very rudimentary state of the organs and their depth from the surface, it is evident that their function must be excessively limited.

The organs of Sight.—These are placed immediately beneath the skin, behind the

dorsal tentacles, but in the adult animal are not visible (with the exception of the case of a single species, as far as we know,) from the exterior.

When the skin is removed, they are seen as minute black dots, placed at the outer sides of the supra-œsophageal ganglia. They are thereto attached by a minute pedicle of variable length, which is so short in some as to make the eyes appear sessile on the ganglia. On closer examination the pedicle is found to have at its base a roundish or oval ganglion, which we have in a former part of this paper called the optic.

The optic nerve emanating in a forward direction from this ganglion enters the base of a well-formed eyeball, consisting of, first, a delicate transparent investing membrane, within which rests a pretty regularly formed cup of black pigmentary matter, the choroid, having projecting from its mouth a globular, bright, crystalline lens, in front of which is a firm, transparent, well-arched membrane attached to the lips of the cup of pigment, the cornea, or perhaps the capsule of the lens. The eye of *Doris* is fully equal in development to that of *Eolis*; but in the former, as it cannot be seen through the skin, we infer that there is no perception of external objects, but that at most the creature can only distinguish between light and darkness. This appears to be the necessary extent of visual power for the preservation from external violence of the individual in animals of such limited locomotive powers, the tactile property of the oral tentacles, assisted by the lips, possibly the seat of taste, being all that is requisite for the selection of matters fitted for the sustentation of the organism.

The organs of Smell.—The dorsal tentacles, which have never been observed to be used as tactile organs, we believe to be the seat of the sense of smell; and this belief is strengthened when we reflect that these organs are most highly developed and minutely laminated; that they are most plentifully supplied with nerves from ganglia placed in front of all the rest of the cerebral masses; that they are externally covered with vibratile cilia, and so placed on the head as easily to receive impressions from any odorous particles that may be mingled with the circumambient water.

It is generally admitted now that snails have the sense of smell; and *Doris*, which is certainly not inferior to them in organization, can scarcely be denied the possession of that endowment, particularly as we find in it a highly developed, conspicuous, sensitive and therefore important organ, to which no other use can properly be assigned, but which appears to correspond, in arrangement of parts and position, to the laminated antennæ of insects, to which olfaction has been commonly attributed. In the Pearly Nautilus, certain laminæ within the oral sheath, plentifully supplied with ganglia and nerves, have been by Professor Owen pointed out as the olfactory organ in that mollusk. It must be borne in mind that the laminated form of the organ of smell, and its supply of ganglia and nerves from the very front of the cerebro-spinal nervous centres, are universal in Fishes and the higher Vertebrata. In Fishes the organ presents, as is well known, a beautiful doubly laminated arrangement, the stem bearing the laminæ being fixed longitudinally to the bottom of the cavity of the nose.

In *Doris* we have a similar form, that of laminæ, attached to a central stem, which is ordinarily erect and exposed, but is capable in most species of being retracted within a cavity. These tentacles are placed on each side of the median line, at the anterior extremity of the body, over the head.

The subject of the nature and signification of the dorsal tentacles has been more fully discussed in our paper on *Eolis*, before alluded to; it is therefore perhaps unnecessary for us here to dwell more at length upon it.

The organ of Taste.—Is doubtful as to its existence; its use may be in some measure subserved by touch, it probably resides in the different lips and the passage to the buccal cavity, or may have its special seat in that cavity itself. The tongue, which is peculiarly a prehensile organ, seems very ill adapted for an organ of taste.

The organ of Touch.—This is the skin, over which the sense is universally diffused; but the faculty is undoubtedly specialized also in the oral appendages, whether these have the ordinary linear form of tentacles, or exist as a veil-like expansion partially encircling the mouth. The veil and the tentacles are alike the homologues of the oral tentacles of *Eolis*, and are supplied like them with nerves coming off from the anterior part of the cerebroid ganglia. The tentacles or veil are so placed that they can with perfect ease be applied to the surface of substances on which the animal crawls or seeks its food, and to the food itself; and there is every reason to believe that the sense of touch residing in these organs is exquisitely delicate.

We have not yet been able to examine with sufficient care the minute structure of the skin; at present we can only offer the following particulars. The skin in most species is tough and coriaceous, and is of a spongy or cellular structure within: the cloak, in all the British species examined, is stiffened with numerous imbedded spicula, having a more or less symmetrical arrangement. Spicula are also observed in the foot, in the tentacles, and in the roots of the branchial plumes. The under surface of the foot and the upper surface of the cloak, as before stated, are covered with vibratile cilia. The branchial plumes and dorsal tentacles are also furnished with them.

Explanation of the Plates.

PLATE XI.

Fig. 1. General view of viscera of *Doris pilosa* seen from above, the dorsal skin and peritoneal membrane having been laid open. *a*, buccal mass; *b*, upper surface of stomach; *c*, pancreas; *d*, intestine; *e, e*, liver; *f*, mucus-gland belonging to female organs; *g*, portion of testis; *h*, spermatheca; *i*, accessory spermatheca; *j*, pericardium, exhibiting through its transparent walls the ventricle and auricle; *k*, ventricle; *l*, auricle; *m, m*, two lateral venous

trunks passing from the skin to the auricle; *n*, efferent or branchio-cardiac vein; *o*, portal heart lying below pericardium, and seen through it; *p*, *p*, upper wall of renal organ, exhibiting, by the aid of injection, a fine network of arterial twigs in connection with the aorta; *q*, a gland-like body in connection with the vascular system, and overlying the cerebral ganglia; *r*, branchial plumes.

Fig. 2. Dorsal view of viscera of *Doris bilamellata.* *a*, buccal mass; *b*, buccal gizzard; *c*, portion of salivary gland; *d*, upper aspect of stomach; *e*, pancreas; *f*, intestine; *g*, *g*, liver; *h*, mucus-gland of female organs; *i*, spermatheca; *j*, *j*, testis; *k*, pericardium; *l*, ventricle; *m*, auricle; *n*, *n*, lateral vein from skin, entering auricle; *o*, portal heart; *p*, *p*, upper wall of renal organ, exhibiting a fine arterial network; *q*, branchial plumes; *r*, gland-like organ in connection with vascular system.

Fig. 3. Dorsal view of viscera of *Doris repanda*, the heart having been removed. *a*, buccal mass; *b*, upper aspect of stomach; *c*, pancreas; *d*, *d*, intestine; *e*, *e*, liver; *f*, *f*, renal organ, exhibiting a symmetrical arrangement of arterial twigs on its upper wall; *g*, aorta with apex of ventricle attached; *h*, portal heart appearing through the upper wall of renal organ; *i*, branchiæ; *j*, generative organs; *k*, glandular organ in connection with aorta, overlying supra-œsophageal ganglia, from which nerves may be seen to radiate.

Fig. 4. Dorsal view of viscera of *D. tuberculata.* *a*, buccal mass; *b*, stomach, upon which part of the gastric plexus of sympathetic nervous system is visible; *c*, intestine; *d*, *d*, liver with the principal arterial trunks of the renal organ seen on the surface; *e*, retracted penis; *f*, mucus-gland of female channel; *g*, spermatheca; *h*, vagina, leading to same from external orifice; *i*, pericardium; *j*, ventricle; *k*, auricle; *l*, *l*, lateral trunk veins from skin to auricle; *m*, efferent or branchio-cardiac vein; *n*, portal heart seen through the transparent membranes of heart; *o*, branchial plumes; *p*, glandular organ in connection with vascular system; *q*, supra-œsophageal ganglia with numerous nerves radiating from them; *r*, branchial ganglia of sympathetic system; *s*, retractor muscles of channel of mouth.

PLATE XII.

Fig. 1. General view of viscera partially separated, arterial system injected, *D. tuberculata.* *a*, buccal mass; *b*, muscles for retracting channel of mouth; *c*, œsophagus; *d*, salivary glands; *e*, stomach; *f*, pancreas; *g*, hepatic duct; *h*, liver; *i*, *i*, intestine; *j*, anus; *k*, renal orifice, with a bristle passed through into renal cavity; *l*, renal organ having the hepatic arteries imbedded in its superior wall; *m*, portal heart, partially contained within renal organ; *n*, apex of ventricle, the heart having been removed; *o*, re-

tracted penis; *p, p,* mucus-gland in connection with female channel; *q,* spermatheca; *r,* vagina leading from external orifice to same; *s,* accessory spermatheca; *t,* supra-œsophageal ganglia giving off numerous nerves; *u,* gland-like body connected with vascular system.

Fig. 2. Alimentary system of *D. Johnstoni*. *a,* buccal mass; *b,* retractor muscles of channel of mouth; *c, c,* salivary glands; *d,* dilated portion of œsophagus; *e,* true stomach; *f,* intestine; *g,* liver.

Fig. 3. Portion of alimentary system, *D. verrucosa*. *a,* œsophagus; *b,* stomach; *c,* pancreas; *d,* hepatic duct; *e,* intestine; *f,* liver.

Fig. 4. Alimentary system, *D. pilosa*. *a,* channel of mouth; *b,* anterior or gizzard-like portion of buccal mass; *c,* crop at commencement of œsophagus; *e,* dilated portion of œsophagus; *f,* true stomach laid open, exposing on its interior surface the orifices of the hepatic duct; *g,* pancreas; *h, h,* intestine; *i,* anus; *k,* liver.

Fig. 5. Alimentary system, &c., *D. bilamellata*, exhibiting renal organ laid open. *a,* channel of mouth; *b,* buccal mass; *c,* gizzard, opening into same; *d,* salivary gland; *e,* tubular portion of tongue; *f,* dilated portion of œsophagus; *g,* liver; *h, h,* intestine; *i,* pancreas; *j,* anus; *k,* small orifice leading into renal cavity, a bristle is passed through this orifice into cavity; *l, l,* wall of renal organ laid open longitudinally, exhibiting a minute network of aortic twigs anastomosing with other branches from the portal trunk; *m,* ventricle; *n,* auricle; *o,* portal heart; *p,* portion of the floor of pericardium attached to same.

Fig. 6. Tongue of *D. pilosa* removed from muscular support. *a,* anterior or exposed portion; *b,* tubular portion.

Fig. 7. Portion of same more highly magnified.

Fig. 8. Central portion of two rows of spines from tongue of *D. tuberculata*.

Fig. 9. One of the spines from same tongue more highly magnified.

Fig. 10. Outer portion of a row of spines from tongue of *D. Johnstoni*.

Fig. 11. Central portion of two rows of spines from tongue of *D. repanda*. *a,* central, *b,* lateral spines.

Fig. 12. An outer spine of same.

Fig. 13. A spine next median line of same.

Fig. 14. Two spines from tongue of *D. aspera*.

PLATE XIII.

Fig. 1. Side view of buccal mass, *D. tuberculata*. *a,* channel of mouth; *b,* buccal mass; *c,* œsophagus; *d,* lingual sac; *e, e,* salivary glands; *f, f,* muscles for retraction of channel of mouth; *g, g,* muscles for retraction of buccal mass; *g′,* belt to which these are attached; *h,* muscles for advancing buccal mass.

Fig. 2. Buccal mass, *D. tuberculata*, laid open above. *a*, channel leading to mouth; *b*, exterior orifice; *c*, outer lip; *d*, inner lip; *e*, buccal lip; *f,f*, walls of buccal cavity; *g*, muscular support of tongue; *h*, tongue; *i*, fleshy septum passing down tubular portion of tongue, and supporting in front a membrane which divides anterior from posterior portion of lingual organ; *j*, œsophagus; *k, k*, retractor muscles of channel of mouth; *l, l*, retractor muscles of buccal mass.

Fig. 3. Tongue of *D. tuberculata*, removed from its fleshy support. *a*, anterior exposed portion; *b*, posterior tubular portion with edges separated; *c*, membranous septum dividing the two portions; *d*, portion of mucous membrane of mouth.

Fig. 4. Upper view of muscular apparatus of tongue. *a*, buccal lip, exhibiting circular and transverse muscles; *b*, œsophagus; *c*, lingual sac; *d,d*, exposed portion of firm central nucleus; *e,e*, radiating and circular muscles for moving the tongue; *f*, channel in continuity with lingual sac; *g*, fleshy septum passing down tubular portion of tongue; *h, h*, upper bundle of muscles; *i*, lower ditto for advancing and rotating muscular support of tongue.

Fig. 5. Side view of muscular apparatus of tongue. *a*, buccal lip, composed principally of circular fibres; *b*, lingual sac; *c*, exposed portion of the firm central nucleus; *d*, radiating muscles for the eversion and retraction of the spinous membrane; *e*, circular muscle to assist in this action; *f*, upper muscles for advance of fleshy support of tongue; *g*, middle ditto to assist in this action; *h*, lower ditto, likewise for the same purpose.

Fig. 6. Upper view of fleshy support of tongue, the upper part of one side being removed. *a*, channel through which tubular portion of tongue passes; *b*, fleshy septum passing to bottom of lingual sac; *c*, lingual sac; *d*, radiating muscles cut through; *e*, central nucleus; *f* and *g*, upper and middle bundles of muscles for advancing lingual support.

Fig. 7. Cavity of buccal mass, *D. pilosa*, exposed from above. *a*, channel of mouth; *b*, outer lip; *c*, inner lip; *d*, wrinkling of mucous membrane; *e*, buccal lip; *f,f*, the two lateral portions of spinous collar with the rudimentary jaws between them; *g*, anterior gizzard-like portion of the buccal mass; *h*, œsophagus; *i*, crop at origin of same; *j*, tongue, resting on fleshy support; *k*, lingual sac.

Fig. 8. Front view of spinous prehensile collar of *D. pilosa*, channel of mouth having been removed. *a, a*, buccal lip; *b*, prehensile collar; *c*, front portion of corneous jaws; *d*, membrane extending from same along margin of prehensile collar; *e, e*, portion of mucous membrane lining channel of mouth.

Fig. 9. Portion of prehensile collar more highly magnified, showing the arrangement of spines.

Fig. 10. Group of spines from same, still more highly magnified.

Fig. 11. Two of same spines, yet more highly enlarged.

Fig. 12. Corneous jaws removed from buccal lip. *a*, anterior, free, exposed points; *b*, extremity buried in buccal lip.

Fig. 13. Stomach and hepatic duct of *D. tuberculata* laid open. *a*, stomach; *b*, œsophagus; *c*, pancreas; *d*, liver; *e*, great biliary duct laid open exposing numerous orifices leading into it from the liver; *f*, cardiac orifice; *g*, opening of pancreatic duct.

PLATE XIV.

Fig. 1. Lingual sac of *D. tuberculata* projecting from buccal mass behind. *a*, posterior extremity of same; *b*, line of section *b*; *c*, line of section *c*.

Fig. 2. Section *b* of lingual sac, explanatory of growth of spines. *a*, fleshy septum extending the whole length of tubular portion of tongue; *b*, membrane of sac; *c*, membrane supporting the spines; *d*, inner or lining membrane; *e*, the spines lying between the two last membranes and imbedded in a soft flocculent matter; *f*, point where septum joins bottom of sac; *g*, vacant space between lining membrane of tongue and fleshy septum.

Fig. 3. Section *c* of lingual sac. *a*, fleshy septum; *b*, muscles of buccal mass; *c*, membrane of sac; *d*, dentigerous membrane; *e*, teeth or spines; *f*, lining membrane; *g*, space between lining membrane and septum.

Fig. 4. A portion of membrane from root of renal organ, *D. pilosa*, exhibiting injected a fine arterial plexus.

Fig. 5. Inner surface of a portion of same membrane of renal organ, *D. repanda*, considerably magnified, exhibiting the tracks of the arterial branches thickly covered with a glandular substance.

Fig. 6. Portion of same membrane more highly magnified, showing the structure of the glandular substance, which is extended over it as well as over the vessels.

Fig. 7. Generative organs of *D. tuberculata* spread out to show the connection of the parts. *a*, inner sac of penis leading to external orifice; *b*, retracted penis; *c*, fine tube passing from testis to extremity of intromittent organ; *d*, testis; *e*, ovary spread over liver; *f*, oviduct; *g*, dilated portion of same; *h*, point where it is united to testis; *i*, point of its union with duct from spermatheca; *j*, female channel leading to external opening; *k*, opake portion of mucus-gland; *l*, semipellucid portion of same; *m*, vagina leading from external orifice to spermatheca *n*; *o*, duct from spermatheca to oviduct; *p*, accessory spermatheca connected to duct of spermatheca.

Fig. 8. Generative organs, *D. bilamellata*. *a*, retracted intromittent organ; *b*, testis; *c*, oviduct as it leaves ovary; *d*, dilated portion of oviduct; *e*, contracted

portion of same at the point where it receives duct from testis; *f*, opake portion of mucus-gland connected with female channel; *g*, semipellucid portion of same gland; *h*, female channel leading to external orifice; *i*, vagina leading from external orifice to spermatheca; *j*, spermatheca; *k*, duct from the same, which after passing into the mucus-gland, is joined to oviduct.

Fig. 9. Generative organs, *D. Johnstoni*. *a*, testis; *b*, vas deferens; *c*, retracted penis; *d*, oviduct as it leaves ovary; *e*, dilated portion of oviduct; *f*, point where it receives duct from spermatheca; *g*, opake portion of mucus-gland; *h*, semipellucid portion of same; *i*, female channel leading to external orifice; *j*, vagina leading from external orifice to spermatheca; *k*, spermatheca; *l*, duct from the same; *m*, accessory spermatheca connected with ditto.

Fig. 10. Generative organs, partially exserted, *D. Johnstoni*. *a*, lip of common orifice; *b*, male intromittent organ; *c*, orifice leading to female parts; *d*, vaginal orifice leading to spermatheca; *e*, large penis-like organ furnished with a stiletto; *f*, sac containing stiletto.

PLATE XV.

Fig. 1. Stiletto in its pouch, *D. Johnstoni*, seen in the compressor. *a*, portion of penis-like organ with which the pouch is connected; *b*, outer pouch of stiletto; *c*, inner pouch of same; *d*, stiletto; *e*, sheath of same; *f*, duct leading to large glandular sac in connection with stiletto.

Fig. 2. Extremity of penis-like organ with stiletto partially exserted. *a*, apex of penis-like organ; *b*, stiletto; *c*, *c*, sheath of same.

Fig. 3. Generative organs, *D. coccinea*. *a*, testis; *b*, vas deferens; *c*, retracted penis; *d*, oviduct as it leaves ovary; *e*, dilated portion of same; *f*, contracted portion of same where it is united to testis; *g*, point at which it is united to duct of spermatheca; *h*, opake part of mucus-gland; *i*, semipellucid part of same; *j*, female channel leading to external orifice; *k*, vagina; *l*, spermatheca; *m*, duct from same; *n*, accessory spermatheca.

Fig. 4. Generative organs, *D. tuberculata*, VERANY. *a*, compact portion of testis; *b*, tubular portion of same; *c*, vas deferens; *d*, retracted penis; *e*, dilated portion of oviduct; *f*, the point of its union with duct from compact portion of testis, and near to which it likewise receives duct from androgynous organs; *g*, opake portion of mucus-gland; *h*, semipellucid portion of same; *i*, female channel leading to external orifice; *j*, vagina leading from exterior to spermatheca *l*; *m*, duct from spermatheca; *n*, accessory spermatheca.

Fig. 5. Generative organs, *D. repanda*. *a*, testis; *b*, vas deferens; *c*, retracted penis; *d*, oviduct as it leaves ovary; *e*, dilated portion of same; *f*, contracted

portion where it receives duct from testis; *g*, point where it receives duct from androgynous apparatus; *h*, opake part of mucus-gland; *i*, semipellucid portion of same; *j*, female channel leading to external opening; *k*, vagina leading to spermatheca *l*; *m*, duct from same to accessory spermatheca *n*; *o*, duct from accessory spermatheca to oviduct.

Fig. 6. Supposed immature spermatophora from testis of *D. tuberculata.*

Fig. 7. Two spermatophora from spermatheca. *a*, one almost filled with irregularly-shaped cells; *b*, another in which the cells have nearly all disappeared, containing innumerable spermatozoa.

Fig. 8. Mature spermatozoa from spermatheca.

PLATE XVI.

Fig. 1. Side view of contracted heart of *D. pilosa*. *a*, ventricle; *b*, auricle; *c*, upper or dorsal wall of pericardium; *d*, floor of same; *e*, aorta; *f*, branchio-cardiac or efferent branchial vein; *g*, portal heart.

Fig. 2. Upper view of heart inflated, *D. tuberculata*, the pericardium laid open, with diagram of branchial circulation appended. *a*, ventricle; *b*, auricle; *c,c*, walls of pericardium; *d*, aorta; *e, e*, two lateral veins coming from skin to auricle; *f*, branchio-cardiac or efferent branchial vein; *g*, outer or efferent circle, venous; *h*, inner or afferent circle, arterial, at root of branchiæ, within which latter circle are situated the anal and the renal orifices; *i*, vessel passing from liver to branchiæ; *j*, great efferent vessel from a branchial plume; *k*, afferent ditto; *l*, anal orifice; *m*, renal ditto.

Fig. 3. Portal heart laid open, *D. pilosa*, showing pectinated laminæ placed round the orifice leading to pericardium. *a*, lip of this orifice.

Fig. 4. Side view of a pectinated lamina.

Fig. 5. View of orifice leading into portal heart from pericardium, showing the inner margin beset with ends of pectinated laminæ.

Fig. 6. Side view of stem of branchial plume, *D. pilosa*. *a*, afferent branchial vessel; *b*, efferent ditto; *c*, peculiar apparatus for the purpose of giving elasticity to breathing organ; *d* and *e*, branchial leaflets. The arrows show the course of the blood.

Fig. 7. Front view of portion of branchial plume showing its peculiar central apparatus, the vessels having been removed.

Fig. 8. Cerebral nervous system, *D. tuberculata*. *a,a*, cerebroid ganglia; *b,b*, branchial; *c, c*, pedial; *d*, olfactory; *e*, buccal; *f*, gastro-œsophageal. 1, olfactory nerves; 2, nerves supplying upper portion of channel of mouth and lip; 3, those to oral tentacles; 4 and 5, to the sides and lower portions of channel of mouth and lip; 6, optic nerves, each having at its origin a small ganglion; 7, auditory capsules, the nerves being invisible; 8 and 9, nerves

supplying anterior portions of mantle; 10, nerves to posterior portions of same, sending a branch of communication; 10′, to branchial ganglia of the sympathetic system; 11 and 12, nerves to side of body; 13, 14 and 15, to foot; 16 and 17, nerves to buccal mass; 18, to tongue; 19, to salivary glands; 20, to top of œsophagus; 21, nerves passing down that tube and united to two large ganglia of sympathetic system of stomach; *g, h, i, j,* four nerves arising from visceral ganglion which is attached to lower surface ot branchial; these go to be united with ganglia of the sympathetic system of the various organs; *k,* first or anterior collar; *l,* middle or second collar or commissure uniting the supra- and infra-œsophageal ganglia; *m,* the third or posterior or great œsophageal collar.

Fig. 9. Transparent capsule of supra-œsophageal ganglia, *D. tuberculata,* exhibiting from below the perforated septa dividing them. *a, a,* cerebroid ganglia; *b, b,* branchial; *c, c,* pedial; *d,* visceral; *e, e,* the extremities of the great œsophageal collar.

Fig. 10. Eye, *D. repanda.* *a,* optic ganglion; *b,* optic nerve; *c,* choroid; *d,* lens; *e,* cornea; *f,* general capsule.

Fig. 11. Auditory capsule, *D. aspera.* *a,* outer capsule; *b,* inner capsule containing numerous otolithes.

Fig. 12. Two otolithes highly magnified showing central nucleus.

PLATE XVII.

Fig. 1. View of under surface of cerebral ganglia showing their connections with the sympathetic system, *D. tuberculata.* *a, a,* cerebroid; *b, b,* branchial; *c, c,* pedial; *d,* olfactory; *e,* buccal; *f,* gastro-œsophageal; *g′,* visceral; *g,* anterior or first nerve from visceral ganglion giving twigs to aorta, to ganglia at apex of ventricle, to pericardium, to portal heart, to renal plexus, and terminating in a ganglion of branchial plexus; *h,* second nerve from visceral ganglion terminating in principal ganglion of genital plexus; *i,* third nerve from visceral ganglion giving branches to a plexus on female generative organs, and ending in the chief ganglion of gastro-hepatic plexus; *j,* fourth or last nerve from visceral ganglion passing to lower portion of intestine, to which it gives branches, most probably, to the ganglia thereon, and terminates in a ganglion of branchial plexus; *k,* first or anterior collar of cerebral system; *l,* second or middle ditto; *m,* third or great œsophageal ditto; 18, lingual nerves; 19, nerves to salivary glands; 20, to upper portion of œsophagus; 21, nerves (par vagum) passing down œsophagus, united with the plexus thereon, and terminating in two of the principal ganglia of gastro-hepatic plexus; 22, nerves in connection with ganglia and nerves on buccal mass; *n,* minute ganglion

on aorta; *o*, ditto on a branch from same; *p*, two ganglia at apex of ventricle; *q*, branchial plexus connected by branches with nerve 10 from branchial ganglia; *r*, principal ganglion of genital plexus; *s*, plexus of female organs; *t*, principal ganglion of gastro-hepatic plexus; *u*, pyloric; *v*, renal; *w*, *w*, intestinal plexus; *α*, retracted penis; *β*, testis; *γ*, mucus-gland; *δ*, stomach; *ε*, hepatic duct; *ζ*, pancreas; *η*, œsophagus; *θ*, intestine; *ι*, anus.

Fig. 2. Cerebral ganglia, *D. Johnstoni*, seen from above. *a*, cerebroid, supporting optic ganglia and auditory capsules; *b*, *b*, branchial ganglia; *c*, *c*, pedial; *d*, olfactory; *e*, buccal; *f*, gastro-œsophageal; 1, nerves to dorsal tentacles; 2 and 3, to lips and channel of mouth; 4, 5 and 6, to mantle; 7, 8 and 9, supply the foot; 10, 11 and 12, visceral nerves; 13, pass down œsophagus; 14, supply salivary glands; 15, nerves to top of œsophagus, 16, to tongue.

Fig. 3. Under view of one side of cerebral ganglia, *D. Johnstoni*. *a*, cerebroid; *b*, branchial; *c*, pedial; *e*, visceral, giving off visceral nerves, 10, 11, 12.

Fig. 4. Dorsal view of cerebral ganglia, *D. verrucosa*, giving off nerves in the usual manner. *a*, *a*, cerebroid; *b*, *b*, branchial; *c*, pedial; *d*, olfactory; *e*, buccal; *f*, gastro-œsophageal.

Fig. 5. Pancreas laid open, *D. tuberculata*, to show transverse laminæ.

Fig. 6. Dorsal view of cerebral ganglia, *D. bilamellata*. *a*, *a*, cerebroid; *b*, *b*, branchial; *c*, *c*, pedial; *d*, olfactory; *e*, buccal; *f*, gastro-œsophageal; *g*, optic; 1, 2, 3, visceral nerves.

Fig. 7. Under view of same ganglia. *a*, *a*, cerebroid; *b*, *b*, branchial; *c*, *c*, pedial; *d*, olfactory; *e*, visceral, giving off three visceral nerves.

Fig. 8. Dorsal view of cerebral ganglia, *D. pilosa*. *a*, *a*, cerebroid; *b*, *b*, branchial; *c*, *c*, pedial; 1, 2, visceral nerves.

Fig. 9. Dorsal view of cerebral ganglia, *D. repanda*: these ganglia are arranged as in the last species; 1, 2, visceral nerves; 2 exhibits a ganglionic swelling.

PLATE XVIII.

Fig. 1. View of cardiac or posterior portion of stomach, *D. tuberculata*, showing gastro-hepatic plexus, and collar or chain of ganglia of sympathetic nervous system. *a*, œsophagus; *b*, hepatic duct; *c*, pancreas; *d*, *d*, nerves of par vagum; *e*, third nerve (*i*) from visceral ganglion; *f*, *f*, great chain or collar of ganglia; *g*, principal ganglion of same; *h*, *h*, two of the principal nerves of gastric plexus connecting gastro-hepatic with pyloric plexus.

Fig. 2. Pyloric extremity of stomach, and small portion of intestine displaying part of pyloric and intestinal plexuses of sympathetic system. *a*, *a*, pyloric plexus; *b*, commencement of intestinal ditto.

Fig. 3. Portion of œsophagus with network of nerves and ganglia of sympathetic system. *a, a,* the two nerves of par vagum.

Fig. 4. Upper portion of intestine with intestinal sympathetic plexus.

Figs. 5, 6 and 7. Ganglia of sympathetic system from stomach highly magnified, demonstrating nerve-globules in relation with nerve-tubules. *a,* neurilemma containing tubules; *b,* ganglionic enlargement containing nerve-globules.

XIV. *Analytical Researches connected with* STEINER's *Extension of* MALFATTI's *Problem.* *By* ARTHUR CAYLEY, *M.A., Fellow of Trinity College, Cambridge. Communicated by* J. J. SYLVESTER, *Esq., F.R.S.*

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THE problem, in a triangle to describe three circles each of them touching the two others and also two sides of the triangle, has been termed after the Italian geometer by whom it was proposed and solved, MALFATTI's problem. The problem which I refer to as STEINER's extension of MALFATTI's problem is as follows:—"To determine three sections of a surface of the second order, each of them touching the two others, and also two of three given sections of the surface of the second order," a problem proposed in STEINER's memoir, 'Einige geometrische Betrachtungen,' Crelle, t. i. The geometrical construction of the problem in question is readily deduced from that given in the memoir just mentioned for a somewhat less general problem, viz. that in which the surface of the second order is replaced by a sphere; it is for the sake of the analytical developments to which the problem gives rise, that I propose to resume here the discussion of the problem. The following is an analysis of the present memoir:—

§ 1. Contains a lemma which appears to me to constitute the foundation of the analytical theory of the sections of a surface of the second order.

§ 2. Contains a statement of the geometrical construction of STEINER's extension of MALFATTI's problem.

§ 3. Is a verification, founded on a particular choice of coordinates, of the construction in question.

§ 4. In this section, referring the surface of the second order to absolutely general coordinates, and after an incidental solution of the problem to determine a section touching three given sections, I obtain the equations for the solution of STEINER's extension of MALFATTI's problem.

§ 5. Contains a separate discussion of a system of equations, including as a particular case the equations obtained in the preceding section.

§§ 6 and 7. Contain the application of the formulæ for the general system to the equations in § 4, and the development and completion of the solution.

§ 8. Is an extension of some preceding formulæ to quadratic functions of any number of variables.

§ 1. *Lemma relating to the sections of a surface of the second order.*

If

$$ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw=0$$

be the equation of a surface of the second order, and

$$\mathfrak{A}x^2+\mathfrak{B}y^2+\mathfrak{C}z^2+\mathfrak{D}w^2+2\mathfrak{F}yz+2\mathfrak{G}zx+2\mathfrak{H}xy+2\mathfrak{L}xw+2\mathfrak{M}yw+2\mathfrak{N}zw=0$$

the reciprocal equation, the condition that the two sections

$$\lambda x+\mu y+\nu z+\varrho w=0$$
$$\lambda' x+\mu' y+\nu' z+\varrho' w=0$$

may touch, is

$$\begin{aligned}&(\mathfrak{A}\lambda^2+\mathfrak{B}\mu^2+\mathfrak{C}\nu^2+\mathfrak{D}\varrho^2+2\mathfrak{F}\mu\nu+2\mathfrak{G}\nu\lambda+2\mathfrak{H}\lambda\mu+2\mathfrak{L}\lambda\varrho+2\mathfrak{M}\mu\varrho+2\mathfrak{N}\nu\varrho)^{\frac{1}{2}}\\ &\quad\times(\mathfrak{A}\lambda'^2+\mathfrak{B}\mu'^2+\mathfrak{C}\nu'^2+\mathfrak{D}\varrho'^2+2\mathfrak{F}\mu'\nu'+2\mathfrak{G}\mu'\lambda'+2\mathfrak{H}\lambda'\mu'+2\mathfrak{L}\lambda'\varrho'+2\mathfrak{M}\mu'\varrho'+2\mathfrak{N}\nu'\varrho')^{\frac{1}{2}}.\\ &\quad=\big(\mathfrak{A}\lambda\lambda'+\mathfrak{B}\mu\mu'+\mathfrak{C}\nu\nu'+\mathfrak{D}\varrho\varrho'+\mathfrak{F}(\mu\nu'+\mu'\nu)+\mathfrak{G}(\nu\lambda'+\nu'\lambda)+\mathfrak{H}(\lambda\mu'+\lambda'\mu)+\mathfrak{L}(\lambda\varrho'+\lambda'\varrho)\\ &\qquad\qquad+\mathfrak{M}(\mu\varrho'+\mu'\varrho)+\mathfrak{N}(\nu\varrho'+\nu'\varrho)\big).\end{aligned}$$

And in particular if the equation of the surface be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+pw^2=0,$$

the condition of contact is

$$\begin{aligned}&\Big(\mathfrak{A}\lambda^2+\mathfrak{B}\mu^2+\mathfrak{C}\nu^2+2\mathfrak{F}\mu\nu+2\mathfrak{G}\nu\lambda+2\mathfrak{L}\lambda\mu+\frac{\mathrm{K}}{p}\varrho^2\Big)^{\frac{1}{2}}\\ &\qquad\Big(\mathfrak{A}\lambda'^2+\mathfrak{B}\mu'^2+\mathfrak{C}\nu'^2+2\mathfrak{F}\mu'\nu'+2\mathfrak{G}\nu'\lambda'+2\mathfrak{H}\lambda'\mu'+\frac{\mathrm{K}}{p}\varrho'^2\Big)^{\frac{1}{2}}\\ &\quad=\Big(\mathfrak{A}\lambda\lambda'+\mathfrak{B}\mu\mu'+\mathfrak{C}\nu\nu'+\mathfrak{F}(\mu\nu'+\mu'\nu)+\mathfrak{G}(\nu\lambda'+\nu'\lambda)+\mathfrak{H}(\lambda\mu'+\lambda'\mu)+\frac{\mathrm{K}}{p}\varrho\varrho'\Big),\end{aligned}$$

in which last formula

$$\mathfrak{A}=bc-f^2,\quad \mathfrak{B}=ca-g^2,\quad \mathfrak{C}=ab-h^2,$$
$$\mathfrak{F}=gh-af,\quad \mathfrak{G}=hf-bg,\quad \mathfrak{H}=fg-ch,$$
$$\mathrm{K}=abc-af^2-bg^2-ch^2+2fgh.$$

§ 2.

In order to state in the most simple form the geometrical construction for the solution of STEINER's extension of MALFATTI's problem, let the given sections be called for conciseness the determinators*; any two of these sections lie in two different cones, the vertices of which determine with the line of intersection of the planes of the determinators, two planes which may be termed bisectors; the six bisectors pass three and three through four straight lines; and it will be convenient to use the term bisectors to denote, not the entire system, but any three bisectors passing through the same line. Consider three sections, which may be termed tactors, each of them touching a determinator and two bisectors, and three other sections (which may be termed separators) each of them passing through the point of contact

* I use the words 'determinators,' &c. to denote indifferently the sections or the planes of the sections; the context is always sufficient to prevent ambiguity.

of a determinator and tactor and touching the other two tactors; the separators will intersect in a line which passes through the point of intersection of the determinators. The three required sections, or as I shall term them the resultors, are determined by the conditions that each resultor touches two determinators and two separators, the possibility of the construction being implied as a theorem. The *à posteriori* verification may be obtained as follows:—

§ 3.

Let $x=0, y=0, z=0$ be the equations of the resultors, $w=0$ the equation of the polar of the point of intersection of the resultors. Since the resultors touch two and two, the equation of the surface is easily seen to be of the form

$$2yz+2zx+2xy+w^2=0\,*.$$

The determinators are sections each of them touching two resultors, but otherwise arbitrary; their equations are

$$-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0$$

$$\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w=0$$

$$\frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w=0.$$

The separators are sections each of them touching two resultors at their point of contact (or what is the same thing, passing through the line of intersection of two resultors), and all of them having a line in common. Their equations may be taken to be

$$cy-bz=0,\quad az-cx=0,\quad bx-ay=0,$$

the values of a, b, c remaining to be determined. Now before fixing the values of these quantities, we may find three sections each of them touching a determinator at a point of intersection with the section which corresponds to it of the sections $cy-bz=0$, $az-cx=0$, $bx-ay=0$, and touching the other two of the last-mentioned sections; and when a, b, c have their proper values the sections so found are the tactors. For, let $\lambda x+\mu y+\nu z+\varrho w=0$ be the equation of a section touching the determinator $-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0$, and the two sections $bx-ay=0$, $az-cx=0$, and suppose

$$\Delta^2=\lambda^2+\mu^2+\nu^2-2\mu\nu-2\nu\varrho-2\lambda\mu-2\varrho^2,$$

the conditions of contact with the sections $bx-ay=0$, $az-cx=0$ are found to be

$$(b+a)\Delta=(b+a)\lambda-(b+a)\mu-(b-a)\nu$$
$$(c+a)\Delta=(c+a)\lambda-(c-a)\mu-(c+a)\nu,$$

values, however, which suppose a correspondence in the signs of the radicals. Thence

* The reciprocal form is, it should be noted,

$$x^2+y^2+z^2-2yz-2zx-2xy-2w^2=0.$$

$(b+a)\mu=(c+a)\nu$; or since the ratios only of the quantities λ, μ, ν, ϱ are material, $\mu=c+a$, $\nu=b+a$, and therefore

$$\Delta^2=\lambda^2-2(2a+b+c)\lambda+(b-c)^2-2\varrho^2=(\lambda-b-c)^2,$$

or

$$\varrho^2=-2(a\lambda+bc).$$

Whence the equation to a section touching $bx-ay=0$, $az-cx=0$ is

$$\lambda x+(c+a)y+(b+a)z+\sqrt{-2(a\lambda+bc)}\,w=0.$$

And to express that this touches the determinator in question, we have

$$\pm\alpha(\lambda-b-c)=\left(\alpha+\frac{1}{\alpha}\right)\lambda-\alpha(2a+b+c)+2\sqrt{-2(a\lambda+bc)}\,;$$

and selecting the upper sign,

$$\frac{1}{\alpha}\lambda-2a\alpha=-2\sqrt{-2(a\lambda+bc)}\,;$$

whence

$$\lambda=-2\alpha(a\alpha-\sqrt{-2bc}),\qquad \sqrt{-2(a\lambda+bc)}=(2a\alpha-\sqrt{-2bc})\,;$$

or the section touching the determinator and the sections $bx-ay=0$, $az-cx=0$ is

$$-2\alpha(a\alpha-\sqrt{-2bc})x+(c+a)y+(b+a)z+(2a\alpha-\sqrt{-2bc})w=0\,;$$

and at the point of contact with the determinator

$$-\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z+w=0$$

$$2yz+2zx+2xy+w^2=0.$$

Eliminating w between the first and second equations and between the second and third equations,

$$\sqrt{-2bc}\left(\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z\right)+cy+bz=0,$$

$$\left(\alpha x+\frac{1}{2\alpha}y+\frac{1}{2\alpha}z\right)^2+2yz=0\,;$$

and from these equations $(cy-bz)^2=0$, or the point of contact lies in the section $cy-bz=0$. It follows that the equations of the tactors are

$$\begin{aligned}
&-2\alpha(a\alpha-\sqrt{-2bc})x+(c+a)y+(b+a)z+(2a\alpha-\sqrt{-2bc})w=0\\
&(c+b)x-2\beta(b\beta-\sqrt{-2ca})y+(a+b)z+(2b\beta-\sqrt{-2ca})w=0\\
&(b+c)x+(a+c)y-2\gamma(c\gamma-\sqrt{-2ab})z+(2c\gamma-\sqrt{-2ab})w=0,
\end{aligned}$$

where a, b, c still remain to be determined.

Now the separators pass through the point of intersection of the determinators; the equations of these give for the point in question,

$$\begin{aligned}
x:y:z:w=&(2\beta\gamma+1)(-\alpha+\beta+\gamma+2\alpha\beta\gamma)\\
:\ &(2\gamma\alpha+1)(\alpha-\beta+\gamma+2\alpha\beta\gamma)\\
:\ &(2\alpha\beta+1)(\alpha+\beta-\gamma+2\alpha\beta\gamma)\\
:\ &4\alpha^2\beta^2\gamma^2-1+\alpha^2+\beta^2+\gamma^2\,;
\end{aligned}$$

and the values of a, b, c are therefore

$$\begin{aligned} a:b:c=&(2\beta\gamma+1)(-\alpha+\beta+\gamma+2\alpha\beta\gamma)\\ &:(2\gamma\alpha+1)(\alpha-\beta+\gamma+2\alpha\beta\gamma)\\ &:(2\alpha\beta+1)(\alpha+\beta-\gamma+2\alpha\beta\gamma),\end{aligned}$$

which are to be substituted for a, b, c in the equations of the separators and tactors respectively.

Now proceeding to find the bisectors, let $\lambda x+\mu y+\nu z+\varrho w=0$ be the equation of a section touching the determinators,

$$\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w=0, \qquad \frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w=0.$$

And suppose, as before, $\Delta^2=\lambda^2+\mu^2+\nu^2-2\mu\nu-2\nu\lambda-2\lambda\mu-2\varrho^2$; the conditions of contact are

$$\pm\beta\Delta=\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\varrho$$

$$\mp\gamma\Delta=\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\varrho,$$

where it is necessary, for the present purpose, to give opposite signs to the radicals. For if the radicals had the same sign, it would follow that

$$\frac{1}{\beta}\left[\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\varrho\right]-\frac{1}{\gamma}\left[\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\varrho\right]=0;$$

or the equation $\lambda x+\mu y+\nu z+\varrho w=0$ would pass through the point

$$x:y:z:w=0;\ \frac{1}{\beta^2};\ \frac{1}{\gamma^2}:-\frac{2}{\beta}+\frac{2}{\gamma};$$

or the section would be a tangent section of the two determinators of the same class with the resultor $x=0$, which ought not to be the case. The proper formula is

$$\frac{1}{\beta}\left[\beta\lambda-\left(\beta+\frac{1}{\beta}\right)\mu+\beta\nu-2\varrho\right]+\frac{1}{\gamma}\left[\gamma\lambda+\gamma\mu-\left(\gamma+\frac{1}{\gamma}\right)\nu-2\varrho\right]=0.$$

And this equation being satisfied, the section

$$\lambda x+\mu y+\nu z+\varrho w=0$$

passes through a point

$$x:y:z:w=2:-\frac{1}{\beta^2}:-\frac{1}{\gamma^2}:-\frac{2}{\beta}-\frac{2}{\gamma}.$$

The bisector passes through this point and the line of intersection of the determinators; its equation is

$$\frac{1}{\beta}\left(\frac{1}{2\beta}x-\beta y+\frac{1}{2\beta}z+w\right)-\frac{1}{\gamma}\left(\frac{1}{2\gamma}x+\frac{1}{2\gamma}y-\gamma z+w\right)=0;$$

or reducing and completing the system, the equations of the bisectors are

$$\left(\frac{1}{2\beta^2}-\frac{1}{2\gamma^2}\right)x-\left(1+\frac{1}{2\gamma^2}\right)y+\left(1+\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)w=0,$$

$$\left(1+\frac{1}{2\gamma^2}\right)x+\left(\frac{1}{2\gamma^2}-\frac{1}{2\alpha^2}\right)y-\left(1+\frac{1}{2\alpha^2}\right)z+\left(\frac{1}{\gamma}-\frac{1}{\alpha}\right)w=0,$$

$$-\left(1+\frac{1}{2\beta^2}\right)x+\left(1+\frac{1}{2\alpha^2}\right)y+\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)w=0.$$

And in order to verify the geometrical construction, it only remains to show that each bisector touches two tactors. Consider the bisector and tactor

$$-\left(1+\frac{1}{2\beta^2}\right)x+\left(1+\frac{1}{2\alpha^2}\right)y+\left(\frac{1}{2\alpha^2}-\frac{1}{2\beta^2}\right)z+\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)w=0,$$

$$-2\alpha(a\alpha-\sqrt{-2bc})x+(c+a)y+(b+a)z+(2a\alpha-\sqrt{-2bc})w=0;$$

and represent these for a moment by

$$\lambda x+\mu y+\nu z+\varrho w=0,\quad \lambda' x+\mu' y+\nu' z+\varrho' w=0.$$

If Δ be the same as before, and Δ' the like function of λ', μ', ν', ϱ', also if

$$\Phi=\lambda\lambda'+\mu\mu'+\nu\nu'-(\mu\nu'+\mu'\nu)-(\nu\lambda'+\nu'\lambda)-(\lambda\mu'+\lambda'\mu)-2\varrho\varrho',$$

then

$$\Delta^2=\left(2+\frac{1}{\alpha\beta}\right)^2,$$

$$\Delta'^2=(2a\alpha^2-2\alpha\sqrt{-2bc}+b+c)^2,$$

$$\Phi=a\alpha^2\left(2+\frac{1}{\alpha\beta}\right)^2-2\alpha\sqrt{-2bc}\left(2+\frac{1}{\alpha\beta}\right)+c\left(2+\frac{1}{\beta^2}\right);$$

and the condition of contact $\Delta\Delta'=\Phi$ (taking the proper sign for the radicals) becomes

$$\left(2+\frac{1}{\alpha\beta}\right)(2a\alpha^2-2\alpha\sqrt{-2bc}+b+c)=a\alpha^2\left(2+\frac{1}{\alpha\beta}\right)^2-2\alpha\sqrt{-2bc}\left(2+\frac{1}{\alpha\beta}\right)+c\left(2+\frac{1}{\beta^2}\right);$$

or reducing,

$$a\alpha-b\beta+c\frac{\alpha-\beta}{2\alpha\beta+1}=0,$$

an equation which is evidently not altered by the interchange of a, α and b, β. The conditions, in order that each bisector may touch two tactors, reduce themselves to the three equations,

$$a\alpha-b\beta+c\frac{\alpha-\beta}{2\alpha\beta+1}=0,$$

$$a\frac{\beta-\gamma}{2\beta\gamma+1}+b\beta-c\gamma=0,$$

$$-a\alpha+b\frac{\gamma-\alpha}{2\alpha\gamma+1}+c\gamma=0,$$

which are satisfied by the values above found for the quantities a, b, c. The possibility and truth of the geometrical construction are thus demonstrated.

§ 4.

Let it be in the first instance proposed to find the equation of a section touching all or any of the sections $x=0$, $y=0$, $z=0$ of the surface of the second order,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+pw^2=0.$$

Any section whatever of this surface may be written in the form

$$(a\lambda+h\mu+g\nu)x+(h\lambda+b\mu+f\nu)y+(g\lambda+f\mu+c\nu)z+\sqrt{-p}\nabla w=0,$$

where

$$\nabla^2=a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu-K,$$

where λ, μ, ν are indeterminate. And considering any other section represented by a like equation,

$$(a\lambda'+h\mu'+g\nu')x+(h\lambda'+b\mu'+f\nu')y+(g\lambda'+f\mu'+c\nu')z+\sqrt{-p}\nabla'w=0,$$

where

$$\nabla'=a\lambda'^2+b\mu'^2+c\nu'^2+2f\mu'\nu'+2g\nu'\lambda'+2h\lambda'\mu'-K,$$

it may be shown by means of the lemma previously given, that the condition of contact is

$$a\lambda\lambda'+b\mu\mu'+c\nu\nu'+f(\mu\nu'+\mu'\nu)+g(\nu\lambda'+\nu'\lambda)+h(\lambda\mu'+\lambda'\mu)\pm K=\nabla\nabla'.$$

Suppose that λ', μ', ν' satisfy the equations

$$\nabla'=0,$$
$$h\lambda'+b\mu'+f\nu'=0,$$
$$g\lambda'+f\mu'+c\nu'=0,$$

so that the last mentioned section becomes $x=0$; and observing that the first of the above equations may be transformed into

$$a\lambda'+h\mu'+g\nu'=\frac{K}{\lambda'},$$

it is easy to obtain $\lambda'=\sqrt{\mathfrak{A}}$, $\mu'=\frac{\mathfrak{H}}{\sqrt{\mathfrak{A}}}$, $\nu'=\frac{\mathfrak{G}}{\sqrt{\mathfrak{A}}}$. The condition of contact becomes

$$\frac{K}{\sqrt{\mathfrak{A}}}\lambda\pm K=0.$$

And taking the under sign, $\lambda=\sqrt{\mathfrak{A}}$, so that if in the above written equation we establish all or any of the equations $\lambda=\sqrt{\mathfrak{A}}$, $\mu=\sqrt{\mathfrak{B}}$, $\nu=\sqrt{\mathfrak{C}}$, we have the equation of a section touching all or the corresponding sections of the sections

$$x=0,\ y=0,\ z=0.$$

In particular we have for a solution of the problem of tactions, the following equation of the section touching $x=0$, $y=0$, $z=0$, viz.

$$(a\sqrt{\mathfrak{A}}+h\sqrt{\mathfrak{B}}+g\sqrt{\mathfrak{C}})x+(h\sqrt{\mathfrak{A}}+b\sqrt{\mathfrak{B}}+f\sqrt{\mathfrak{C}})y+(g\sqrt{\mathfrak{A}}+f\sqrt{\mathfrak{B}}+c\sqrt{\mathfrak{C}})z$$
$$+\frac{\sqrt{-p}}{\sqrt{K}}\sqrt{2(\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F})(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})}\,w=0.$$

Anticipating the use of a notation the value of which will subsequently appear, or putting

$$\mathrm{f}=\sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F}},\ \mathrm{g}=\sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G}},\ \mathrm{h}=\sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}},\ \mathrm{J}=\sqrt{2}\sqrt[4]{\mathfrak{A}\mathfrak{B}\mathfrak{C}},$$

values which give

$$K^2=-\mathrm{f}^4-\mathrm{g}^4-\mathrm{h}^4+2\mathrm{g}^2\mathrm{h}^2+2\mathrm{h}^2\mathrm{f}^2+2\mathrm{f}^2\mathrm{g}^2-\frac{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2},$$

the equation of the section in question is

$$\frac{\mathrm{f}^2}{\sqrt{\mathfrak{A}}}(-\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2)x+\frac{\mathrm{g}^2}{\sqrt{\mathfrak{B}}}(\mathrm{f}^2-\mathrm{g}^2+\mathrm{h}^2)y+\frac{\mathrm{h}^2}{\sqrt{\mathfrak{C}}}(\mathrm{f}^2+\mathrm{g}^2-\mathrm{h}^2)z+\frac{\mathrm{fgh}\sqrt{-p}\sqrt{K}}{\mathrm{J}}w=0.$$

I proceed to investigate a transformation of the equation for the section with an indeterminate parameter λ, which touches the two sections $y=0$, $z=0$. We have

$$a\nabla^2=(a\lambda+h\mu+g\nu)^2+(\mathfrak{C}\mu^2+\mathfrak{B}\nu^2-2\mathfrak{F}\mu\nu)-\mathfrak{B}\mathfrak{C}+\mathfrak{F}^2;$$

or putting for μ and ν their values $\sqrt{\mathfrak{B}}$, $\sqrt{\mathfrak{C}}$ in the second term,

$$a\nabla^2=(a\lambda+h\mu+g\nu)^2+(\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F})^2;$$

or introducing instead of λ an indeterminate quantity X, such that

$$a\lambda+h\mu+g\nu=(\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F})\mathrm{X},$$

we have

$$a\nabla^2=(\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F})\sqrt{1+\mathrm{X}^2}.$$

Also introducing throughout X instead of λ, and completing the substitution of $\sqrt{\mathfrak{B}}$, $\sqrt{\mathfrak{C}}$ for μ, ν, the equation of the section touching $y=0$, $z=0$, becomes

$$(ax+hy+gz)\mathrm{X}+y\sqrt{\mathfrak{C}}+z\sqrt{\mathfrak{B}}+\sqrt{-ap}\sqrt{1+\mathrm{X}^2}.w=0.$$

And it may be remarked in passing, that this is a very convenient form for the demonstration of the theorem; "If two sections of a surface of the second order touch each other, and are also tangent sections (of the same class) to two fixed sections, then considering the planes through the axis of the fixed sections and the poles of the tangent sections, and also the tangent planes through this axis, the anharmonic ratio of the four planes is independent of the position of the moveable tangent sections;" where by the axis of the fixed sections is to be understood the line joining their poles.

The sections touching $z=0$, $x=0$, and $x=0$, $y=0$, are of course

$$x\sqrt{\mathfrak{C}}+(hx+by+fz)\mathrm{Y}+z\sqrt{\mathfrak{A}}+\sqrt{-bp}\sqrt{1+\mathrm{Y}^2}.w=0$$
$$x\sqrt{\mathfrak{B}}+y\sqrt{\mathfrak{A}}+(gx+fy+cz)\mathrm{Z}+\sqrt{-cp}\sqrt{1+\mathrm{Z}^2}.w=0,$$

where

$$h\lambda'+b\mu'+f\nu'=(\sqrt{\mathfrak{C}\mathfrak{A}}-\mathfrak{G})\mathrm{Y},\quad \lambda'=\sqrt{\mathfrak{A}},\ \mu'=\mu',\ \nu'=\sqrt{\mathfrak{C}}$$
$$g\lambda''+f\mu''+c\nu''=(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\mathrm{Z},\quad \lambda''=\sqrt{\mathfrak{A}},\ \mu''=\sqrt{\mathfrak{B}},\ \nu''=\nu''.$$

The conditions of contact of the sections represented by the above written equations would be perhaps most simply obtained directly from the lemma, but it is proper to deduce it from the formula for contact used in the present memoir. If for shortness

$$\Phi(\pm)=a\lambda'\lambda''+b\mu'\mu''+c\nu'\nu''+f(\mu'\nu''+\mu''\nu')+g(\nu'\lambda''+\nu''\lambda')+h(\lambda'\mu''+\lambda''\mu')\pm\mathrm{K},$$

where the symbol Φ $(\pm)$ is used in order to mark the essentially different character of the results corresponding to the different values of the ambiguous sign, then

$$\begin{aligned} bc\Phi(-)=&f(h\lambda'+b\mu'+f\nu')(g\lambda''+f\mu''+c\nu''),\\ &+(\mathfrak{A}\nu'-\mathfrak{G}\lambda')(g\lambda''+f\mu''+c\nu''),\\ &+(\mathfrak{A}\mu''-\mathfrak{H}\lambda'')(h\lambda'+b\mu'+f\nu'),\\ &+\nu'\mu''(-\mathfrak{A}f)+\nu'\lambda''f\mathfrak{H}+\lambda'\mu''f\mathfrak{G}+\lambda'\lambda''(\mathrm{K}-f\mathfrak{F})\\ &-\mathfrak{A}\mathrm{K}-f^2\mathrm{K}.\end{aligned}$$

$$\begin{aligned}
&=f(h\lambda'+b\mu'+f\nu')(g\lambda''+f\mu''+c\nu'')\\
&\quad+\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(g\lambda''+f\mu''+c\nu'')\\
&\quad+\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})(h\lambda'+b\mu'+f\nu')\\
&\quad+f\big(-\mathfrak{A}\sqrt{\mathfrak{B}\mathfrak{C}}+\mathfrak{H}\sqrt{\mathfrak{C}\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{A}\mathfrak{F}-(\mathfrak{G}\mathfrak{H}-\mathfrak{A}\mathfrak{F})\big)\\
&=f(h\lambda'+b\mu'+f\nu')(g\lambda''+f\mu''+c\nu'')\\
&\quad+\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(g\lambda''+f\mu''+c\nu'')\\
&\quad+\sqrt{\mathfrak{A}}(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})(h\lambda'+b\mu'+f\nu')\\
&\quad-f(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}),
\end{aligned}$$

i. e.

$$bc\Phi(-)=(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\{f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})-f\}.$$

What, however, is really required*, is the value of $\Phi(+)$; to find this,

$$\begin{aligned}
bc\Phi(+)&=bc\Phi(-)+2bc\mathrm{K}\\
&=(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\{f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f\}\\
&\quad+2bc\mathrm{K}-2f(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}),
\end{aligned}$$

the second line of which is

$$\begin{aligned}
&2(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\left\{\frac{bc\mathrm{K}}{\mathrm{K}^2bc}(\sqrt{\mathfrak{A}\mathfrak{C}}+\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}+\mathfrak{H})-f\right\}\\
&\quad=\frac{2(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})}{\mathrm{K}}\{(\sqrt{\mathfrak{A}\mathfrak{C}}+\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}+\mathfrak{H})-\mathfrak{G}\mathfrak{H}+\mathfrak{A}\mathfrak{F}\}\\
&\quad=2(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{\mathfrak{A}}\,\theta,
\end{aligned}$$

where

$$\theta=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{A}\mathfrak{B}\mathfrak{C}}+\mathfrak{F}\sqrt{\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{B}}+\mathfrak{H}\sqrt{\mathfrak{C}});$$

and consequently

$$bc\Phi(+)=(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\{f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f+2\theta\sqrt{\mathfrak{A}}\},$$

a reduction, which on account of its peculiarity, I have thought right to work out in full.

The condition of contact is

$$\Phi(+)=\nabla'\nabla''=\frac{1}{\sqrt{bc}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}.$$

* It may be shown without difficulty that the $(-)$ sign would imply that the sections touching $z=0$, $x=0$ and $x=0$, $y=0$ were sections touching $x=0$ at the same point. By taking the $(-)$ sign in each equation we should have the solution of the problem "to determine three sections of a surface of the second order, the two sections of each pair touching one of three given sections at the same point," which is not without interest; the solution may be completed without any difficulty.

Or finally, the condition in order that the sections

$$x\sqrt{\mathfrak{C}}+(hx+by+fz)\mathrm{Y}+z\sqrt{\mathfrak{A}}+\sqrt{-bp}\sqrt{1+\mathrm{Y}^2}w=0$$

$$x\sqrt{\mathfrak{B}}+y\sqrt{\mathfrak{A}}+(gx+fy+cz)\mathrm{Z}+\sqrt{-cp}\sqrt{1+\mathrm{Z}^2}w=0$$

(the former of which is a section touching $z=0$, $x=0$, and the latter a section touching $x=0$, $y=0$) may touch, is

$$f\mathrm{YZ}+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+(f+2\theta\sqrt{\mathfrak{A}})-\sqrt{bc}\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0.$$

The preceding researches show that the solution of STEINER's extension of MALFATTI's problem depends on a system of equations, such as the system mentioned at the commencement of the following section.

§ 5.

Consider the system of equations

$$\alpha+\beta(\mathrm{Y}+\mathrm{Z})+\gamma\mathrm{YZ}+\delta\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0$$

$$\alpha'+\beta'(\mathrm{Z}+\mathrm{X})+\gamma'\mathrm{ZX}+\delta'\sqrt{1+\mathrm{Z}^2}\sqrt{1+\mathrm{X}^2}=0$$

$$\alpha''+\beta''(\mathrm{X}+\mathrm{Y})+\gamma''\mathrm{XY}+\delta''\sqrt{1+\mathrm{X}^2}\sqrt{1+\mathrm{Y}^2}=0.$$

These equations may, it will be seen, be solved by quadratics only, when the coefficients satisfy the relations

$$\frac{\beta}{\gamma-\alpha}=\frac{\beta'}{\gamma'-\alpha'}=\frac{\beta''}{\gamma''-\alpha''}$$

$$\frac{\beta^2+\gamma^2-\delta^2}{\gamma^2-\alpha^2}=\frac{\beta'^2+\gamma'^2-\delta'^2}{\gamma'^2-\alpha'^2}=\frac{\beta''^2+\gamma''^2-\delta''^2}{\gamma''^2-\alpha''^2},$$

equations which it should be remarked are satisfied by

$$\beta=0,\ \beta'=0,\ \beta''=0,\ \gamma=\delta,\ \gamma'=\delta',\ \gamma''=\delta''.$$

Or if we write

$$\frac{\alpha}{\gamma}=-l,\ \frac{\alpha'}{\gamma'}=-m,\ \frac{\alpha''}{\gamma''}=-n,$$

the equations become by a simple reduction,

$$\mathrm{Y}^2+\mathrm{Z}^2+2l\mathrm{YZ}=l^2-1$$

$$\mathrm{Z}^2+\mathrm{X}^2+2m\mathrm{ZX}=m^2-1$$

$$\mathrm{X}^2+\mathrm{Y}^2+2n\mathrm{XY}=n^2-1,$$

which are equivalent to the equations discussed in my paper "On a System of Equations connected with MALFATTI's Problem and on another Algebraical System," Cambridge and Dublin Mathematical Journal, t. iv. p. 270; the solution might have been effected by the direct method, which I shall here adopt, of eliminating any one of the variables between the two equations into which it enters, and combining the result with the third equation.

Writing the second and third equations under the form

$$A'+B'X+C'\sqrt{1+X^2}=0$$

$$A''+B''X+C''\sqrt{1+X^2}=0,$$

the result of the elimination may be presented in the form

$$A'A''+B'B''-C'C''=\sqrt{A'^2+B'^2-C'^2}\sqrt{A''^2+B''^2-C''^2},$$

which is most easily obtained by writing $X=\tan\phi$ and operating with the symbol $\cos^{-1}$; but if the rationalized equations be represented by

$$\lambda'+2\mu'X+\nu'X^2=0 \text{ and } \lambda''+2\mu''X+\nu''X^2=0,$$

the form

$$4(\lambda'\nu'-\mu'^2)(\lambda''\nu''-\mu''^2)=(\lambda'\nu''+\lambda''\nu'-2\mu'\mu'')^2$$

leads easily to the result in question. The values which enter are

$$A'=\alpha'+\beta'Z \qquad A''=\alpha''+\beta''Y$$

$$B'=\beta'+\gamma'Z \qquad B''=\beta''+\gamma''Y$$

$$C'=\delta'\sqrt{1+Z^2} \quad C''=\delta''\sqrt{1+Y^2};$$

whence, in the first place, by the equation connecting Y, Z,

$$C'C''=-\frac{\delta'\delta''}{\delta}\{\alpha+\beta(Y+Z)+\delta YZ\}.$$

It is obviously convenient that $A'A''+B'B''$ should be symmetrical with respect to Y and Z, and this will be the case if

$$\alpha'\beta''+\beta'\gamma''=\alpha''\beta'+\beta''\gamma', \quad i.e. \text{ if } \beta'(\gamma''-\alpha'')=\beta''(\gamma'-\alpha').$$

Or assuming that the equations are symmetrically related to the system, we have the first set of relations between the coefficients, relations which are satisfied by

$$\alpha=\gamma+2\phi\beta,\ \alpha'=\gamma'+2\phi\beta',\ \alpha''=\gamma''+2\phi\beta'',$$

and the values of α', α', α'' will be considered henceforth as given by these conditions. We have

$$A'A''+B'B''-C'C''=\alpha'\alpha''+\beta'\beta''+(\gamma'\beta''+\gamma''\beta'+2\phi\beta'\beta'')(Y+Z)+(\beta'\beta''+\gamma'\gamma'')YZ$$
$$+\frac{\delta'\delta''}{\delta}\{\alpha+\beta(Y+Z)+\gamma YZ\}.$$

The quantities $A'^2+B'^2-C'^2$, $A''^2+B''^2-C''$ are quadratic functions of Z and Y respectively, and with proper relations between the coefficients, we may assume

$$(A'^2+B'^2-C'^2)(A''^2+B''^2-C''^2)=l^2s^2\{U^2+k[(\alpha+\beta(Y+Z)+\gamma YZ)^2-\delta^2(1+Y^2)(1+Z^2)]\},$$

in which U is a linear function of $Y+Z$ and YZ, and k and ls are constants. The first side must, in the first place, be symmetrical with respect to Y and Z, or

$$\alpha'^2+\beta'^2-\delta'^2,\ (\alpha'+\gamma')\beta',\ \beta'^2+\gamma'^2-\delta'^2$$

must be proportional to

$$\alpha''^2+\beta''^2-\delta''^2,\ (\alpha''+\gamma'')\delta'',\ \beta''^2+\gamma''^2-\delta''^2.$$

But since

$$(\alpha'+\gamma')\beta',\ (\alpha''+\gamma'')\beta''$$

are proportional to

$$\gamma'^2-\alpha'^2,\ \gamma''^2-\alpha''^2,$$

it is only necessary that

$$\beta'^2+\gamma'^2-\delta'^2,\ \beta''^2+\gamma''^2-\delta''$$

should be proportional to

$$\gamma'^2-\alpha'^2,\ \gamma''^2-\alpha''^2.$$

Or since the equations are supposed symmetrically related to the system, we must have the second set of relations between the coefficients. Suppose

$$\frac{\beta^2+\gamma^2-\delta^2}{\gamma^2-\alpha^2}=\frac{\beta'^2+\gamma'^2-\delta'^2}{\gamma'^2-\alpha'^2}=\frac{\beta''^2+\gamma''^2-\delta''^2}{\gamma''^2-\alpha''^2}=-\frac{s}{\varphi},$$

then since

$$\gamma^2-\alpha^2=-4(\gamma+\varphi\beta)\varphi\beta,\ \&c.,$$

we have

$$\delta^2=\beta^2+\gamma^2-4s(\gamma+\varphi\beta)\beta$$
$$\delta'^2=\beta'^2+\gamma'^2-4s(\gamma'+\varphi\beta')\beta'$$
$$\delta''^2=\beta''^2+\gamma''^2-4s(\gamma''+\varphi\beta'')\beta'',$$

and δ, δ', δ'' will be supposed henceforth to satisfy these equations.

We have next

$$A'^2+B'^2-C'^2=4(\gamma'+\varphi\beta')\beta'(s+\varphi+Z+sZ^2)$$
$$A''^2+B''^2-C''^2=4(\gamma''+\varphi\beta'')\beta''(s+\varphi+Y+sY^2),$$

which may be simplified by writing

$$s=\frac{\mu-\varphi}{1+\mu^2},\qquad \nu=\frac{1+\mu\varphi}{\mu-\varphi},$$

where μ, ν are to be considered as given functions of s and φ. These values give

$$A'^2+B'^2-C'^2=4(\gamma'+\varphi\beta')\beta's(Z+\mu)(Z+\nu)$$
$$A''^2+B''^2-C''^2=4(\gamma''+\varphi\beta'')\beta''s\,(Y+\mu)(Y+\nu).$$

Hence, putting for simplicity

$$l^2=4(\gamma'+\varphi\beta')(\gamma''+\varphi\beta'')\beta'\beta'',$$

we have

$$4(Z+\mu)(Z+\nu)(Y+\mu)(Y+\nu)=U^2+k[(\alpha+\beta(Y+Z)+\gamma YZ)^2-\delta^2(1+Y^2)(1+Z^2)].$$

And the two sides have next to be expressed in terms of $Y+Z$ and YZ.

If for symmetry we write

$$\xi=1,\ \eta=Y+Z,\ \zeta=YZ,$$

then

$$4(\mu^2\xi+\mu\eta+\zeta)(\nu^2\xi+\nu\eta+\zeta)+k\delta^2[(\xi-\zeta)^2+\eta^2]=U^2+k(\alpha\xi+\beta\eta+\gamma\zeta)^2.$$

And U is now to be considered a linear function of ξ, η, ζ.

The condition that the first side of the equation may divide into factors, gives an equation for determining k; since the condition is satisfied for $k=0$ and $k=\infty$, the

equation will be linear, and it is easily seen that the value is $k=\frac{1}{\delta^2}(\mu-\nu)^2$. In fact

$$4(\mu^2\xi+\mu\eta+\zeta)(\nu^2\xi+\nu\eta+\zeta)^2+(\mu-\nu)^2[(\xi-\zeta)^2+\eta^2]$$
$$=(2\mu\nu\xi+(\mu+\nu)\eta+2\zeta)^2+(\mu-\nu)^2(\xi+\zeta)^2.$$

Hence

$$\{2\mu\nu\xi+(\mu+\nu)\eta+2\zeta\}^2-\mathrm{U}^2=\frac{(\mu-\nu)^2}{\delta^2}\{(\alpha\xi+\beta\eta+\gamma\zeta)^2-\delta^2(\xi+\zeta)^2\}.$$

And we may assume

$$2\mu\nu\xi+(\mu+\nu)\eta+2\zeta+\mathrm{U}=\frac{\mu-\nu}{\delta}\Lambda\{(\alpha\xi+\beta\eta+\gamma\zeta)-\delta(\xi+\zeta)\}$$

$$2\mu\nu\xi+(\mu+\nu)\eta+2\zeta-\mathrm{U}=\frac{\mu-\nu}{\delta}\frac{1}{\Lambda}\{(\alpha\xi+\beta\eta+\gamma\zeta)+\delta(\xi+\zeta)\},$$

subject to its being shown that

$$4\mu\nu\xi+2(\mu+\nu)\eta+4\zeta=\frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)(\alpha\xi+\beta\eta+\gamma\zeta)-\delta\left(\Lambda-\frac{1}{\Lambda}\right)(\xi+\zeta)\right\}$$

gives a constant value for Λ. The comparison of coefficients gives

$$4\mu\nu=\frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)\alpha-\left(\Lambda-\frac{1}{\Lambda}\right)\delta\right\}$$

$$2\mu+2\nu=\frac{\mu-\nu}{\delta}\left(\Lambda+\frac{1}{\Lambda}\right)\beta$$

$$4=\frac{\mu-\nu}{\delta}\left\{\left(\Lambda+\frac{1}{\Lambda}\right)\gamma-\left(\Lambda-\frac{1}{\Lambda}\right)\delta\right\},$$

the first and third of which give

$$4(1-\mu\nu)=\frac{\mu-\nu}{\delta}\left(\Lambda+\frac{1}{\Lambda}\right)(\gamma-\alpha),$$

which will be identical with the second, if

$$\frac{2(1-\mu\nu)}{\mu+\nu}=\frac{\beta}{\gamma-\alpha}=-2\phi,$$

which follows at once from the equation

$$\nu=\frac{1+\mu\phi}{\mu-\phi}.$$

Forming next the two equations

$$\Lambda+\frac{1}{\Lambda}=\frac{2}{(\mu-\nu)\beta}(\mu+\nu)\delta$$

$$\Lambda-\frac{1}{\Lambda}=\frac{2}{(\mu-\nu)\beta}\{(\mu+\nu)\gamma-2\beta\}$$

these will be equivalent to a single equation if

$$(\mu+\nu)^2\delta^2=\{(\mu+\nu)\gamma-2\beta\}^2+(\mu-\nu)^2\beta^2,$$

i. e. if

$$(\mu+\nu)^2\delta^2=(\mu+\nu)^2(\beta^2+\gamma^2)-4(\mu+\nu)\beta\gamma-4(\mu\nu-1)\beta^2;$$

or finally, if

$$\delta^2=\beta^2+\gamma^2-4s\beta\left(\gamma+\frac{\mu\nu-1}{\mu+\nu}\beta\right)=\beta^2+\gamma^2-4s(\gamma+\phi\beta)\beta,$$

which is in fact the case.

Writing the equations for $\Lambda+\frac{1}{\Lambda},\ \Lambda-\frac{1}{\Lambda}$

in the form

$$\Lambda+\frac{1}{\Lambda}=\frac{2\delta}{(\mu-\nu)\beta s}$$

$$\Lambda-\frac{1}{\Lambda}=\frac{2}{(\mu-\nu)\beta s}(\gamma-2\beta s),$$

and substituting in $\mathrm{U}=\frac{\mu-\nu}{2\delta}\left\{\left(\Lambda-\frac{1}{\Lambda}\right)(\alpha\xi+\beta\eta+\gamma\zeta)-\left(\Lambda+\frac{1}{\Lambda}\right)\delta^2(\xi+\zeta)\right\}$,

we have

$$\mathrm{U}=\frac{1}{s\beta\delta}\{(\gamma-2\beta s)(\alpha\xi+\beta\eta+\gamma\zeta)-\delta^2(\xi+\zeta)\}$$

$$=\frac{1}{s\beta\delta}\{(-\beta+2s\gamma+2\phi\gamma)\xi+(\gamma-2s\beta)\eta+(-\beta+2s\gamma+4s\phi\beta)\zeta\}.$$

And consequently, multiplying by

$$ls=2\sqrt{(\gamma'+\phi\beta')(\gamma''+\phi\beta'')\beta'\beta''}$$

we have

$$\sqrt{\mathrm{A}'^2+\mathrm{B}'^2-\mathrm{C}'^2}\sqrt{\mathrm{A}''^2+\mathrm{B}''^2-\mathrm{C}''^2}$$

$$=\frac{2}{\delta}\sqrt{(\gamma'+\phi\beta')(\gamma''+\phi\beta'')\beta'\beta''}\{(-\beta+2s\gamma+2\phi\gamma)\xi+(\gamma-2s\beta)\eta+(-\beta+2s\gamma+4s\phi\beta)\zeta\},$$

or collecting the different terms

$$(\alpha'\alpha''+\beta'\beta'')\xi+(\gamma'\beta''+\gamma''\beta'+2\phi\beta'\beta'')\eta+(\beta'\beta''+\gamma'\gamma'')\zeta+\frac{\delta'\delta''}{\delta}(\alpha\xi+\beta\eta+\gamma\zeta)$$

$$-\frac{2}{\delta}\sqrt{(\gamma'+2\phi\beta')(\gamma''+2\phi\beta'')\beta'\beta''}\{(-\beta+2s\gamma+2\phi\gamma)\xi+(\gamma-2s\beta)\eta+(-\beta+2s\gamma+4s\phi\beta)\zeta\}=0,$$

which, combined with the first equation written under the form

$$(\alpha\xi+\beta\eta+\gamma\zeta)^2-\delta^2[(\xi-\zeta)^2+\eta^2]=0,$$

determines the ratios of ξ, η, ζ, *i. e.* the values of $\mathrm{Y}+\mathrm{Z}$ and YZ.

§ 6.

The system of equations

$$(f+2\theta\sqrt{\mathfrak{A}})+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f\mathrm{YZ}-\sqrt{bc}\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0$$

$$(g+2\theta\sqrt{\mathfrak{B}})+\sqrt{\mathfrak{B}}(\mathrm{Z}+\mathrm{X})+g\mathrm{ZX}-\sqrt{ca}\sqrt{1+\mathrm{Z}^2}\sqrt{1+\mathrm{X}^2}=0$$

$$(h+2\theta\sqrt{\mathfrak{C}})+\sqrt{\mathfrak{C}}(\mathrm{X}+\mathrm{Y})+h\mathrm{XY}-\sqrt{ab}\sqrt{1+\mathrm{X}^2}\sqrt{1+\mathrm{Y}^2}=0,$$

where

$$\theta=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{ABC}}+\mathfrak{F}\sqrt{\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{B}}+\mathfrak{H}\sqrt{\mathfrak{C}}),$$

on which depends the solution of STEINER's extension of MALFATTI's problem, is at once seen to belong to the class of equations treated of in the preceding section, and

we have $\phi=\theta$, $s=0$. The equations at the conclusion of the preceding section become

$$\{\sqrt{\mathfrak{BC}}+gh+2\theta(g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}})+4\theta^2\sqrt{\mathfrak{BC}}\}\xi+\{g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}}+2\theta\sqrt{\mathfrak{BC}}\}\eta+\{\sqrt{\mathfrak{BC}}+gh\}\zeta$$

$$-a[(f+2\theta\sqrt{\mathfrak{A}})\xi+\sqrt{\mathfrak{A}}\eta+f\zeta]-\frac{2}{\sqrt{bc}}\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{BC}}}\{(\sqrt{\mathfrak{A}}-2\theta f)\xi-f\eta+\sqrt{\mathfrak{A}}\zeta\}=0$$

$$\{(f+2\theta\sqrt{\mathfrak{A}})\xi+\sqrt{\mathfrak{A}}\eta+f\zeta\}^2-bc\{(\xi-\zeta)^2+\eta^2\}=0,$$

which may also be written

$$(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\xi+\zeta)+(-a\sqrt{\mathfrak{A}}+g\sqrt{\mathfrak{C}}+h\sqrt{\mathfrak{B}}+2\theta\sqrt{\mathfrak{BC}})(\eta+2\theta\xi)$$

$$-\frac{2}{\sqrt{bc}}\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{BC}}}\left((\sqrt{\mathfrak{A}}-2\theta f)\xi-f\eta+\sqrt{\mathfrak{A}}\zeta\right)=0.$$

$$\{f(\xi+\zeta)+\sqrt{\mathfrak{A}}(\eta+2\theta\xi)\}^2-bc\{(\xi-\zeta)^2+\eta^2\}=0.$$

Or observing that

$$g+\theta\sqrt{\mathfrak{B}}=\frac{1}{K}(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\sqrt{\mathfrak{AB}}+\mathfrak{H});\quad h+\theta\sqrt{\mathfrak{C}}=\frac{1}{K}(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\sqrt{\mathfrak{AC}}+\mathfrak{G})$$

$$-a\sqrt{\mathfrak{A}}+h\sqrt{\mathfrak{B}}+g\sqrt{\mathfrak{C}}+2\theta\sqrt{\mathfrak{BC}}=\theta(\sqrt{\mathfrak{BC}}+\mathfrak{F}),$$

and putting for a moment

$$\lambda=\frac{1}{K}\sqrt{(\sqrt{\mathfrak{AC}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H})\sqrt{\mathfrak{BC}}},$$

and therefore

$$\sqrt{(g+\theta\sqrt{\mathfrak{B}})(h+\theta\sqrt{\mathfrak{C}})\sqrt{\mathfrak{BC}}}=(\sqrt{\mathfrak{BC}}+\mathfrak{F})\lambda,$$

the first equation divides by $(\sqrt{\mathfrak{BC}}+\mathfrak{F})$, and the result is

$$(\xi+\zeta)+\theta(\eta+2\theta\xi)-\frac{2\lambda}{\sqrt{bc}}\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}=0.$$

And by an easy transformation the second equation becomes

$$-\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}^2+4bc\zeta\left((\xi+\zeta)+\theta(\eta+2\theta\xi)-(1+\theta^2)\xi\right)=0.$$

Or putting

$$\xi+\zeta+\theta(\eta+2\theta\xi)=\Theta$$

$$\frac{1}{\sqrt{bc}}\left(\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\right)=\Phi$$

$$\zeta \qquad\qquad =\Psi,$$

the equations become

$$\Theta-2\lambda\Phi=0$$

$$-\Phi^2+4\Psi\{\Theta-(1+\theta^2)\Psi\}=0.$$

Whence eliminating Φ,

$$\left(2\Psi-\frac{\Theta}{1+\theta^2}\right)^2=\frac{\Theta^2}{(1+\theta^2)^2}\left(1-\frac{1+\theta^2}{4\lambda^2}\right),$$

or observing that

$$1+\theta^2=\frac{1}{K^2}(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\sqrt{\mathfrak{CA}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H}),$$

and reducing,

$$\Psi=\frac{K^2\Theta}{(\sqrt{\mathfrak{BC}}+\mathfrak{F})(\sqrt{\mathfrak{CA}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H})}\left(1+\frac{\sqrt{(\sqrt{\mathfrak{BC}}-\mathfrak{F})\sqrt{\mathfrak{BC}}}}{\sqrt{2\mathfrak{BC}}}\right).$$

Also $\Theta=2\lambda\Phi$ gives

$$\Phi=\frac{K\Theta}{2\sqrt{(\sqrt{\mathfrak{AC}}+\mathfrak{G})(\sqrt{\mathfrak{AB}}+\mathfrak{H})\sqrt{\mathfrak{BC}}}}.$$

Suppose

$$\sqrt{\mathfrak{BC}}+\mathfrak{F}=\alpha,\ \sqrt{\mathfrak{BC}}-\mathfrak{F}=\alpha_{,}\quad\therefore\ \alpha\alpha_{,}=\mathrm{K}a$$

$$\sqrt{\mathfrak{CA}}+\mathfrak{G}=\beta,\ \sqrt{\mathfrak{CA}}-\mathfrak{G}=\beta_{,}\qquad \beta\beta_{,}=\mathrm{K}b$$

$$\sqrt{\mathfrak{AB}}+\mathfrak{H}=\gamma,\ \sqrt{\mathfrak{AB}}-\mathfrak{H}=\gamma_{,}\qquad \gamma\gamma_{,}=\mathrm{K}c;$$

then substituting

$$\Theta-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{bc}\,\Phi=0$$

$$\Theta-\frac{4bc}{\beta_{,}\gamma_{,}}(\alpha+\alpha_{,})\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)\Psi=0,$$

that is,

$$\xi+\zeta+\theta(\eta+2\theta\xi)-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\{\sqrt{\mathfrak{A}}(\xi+\zeta)-f(\eta+2\theta\xi)\}=0$$

$$\xi+\zeta+\theta(\eta+2\theta\xi)-\frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)\zeta=0,$$

which may be written

$$\mathrm{L}\xi+\mathrm{M}\eta+\mathrm{N}\zeta=0$$

$$\mathrm{L}'\xi+\mathrm{M}'\eta+\mathrm{N}'\zeta=0,$$

where

$$\mathrm{L}=1+2\theta^2-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}(\sqrt{\mathfrak{A}}-2\theta f),\ \mathrm{M}=\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}f,\ \mathrm{N}=1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{\mathfrak{A}},$$

$$\mathrm{L}'=1+2\theta^2-\frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right),\quad \mathrm{M}'=\theta\qquad,\ \mathrm{N}'=1.$$

Or since ξ, η, ζ are equal to 1, Y+Z, YZ respectively,

$$1:\mathrm{Y}+\mathrm{Z}:\mathrm{YZ}=\mathrm{MN}'-\mathrm{M}'\mathrm{N}:\mathrm{NL}'-\mathrm{N}'\mathrm{L}:\mathrm{LM}'-\mathrm{L}'\mathrm{M}$$

$$=-\frac{\sqrt{2}(\alpha+\alpha_{,})}{\sqrt{\beta_{,}\gamma_{,}}}(f+\theta\sqrt{\mathfrak{A}})$$

$$:\frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}\sqrt{\mathfrak{A}}\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}(f+\theta\sqrt{\mathfrak{A}})2\theta$$

$$:-\frac{4bc(\alpha+\alpha_{,})}{\beta_{,}\gamma_{,}}\left(\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}f\right)\left(1-\frac{\sqrt{\alpha_{,}}}{\sqrt{\alpha+\alpha_{,}}}\right)+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{,}}}{\sqrt{\beta_{,}\gamma_{,}}}(f+\theta\sqrt{\mathfrak{A}}).$$

Also

$$f+\theta\sqrt{\mathfrak{A}}=\frac{\beta\gamma}{\mathrm{K}}=\frac{\mathrm{K}bc}{\beta_{,}\gamma_{,}},$$

whence

$$Y+Z=-\frac{2\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}\sqrt{\beta_{\prime}\gamma_{\prime}}}{K}\left(1-\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}\sqrt{\mathfrak{A}}\right)\left(1-\frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}\right)-2\theta$$

$$YZ=\frac{2\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}\sqrt{\beta_{\prime}\gamma_{\prime}}}{K}\left(\theta+\frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}f}{\sqrt{\beta_{\prime}\gamma_{\prime}}}\right)\left(1-\frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}\right)-1.$$

And by forming the analogous expressions for $Z+X$ and ZX, $X+Y$ and XY, the values of X, Y, Z may be determined. But the equations in question simplify themselves in a remarkable manner by the notation before alluded to.

Suppose

$$\mathrm{f}=\sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{B}\mathfrak{C}}-\mathfrak{F}},\ \mathrm{g}=\sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{C}\mathfrak{A}}-\mathfrak{G}},\ \mathrm{h}=\sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}},\ \mathrm{J}=\sqrt{2}\sqrt[4]{\mathfrak{A}\mathfrak{B}\mathfrak{C}},$$

these values give

$$\begin{aligned}
\frac{K\sqrt{\mathfrak{A}}}{\sqrt{\mathfrak{B}\mathfrak{C}}}a &= 2\mathrm{f}^2\left(1-\frac{\mathrm{f}^2}{\mathrm{J}^2}\right)\\
\frac{K\sqrt{bc}}{\sqrt{\mathfrak{A}}} &= 2\mathrm{gh}\sqrt{1-\frac{\mathrm{g}^2}{\mathrm{J}^2}}\sqrt{1-\frac{\mathrm{h}^2}{\mathrm{J}^2}}\\
\frac{Kf}{\sqrt{\mathfrak{A}}} &= \mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+\frac{2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\\
&\vdots\\
K\theta &= -\mathrm{f}^2-\mathrm{g}^2-\mathrm{h}^2+2\mathrm{J}^2\\
K^2 &= -\mathrm{f}^4-\mathrm{g}^4-\mathrm{h}^4+2\mathrm{g}^2\mathrm{h}^2+2\mathrm{h}^2\mathrm{f}^2+2\mathrm{f}^2\mathrm{g}^2-\frac{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}.
\end{aligned}$$

Applying these results to the preceding formulæ and forming for that purpose the equations

$$2\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}\sqrt{\beta_{\prime}\gamma_{\prime}}=4\mathrm{gh},\quad \frac{\sqrt{2}\sqrt{\alpha+\alpha_{\prime}}}{\sqrt{\beta_{\prime}\gamma_{\prime}}}=\frac{\mathrm{J}^2}{\sqrt{\mathfrak{A}}\mathrm{gh}},\quad \frac{\sqrt{\alpha_{\prime}}}{\sqrt{\alpha+\alpha_{\prime}}}=\frac{\mathrm{f}}{\mathrm{J}}$$

$$\mathrm{gh}K\theta+\frac{\mathrm{J}^2}{\sqrt{\mathfrak{A}}}Kf=(\mathrm{J}^2-\mathrm{gh})\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)-2\mathrm{gh}(\mathrm{g}-\mathrm{h})^2,$$

we have

$$\begin{aligned}
K(Y+Z)+2K\theta &= 4(\mathrm{J}^2-\mathrm{gh})\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\\
K^2YZ+K^2 &= 4\{(\mathrm{J}^2-\mathrm{gh})\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)-2\mathrm{gh}(\mathrm{g}-\mathrm{h})^2\}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right);
\end{aligned}$$

the former of which, combined with the similar equations for $Z+X$ and $X+Y$, gives for X, Y, Z the values to be presently stated, and these values will of course verify the second equation and the corresponding equations for ZX and XY.

Recapitulating the preceding notation, if $x=0$, $y=0$, $z=0$ are the equations of the given sections, $w=0$ the equation of the polar plane of their point of intersection with respect to the surface

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+pw^2=0,$$

the equation of the surface, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, K as usual, and

$$\theta=\frac{1}{\mathrm{K}}(\sqrt{\mathfrak{ABC}}+\mathfrak{F}\sqrt{\mathfrak{A}}+\mathfrak{G}\sqrt{\mathfrak{B}}+\mathfrak{H}\sqrt{\mathfrak{C}}),$$

then the equations of the required sections are

$$(ax+hy+gz)\mathrm{X}+y\sqrt{\mathfrak{C}}+z\sqrt{\mathfrak{B}}+\sqrt{-ap}\sqrt{1+\mathrm{X}^2}w=0$$
$$x\sqrt{\mathfrak{C}}+(hx+by+fz)\mathrm{Y}+z\sqrt{\mathfrak{A}}+\sqrt{-bp}\sqrt{1+\mathrm{Y}^2}w=0$$
$$x\sqrt{\mathfrak{B}}+y\sqrt{\mathfrak{A}}+(gx+fy+cz)\mathrm{Z}+\sqrt{-cp}\sqrt{1+\mathrm{Z}^2}w=0,$$

where X, Y, Z are to be determined by the following equations,

$$(f+2\theta\sqrt{\mathfrak{A}})+\sqrt{\mathfrak{A}}(\mathrm{Y}+\mathrm{Z})+f\mathrm{YZ}-\sqrt{bc}\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=0$$
$$(g+2\theta\sqrt{\mathfrak{B}})+\sqrt{\mathfrak{B}}(\mathrm{Z}+\mathrm{X})+g\mathrm{ZX}-\sqrt{ca}\sqrt{1+\mathrm{Z}^2}\sqrt{1+\mathrm{X}^2}=0$$
$$(h+2\theta\sqrt{\mathfrak{C}})+\sqrt{\mathfrak{C}}(\mathrm{X}+\mathrm{Y})+h\mathrm{XY}-\sqrt{ab}\sqrt{1+\mathrm{X}^2}\sqrt{1+\mathrm{Y}^2}=0;$$

and the solution of which, putting

$$\mathfrak{f}=\sqrt[4]{\mathfrak{A}}\sqrt{\sqrt{\mathfrak{BC}}-\mathfrak{F}},\quad \mathfrak{g}=\sqrt[4]{\mathfrak{B}}\sqrt{\sqrt{\mathfrak{CA}}-\mathfrak{G}},\quad \mathfrak{h}=\sqrt[4]{\mathfrak{C}}\sqrt{\sqrt{\mathfrak{AB}}-\mathfrak{H}},\quad \mathrm{J}=\sqrt{2}\sqrt[4]{\mathfrak{ABC}},$$

is given by the equations

$$\mathrm{KX}=\frac{2\mathfrak{fgh}}{\mathrm{J}}+(-\mathfrak{f}+\mathfrak{g}+\mathfrak{h})^2-2(-\mathfrak{f}+\mathfrak{g}+\mathfrak{h})\mathrm{J}$$
$$\mathrm{KY}=\frac{2\mathfrak{fgh}}{\mathrm{J}}+(\mathfrak{f}-\mathfrak{g}+\mathfrak{h})^2-2(\mathfrak{f}-\mathfrak{g}+\mathfrak{h})\mathrm{J}$$
$$\mathrm{KZ}=\frac{2\mathfrak{fgh}}{\mathrm{J}}+(\mathfrak{f}+\mathfrak{g}-\mathfrak{h})^2-2(\mathfrak{f}+\mathfrak{g}-\mathfrak{h})\mathrm{J}\,^*.$$

Instead of the direct but very tedious process by which these values of X, Y, Z have been obtained, we may substitute the following *à posteriori* verification.

We have

$$\mathrm{K}^2(1+\mathrm{X}^2)=4(-\mathfrak{f}+\mathfrak{g}+\mathfrak{h})^2\mathrm{J}^2\left(1+\frac{\mathfrak{f}}{\mathrm{J}}\right)\left(1-\frac{\mathfrak{g}}{\mathrm{J}}\right)\left(1-\frac{\mathfrak{h}}{\mathrm{J}}\right)$$
$$\mathrm{K}^2\sqrt{1+\mathrm{Y}^2}\sqrt{1+\mathrm{Z}^2}=4\left(\mathfrak{f}^2-(\mathfrak{g}-\mathfrak{h})^2\right)\mathrm{J}^2\left(1-\frac{\mathfrak{f}}{\mathrm{J}}\right)\sqrt{1-\frac{\mathfrak{g}^2}{\mathrm{J}^2}}\sqrt{1-\frac{\mathfrak{h}^2}{\mathrm{J}^2}}$$
$$\mathrm{K}^2(1+\mathrm{YZ})=4\left(1-\frac{\mathfrak{f}}{\mathrm{J}}\right)\left\{(\mathrm{J}^2-\mathfrak{gh})\left(\mathfrak{f}^2-(\mathfrak{g}-\mathfrak{h})^2\right)-2\mathfrak{gh}(\mathfrak{g}-\mathfrak{h})^2\right\}$$
$$\mathrm{K}(\mathrm{Y}+\mathrm{Z})-2\mathfrak{f}^2-\mathfrak{g}^2-2\mathfrak{h}^2+4\mathrm{J}^2=4\left(1-\frac{\mathfrak{f}}{\mathrm{J}}\right)(\mathrm{J}^2-\mathfrak{gh}).$$

Putting also

$$\mathfrak{f}^2-\mathfrak{g}^2-\mathfrak{h}^2+\frac{2\mathfrak{g}^2\mathfrak{h}^2}{\mathrm{J}^2}=\left(\mathfrak{f}^2-(\mathfrak{g}-\mathfrak{h})^2\right)-\frac{2\mathfrak{gh}(\mathrm{J}^2-\mathfrak{gh})}{\mathrm{J}^2}$$
$$\mathrm{K}^2=\left(\mathfrak{f}^2-(\mathfrak{g}-\mathfrak{h})^2\right)\left((\mathfrak{g}+\mathfrak{h})^2-\mathfrak{f}^2-\frac{4\mathfrak{g}^2\mathfrak{h}^2}{\mathrm{J}^2}\right)-\frac{4\mathfrak{g}^2\mathfrak{h}^2(\mathfrak{g}-\mathfrak{h})^2}{\mathrm{J}^2},$$

* It is perhaps worth noticing that the value of the quantity λ previously made use of, is

$$\lambda=\frac{\mathfrak{f}^2}{\mathrm{K}a\sqrt{\mathfrak{A}}}\left\{\frac{2\mathfrak{fgh}}{\mathrm{J}}-\mathfrak{g}^2-\mathfrak{h}^2+(\mathrm{J}+\mathfrak{f}-\mathfrak{g}-\mathfrak{h})^2\right\}.$$

we have

$$\left(f^2-g^2-h^2+\frac{2g^2h^2}{J^2}\right)K^2(1+YZ)$$

$$=4\left(1-\frac{f}{J}\right)\left\{\left(f^2-(g-h)^2\right)\left[(J^2-gh)\left(f^2-(g-h)^2\right)-2gh(g-h)^2-2gh\frac{(J^2-gh)^2}{J^2}\right]\right.$$
$$\left.+\frac{4g^2h^2(g-h)^2(J^2-gh)}{J^2}\right\}$$

$$K^2\{K(Y+Z)-2f^2-2g^2-2h^2+4J^2\}$$

$$=4\left(1-\frac{f}{J}\right)\left\{\left(f^2-(g-h)^2\right)\left[(J^2-gh)\left((g+h)^2-f^2-\frac{4g^2h^2}{J^2}\right)\right]-\frac{4g^2h^2(g-h)^2(J^2-gh)}{J^2}\right\}.$$

Also, since

$$\left(f^2-(g-h)^2\right)+\left((g+h)^2-f^2-\frac{4g^2h^2}{J^2}\right)=4gh\frac{(J^2-gh)}{J^2},$$

we have

$$\left(f^2-g^2-h^2+\frac{2g^2h^2}{J^2}\right)K^2(1+YZ)+K^2\{K(Y+Z)-2f^2-2g^2-2h^2\}$$

$$=4\left(1-\frac{f}{J}\right)\left(f^2-(g-h)^2\right)2ghJ^2\left(1-\frac{g^2}{J^2}\right)\left(1-\frac{h^2}{J}\right).$$

And the values obtained above give also

$$2gh\sqrt{1-\frac{g^2}{J^2}}\sqrt{1-\frac{h^2}{J^2}}K^2\sqrt{1+Y^2}\sqrt{1+Z^2}$$

$$=4\left(1-\frac{f}{J}\right)\left(f^2-(g-h)^2\right)2ghJ^2\left(1-\frac{g^2}{J^2}\right)\left(1-\frac{h^2}{J^2}\right),$$

which shows that the relation between Y and Z is verified by the assumed values of these quantities, and the other two equations are of course also verified. The solution of the problem will be rendered more complete if the equations of the required sections and of the auxiliary sections made use of in the geometrical construction are expressed in terms of f, g, h, J.

§ 7.

First, to substitute in the equations of the required sections or resultors. Writing the first equation in the form

$$\frac{K^2}{2\sqrt{\mathfrak{B}\mathfrak{C}}}\left\{a\mathrm{X}x+(h\mathrm{X}+\sqrt{\mathfrak{C}})y+(g\mathrm{X}+\sqrt{\mathfrak{B}})z+\sqrt{-ap}\sqrt{1+\mathrm{X}^2}w\right\}=0,$$

the coefficient of x will be

$$\frac{f^2}{\sqrt{\mathfrak{A}}}\left(1-\frac{f^2}{J^2}\right)\left\{\frac{2fgh}{J}+(-f+g+h)^2-2J(-f+g+h)\right\},$$

or, as it is convenient to write it,

$$\left(1+\frac{f}{J}\right)f(-f+g+h)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)\left\{\frac{2fgh}{J(-f+g+h)}-f+g+h-2J\right\}.$$

The coefficient of y is

$$\frac{1}{2\sqrt{\mathfrak{B}}}\left\{\left(-f^2-g^2+h^2+\frac{2g^2h^2}{J^2}\right)\left(\frac{2fgh}{J}+(-f+g+h)^2-2J(-f+g+h)\right)\right.$$
$$\left.-f^4-g^4-h^4+2g^2h^2+2h^2f^2+2f^2g^2-\frac{4f^2g^2h^2}{J^2}\right\},$$

or, after all reductions,

$$\left(1-\frac{f}{J}\right)f(-f+g+h)\frac{h}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)\left\{\frac{-2fgh}{J(-f+g+h)}+f-g+h+\frac{2J(f^2+g^2-h^2)}{2fg}\right\};$$

and similarly the coefficient of z is

$$\frac{1}{2\sqrt{\mathfrak{C}}}\left\{\left(-f^2+g^2-h^2+\frac{2h^2f^2}{J^2}\right)\left(\frac{2fgh}{J}+(-f+g+h)^2-2J(-f+g+h)\right)\right.$$
$$\left.-f^4-g^4-h^4+2g^2h^2+2h^2f^2+2f^2g^2-\frac{4f^2g^2h^2}{J^2}\right\},$$

or, after all reductions,

$$\left(1+\frac{f}{J}\right)f(-f+g+h)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)\left\{\frac{-2fgh}{J(-f+g+h)}+f+g-h+\frac{2J(f^2-g^2+h^2)}{2fh}\right\};$$

and the coefficient of w is

$$\left(1+\frac{f}{J}\right)f(-f+g+h)2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}.$$

Whence, forming the equation of the resultor in question, and by means of it those of the other resultors, the equations of the resultors are

$$\left(\frac{2fgh}{J(-f+g+h)}-f+g+h-2J\right)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$
$$+\left(\frac{-2fgh}{J(-f+g+h)}+f-g+h+2J\frac{f^2+g^2-h^2}{2fg}\right)\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y$$
$$+\left(\frac{-2fgh}{J(-f+g+h)}+f+g-h+2J\frac{f^2-g^2+h^2}{2fh}\right)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z$$
$$+2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}w=0$$

$$\left(\frac{-2fgh}{J(f-g+h)}-f+g+h+2J\frac{f^2+g^2-h^2}{2fg}\right)\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$
$$+\left(\frac{2fgh}{J(f-g+h)}+f-g+h-2J\right)\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y$$
$$+\left(\frac{-2fgh}{J(f-g+h)}+f+g-h+2J\frac{-f^2+g^2+h^2}{2gh}\right)\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z$$
$$+2\sqrt{K}\sqrt{1-\frac{f}{J}}\sqrt{1-\frac{g}{J}}\sqrt{1-\frac{h}{J}}\sqrt{-p}w=0$$

$$\left(\frac{-2\mathrm{fgh}}{\mathrm{J}(\mathrm{f}+\mathrm{g}-\mathrm{h})}-f+g+h+2\mathrm{J}\frac{\mathrm{f}^2-\mathrm{g}^2+\mathrm{h}^2}{2\mathrm{fh}}\right)\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x$$

$$+\left(\frac{-2\mathrm{fgh}}{\mathrm{J}(\mathrm{f}+\mathrm{g}-\mathrm{h})}+\mathrm{f}-\mathrm{g}+\mathrm{h}+2\mathrm{J}\frac{-\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2}{2\mathrm{gh}}\right)\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y$$

$$+\left(\frac{2\mathrm{fgh}}{\mathrm{J}(\mathrm{f}+\mathrm{g}-\mathrm{h})}+\mathrm{f}+\mathrm{g}-\mathrm{h}-2\mathrm{J}\right)\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z$$

$$+2\sqrt{\mathrm{K}}\sqrt{1-\frac{\mathrm{f}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{g}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{h}}{\mathrm{J}}}\sqrt{-p}\,w=0,$$

values which might be somewhat simplified by writing ξ, η, ζ, ω instead of

$$\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x,\quad \frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y,\quad \frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z,\quad 2\sqrt{1-\frac{\mathrm{f}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{g}}{\mathrm{J}}}\sqrt{1-\frac{\mathrm{h}}{\mathrm{J}}}\sqrt{-p}\,w.$$

And it may be also remarked, that the coefficients as well of these formulæ as of those which follow may be elegantly expressed in terms of the parts of a triangle having f, g, h for its sides.

The equations of the separators are found by taking the differences two and two of the equations of the resultors (this requires to be verified *à posteriori*); thus subtracting the third equation from the second the result contains a constant factor,

$$\frac{1}{\mathrm{J}(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2)\mathrm{gh}}\left\{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2-\mathrm{J}(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2)((\mathrm{g}+\mathrm{h})^2-\mathrm{f}^2)\right\},$$

equivalent to

$$\frac{1}{\mathrm{J}(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2)\mathrm{gh}}\left(4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2-\mathrm{J}^2\left(\mathrm{K}^2+\frac{4\mathrm{f}^2\mathrm{g}^2\mathrm{h}^2}{\mathrm{J}^2}\right)\right)\quad\text{or}\quad\frac{-\mathrm{JK}^2}{(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2)\mathrm{gh}}.$$

Rejecting the factor in question and forming the analogous two equations, the equations of the separators are

$$-\frac{\mathrm{g}-\mathrm{h}}{\mathrm{f}}\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x+\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y-\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z=0$$

$$-\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x-\frac{\mathrm{h}-\mathrm{f}}{\mathrm{g}}\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y+\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z=0$$

$$\frac{\mathrm{f}}{\sqrt{\mathfrak{A}}}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)x-\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y-\frac{\mathrm{f}-\mathrm{g}}{\mathrm{h}}\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z=0;$$

and from the mode of formation of these equations it is obvious that the separators have a line in common.

The equations of the determinators being $x=0$, $y=0$, $z=0$, the equations of the tactors are

$$\sqrt{\mathfrak{B}}z-\sqrt{\mathfrak{C}}y=0,\quad \sqrt{\mathfrak{C}}x-\sqrt{\mathfrak{A}}z=0,\quad \sqrt{\mathfrak{A}}y-\sqrt{\mathfrak{B}}x=0;$$

and if $\alpha x+\beta y+\gamma z+\delta w=0$ be the equation of the tactor touching

$$x=0,\ \sqrt{\mathfrak{C}}x-\sqrt{\mathfrak{A}}z=0\ \text{and}\ \sqrt{\mathfrak{A}}y-\sqrt{\mathfrak{B}}x=0,$$

the conditions of contact are

$$\mathfrak{A}\left(\mathfrak{A}\alpha^2+\ldots\frac{\mathrm{K}}{p}\delta^2\right)=(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)^2,$$

$$2\sqrt{\mathfrak{A}\mathfrak{B}}(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\left(\mathfrak{A}\alpha^2+\ldots\frac{K}{p}\delta^2\right)=\left\{(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})(\alpha\sqrt{\mathfrak{A}}-\beta\sqrt{\mathfrak{B}})+\gamma(\mathfrak{G}\sqrt{\mathfrak{B}}-\mathfrak{H}\sqrt{\mathfrak{C}})\right\}^2,$$

$$2\sqrt{\mathfrak{A}\mathfrak{C}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})\left(\mathfrak{A}\alpha^2+\ldots\frac{K}{p}\delta^2\right)=\left\{(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})(\alpha\sqrt{\mathfrak{A}}-\gamma\sqrt{\mathfrak{C}})+\beta(\mathfrak{H}\sqrt{\mathfrak{C}}-\mathfrak{F}\sqrt{\mathfrak{A}})\right\}^2,$$

whence

$$\frac{1}{\sqrt{\mathfrak{A}}}\sqrt{2\sqrt{\mathfrak{A}\mathfrak{B}}(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})}(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=$$
$$(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{\mathfrak{A}}\alpha-(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})\sqrt{\mathfrak{B}}\beta+(\mathfrak{G}\sqrt{\mathfrak{B}}-\mathfrak{F}\sqrt{\mathfrak{A}})\gamma$$

$$\frac{1}{\sqrt{\mathfrak{A}}}\sqrt{2\sqrt{\mathfrak{A}\mathfrak{C}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})}(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=$$
$$(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})\sqrt{\mathfrak{A}}\alpha+(\mathfrak{H}\sqrt{\mathfrak{C}}-\mathfrak{F}\sqrt{\mathfrak{A}})\beta-(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})\gamma$$

$$c\beta^2+b\gamma^2-2f\beta\gamma+\frac{\mathfrak{A}}{p}\delta^2=0,$$

and putting for a moment

$$\mu=\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G}-\sqrt{2\sqrt{\mathfrak{A}\mathfrak{C}}(\sqrt{\mathfrak{A}\mathfrak{C}}-\mathfrak{G})}$$
$$\nu=\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H}-\sqrt{2\sqrt{\mathfrak{A}\mathfrak{B}}(\sqrt{\mathfrak{A}\mathfrak{B}}-\mathfrak{H})}.$$

After some reductions, and observing that the ratios only of the quantities α, β, γ, δ are material,

$$\alpha=\frac{K}{\sqrt{\mathfrak{A}}}(K+h\nu+g\mu)$$
$$\beta=\frac{K}{\sqrt{\mathfrak{A}}}(\quad b\nu+f\mu)$$
$$\gamma=\frac{K}{\sqrt{\mathfrak{A}}}(\quad f\nu+c\mu)$$
$$\delta=\frac{K}{\sqrt{\mathfrak{A}}}\sqrt{-p(b\nu^2+c\mu^2+2f\mu\nu)}\,;$$

and it is easily seen also that the coordinates of the point of contact are

$$x=0,\quad y=\nu,\quad z=\mu,\quad w=-\frac{\delta}{p}\frac{\sqrt{\mathfrak{A}}}{K}\,;$$

also

$$\mu=-\frac{\mathrm{Jg}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right),\quad \nu=-\frac{\mathrm{Jh}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right).$$

And substituting and introducing throughout the quantities f, g, h, J, also forming the analogous equations, the equations of the tactors are

$$\left\{\mathrm{f}^2(-\mathrm{f}^2+\mathrm{g}^2+\mathrm{h}^2)+(\mathrm{g}+\mathrm{h})\mathrm{J}\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2-\frac{2\mathrm{f}^2\mathrm{gh}}{\mathrm{J}^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{A}}}x$$
$$-\left\{\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2+\frac{2\mathrm{gh}(\mathrm{g}-\mathrm{h})}{\mathrm{J}}\right\}\mathrm{J}\frac{\mathrm{g}}{\sqrt{\mathfrak{B}}}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)y$$
$$-\left\{\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2-\frac{2\mathrm{gh}(\mathrm{g}-\mathrm{h})}{\mathrm{J}}\right\}\mathrm{J}\frac{\mathrm{h}}{\sqrt{\mathfrak{C}}}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)z$$
$$+2\sqrt{K}\sqrt{\mathrm{gh}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)\left(\mathrm{f}^2-(\mathrm{g}-\mathrm{h})^2\right)}\sqrt{-p}\,w=0$$

$$-\left\{g^2-(h-f)^2+\frac{2hf(h-f)}{J}\right\}J\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$

$$+\left\{g^2(f^2-g^2+h^2)+(h+f)J\left(g^2-(h-f)^2-\frac{2fg^2h}{J^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{B}}}y$$

$$-\left\{g^2-(h-f)^2-\frac{2hf(h-f)}{J}\right\}J\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z$$

$$+2\sqrt{K}\sqrt{hf\left(1-\frac{h}{J}\right)\left(1-\frac{f}{J}\right)(g^2-(h-f)^2)}\sqrt{-p}w=0$$

$$-\left\{h^2-(f-g)^2+\frac{2fg(f-g)}{J}\right\}J\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x$$

$$-\left\{h^2-(f-g)^2+\frac{2fg(f-g)}{J}\right\}J\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y$$

$$+\left\{h^2(f^2+g^2-h^2)+(f+g)J\left(h^2-(f-g)^2-\frac{2fgh^2}{J^2}\right)\right\}\frac{1}{\sqrt{\mathfrak{C}}}z$$

$$+2\sqrt{K}\sqrt{fg\left(1-\frac{f}{J}\right)\left(1-\frac{g}{J}\right)(h^2-(f-g)^2)}\sqrt{-p}w=0.$$

It is obvious, from the equations, that each separator passes through the point of contact of a tactor and determinator, it consequently only remains to be shown that each separator touches two tactors. Consider the tactor which has been represented by $\alpha x+\beta y+\gamma z+\delta w=0$, the unreduced values of the coefficients give

$$\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma=K^2\sqrt{\mathfrak{A}}$$

$$\mathfrak{H}\alpha+\mathfrak{B}\beta+\mathfrak{F}\gamma=\frac{K^2}{\sqrt{\mathfrak{A}}}(\mathfrak{H}+\nu)$$

$$\mathfrak{G}\alpha+\mathfrak{F}\beta+\mathfrak{C}\gamma=\frac{K^2}{\sqrt{\mathfrak{A}}}(\mathfrak{G}+\mu)$$

$$\sqrt{\mathfrak{A}\alpha^2+\ldots\frac{K}{p}\delta^2}=\frac{1}{\sqrt{\mathfrak{A}}}(\mathfrak{A}\alpha+\mathfrak{H}\beta+\mathfrak{G}\gamma)=K^2.$$

Represent for a moment the separator

$$\frac{f}{\sqrt{\mathfrak{A}}}\left(1-\frac{f}{J}\right)x-\frac{g}{\sqrt{\mathfrak{B}}}\left(1-\frac{g}{J}\right)y-\frac{f-g}{h}\frac{h}{\sqrt{\mathfrak{C}}}\left(1-\frac{h}{J}\right)z=0$$

by $lx+my+nz+sw=0$. Then putting $\mathfrak{A}l^2+\ldots\frac{K}{p}s^2=\square^2$, since

$$\mathfrak{A}\alpha l+\ldots+\frac{K}{p}\delta s=K^2\left\{l\sqrt{\mathfrak{A}}+\frac{m}{\sqrt{\mathfrak{A}}}(\mathfrak{H}+\nu)+\frac{n}{\sqrt{\mathfrak{A}}}(\mathfrak{G}+\mu)\right\}$$

$$=K^2\left\{f\left(1-\frac{f}{J}\right)-g\left(1-\frac{2h}{J}\right)\left(1-\frac{g}{J}\right)-(f-g)\left(1-\frac{2g}{J}\right)\left(1-\frac{h}{J}\right)\right\}$$

$$=\frac{K^2}{J}\left\{-(f-g)^2+h(f+g)-\frac{2fgh}{J}\right\},$$

the condition of contact becomes

$$\square=\frac{1}{J}\left\{-(f-g)^2+h(f+g)-\frac{2fgh}{J}\right\}.$$

Or, forming the value of $\square^2$ and substituting

$$\mathrm{f}^2\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)^2+\mathrm{g}^2\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)^2+(\mathrm{f}-\mathrm{g})^2\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)^2$$

$$+2\left(1-\frac{2\mathrm{f}^2}{\mathrm{J}^2}\right)(\mathrm{f}-\mathrm{g})\mathrm{g}\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)-2\left(1-\frac{2\mathrm{g}^2}{\mathrm{J}^2}\right)(\mathrm{f}-\mathrm{g})\mathrm{f}\left(1-\frac{\mathrm{h}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{f}}{\mathrm{g}}\right)-2\left(1-\frac{2\mathrm{h}^2}{\mathrm{J}^2}\right)\mathrm{fg}\left(1-\frac{\mathrm{f}}{\mathrm{J}}\right)\left(1-\frac{\mathrm{g}}{\mathrm{J}}\right)$$

$$=\frac{1}{\mathrm{J}^2}\left(-(\mathrm{f}-\mathrm{g})^2+\mathrm{h}(\mathrm{f}+\mathrm{g})-\frac{2\mathrm{fgh}}{\mathrm{J}}\right)^2,$$

which may be verified without difficulty, and thus the construction for the resultors is shown to be true.

§ 8.

Several of the formulæ of the preceding sections of this memoir apply to any number of variables. Consider the surface (*i. e.* hypersurface)

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy\ldots+pt^2=0,$$

and the section (*i. e.* hypersection)

$$(a\lambda+h\mu+g\nu\ldots)x+(h\lambda+b\mu+f\nu\ldots)y+(g\lambda+f\mu+c\nu\ldots)z\ldots+\sqrt{-p}\,\nabla t=0,$$

where

$$\nabla^2=a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu\ldots-\mathrm{K},$$

the condition of contact with any other section represented by a similar equation is

$$a\lambda\lambda'+b\mu\mu'+c\nu\nu'+f(\mu\nu'+\mu'\nu)+g(\nu\lambda'+\nu'\lambda)+h(\lambda\mu'+\lambda'\mu)\ldots\pm\mathrm{K}=\nabla\nabla',$$

where K is the determinant formed with the coefficients $a, b, c, f, g, h, \ldots$ And consequently, by establishing all or any of the equations $\lambda=\sqrt{\mathfrak{A}}$, $\mu=\sqrt{\mathfrak{B}}$, $\nu=\sqrt{\mathfrak{C}}$, ... we have the condition in order that the section in question may touch all or the corresponding sections of the sections $x=0$, $y=0$, $z=0$, ...

Let n be the number of the variables x, y, z.., then $\mathrm{K}^{n-1}=\begin{vmatrix}\mathfrak{A} & \mathfrak{H} & \mathfrak{G} & \ldots\\ \mathfrak{H} & \mathfrak{B} & \mathfrak{F} & \\ \mathfrak{G} & \mathfrak{F} & \mathfrak{C} & \\ \vdots & & & \end{vmatrix}$

also $$\mathrm{K}^{n-2}\{(a\lambda+h\mu+g\nu..)x+(h\lambda+b\mu+f\nu..)y+(g\lambda+f\mu+c\nu..)z\ldots\}$$

$$=-\begin{vmatrix} & x & y & z\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}$$

whence also

$$\mathrm{K}^{n-2}(\nabla^2+\mathrm{K})=-\begin{vmatrix} & \lambda & \mu & \nu\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix} \text{ or } \mathrm{K}^{n-2}\nabla^2=-\begin{vmatrix} 1 & \lambda & \mu & \nu\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{vmatrix}$$

and the equation of the section in question becomes

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \mu & \nu\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\;t=0$$

also the condition of contact with the corresponding section is

$$-\left|\begin{array}{cccc} \mp 1 & \lambda & \mu & \nu\,.. \\ \lambda' & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu' & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu' & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| = \sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \mu & \nu\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\sqrt{-\left|\begin{array}{cccc} 1 & \lambda' & \mu' & \nu'\,.. \\ \lambda' & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu' & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \nu' & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}$$

In particular the equation of the sections which touches all the sections $x=0, y=0, z=0$ .., is

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \sqrt{\mathfrak{A}} & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\;t=0$$

Again, the equations of the section touching $y=0$, $z=0$, ... and the section touching $x=0$, $z=0$, ... are

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\;t=0$$

$$-\left|\begin{array}{cccc} & x & y & z\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| + K^{\frac{1}{2}n-1}\sqrt{-p}\sqrt{-\left|\begin{array}{cccc} 1 & \sqrt{\mathfrak{A}} & \mu & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\;t=0$$

and the condition of contact of these two sections is

$$-\left|\begin{array}{cccc} \mp 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right| = \sqrt{-\left|\begin{array}{cccc} 1 & \lambda & \sqrt{\mathfrak{B}} & \sqrt{\mathfrak{C}}\,.. \\ \lambda & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \sqrt{\mathfrak{B}} & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}\sqrt{-\left|\begin{array}{cccc} 1 & \sqrt{\mathfrak{A}} & \mu & \sqrt{\mathfrak{C}}\,.. \\ \sqrt{\mathfrak{A}} & \mathfrak{A} & \mathfrak{H} & \mathfrak{G} \\ \mu & \mathfrak{H} & \mathfrak{B} & \mathfrak{F} \\ \sqrt{\mathfrak{C}} & \mathfrak{G} & \mathfrak{F} & \mathfrak{C} \\ \vdots & & & \end{array}\right|}$$

It would seem from the appearance of these equations that there should be some simpler method of obtaining the solution than the method employed in the previous part of this memoir.

2 *Stone Buildings*,
April 1852.

XV. *An Experimental Inquiry undertaken with the view of ascertaining whether any, and what signs of current Force are manifested during the organic process of Secretion in living animals* (continued). *By* H. F. Baxter, *Esq. Communicated by Dr.* Todd, *F.R.S.*

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In drawing up the former paper, published in the Transactions for 1848, the author experienced some difficulty, inasmuch as the subject of these researches presents a mixed character; and conscious that strong objections might be reasonably started by the physical philosopher to conclusions deduced from experiments of the following nature, he has naturally felt some hesitation in presenting the following communication to the Society. As no attempt, that he is aware of, has been made to refute the former conclusions by *experimental* evidence, it only remains for him to state that he has treated the subject more as a physiologist, and for physiologists, than for physicists; that he has not considered himself called upon to refute every objection the physical philosopher might be supposed to raise; and that he has endeavoured at the same time to avoid every thing of a controversial character.

Precautions.—Instead of entering fully into all the precautions necessary to be observed in researches of the following description, we shall refer to the works of Faraday*, Matteucci† and Becquerel‡. We must however bear in mind that it is not the mere fact of getting an effect upon the needle of the galvanometer that is to satisfy us; *that* can be readily obtained: we have to point out to what class of phenomena the effects may be referred.

As it appeared desirable to condense the experiments as much as possible, we shall make a few general remarks as to the mode in which the experiments were performed. The whole arrangement was ascertained to be in a good working condition previous to each set of experiments. As an objection might be started to the use of the former electrodes, two others were procured of platinum wire, No. 16 gauge, each a foot in length, and from the same piece of wire. Smaller wooden mercurial cups were also used, and it was at these cups that the making and breaking of contacts were made. The extremities of the electrodes were well cleaned after and previous to the formation of each circuit; moreover, the electrodes were held lightly in the hand, not squeezed.

* Experimental Researches in Electricity, vol. ii. p. 60.

† Traité des Phénomènes Électro-Physiologiques, chap. iii.

‡ Traité de l'Électricité, tome iv. p. 164.

§ 1. *On the manifestation of Current Force during Biliary Secretion.*

Experiment 1.—Rabbit. Prussic acid dropped on the eye. One electrode inserted into the gall-bladder; the other in contact with the blood flowing from the vena cava inferior in the chest; the latter positive* 10°, and by breaking and making contact made to increase. Between the blood in the vena porta and the bile; blood slightly positive. Mucous membrane of stomach and blood in the chest; stomach positive 8°. The circuit between the gall-bladder and the blood in the chest was again formed; blood positive 10°.

Experiment 2.—Rabbit. Prussic acid dropped on the eye. As the results were similar to those obtained in the last experiment we need not repeat them.

The parts were allowed to remain for half an hour, when the following circuits were formed:—Between the surface of the liver and blood in the chest; no effect. The mucous membrane of the gall-bladder and blood in the chest; blood positive 3° or 4°. The mucous membrane of gall-bladder and the wall of the chest, out of the way of the blood; no effect.

Experiment 3.—Cat. Prussic acid, swallowed. Between the bile in the gall-bladder and blood flowing from the vena porta; the latter slightly positive. Between the bile in the gall-bladder and blood flowing from the vena cava inferior, in the chest; the latter positive 5°.

Experiment 4.—Cat. Prussic acid, swallowed. The electrodes were inserted into the gall-bladder and vena cava inferior in the chest; the latter positive 4° or 5°, and as the blood flowed out from the wound the motion of the needle increased. Between the gall-bladder and vena porta, the latter slightly positive.

Several other experiments might be related proving the same conclusions, but as the fact may not be disputed, we shall state, in a general manner, the other circuits that were formed; viz. between clots of blood and pieces of liver; between the mucous membrane of the intestines and the gall-bladder; between the blood in the chest and various parts of the abdomen; one electrode was coated with bile, and then both of the electrodes were dipped into the blood in the chest. It was generally found that the electrode in contact with the blood was *positive*, but not always; sometimes vibrations of the needle only occurred, at other times the needle went as far as 80° or 90° and then stopped. The motions of the needle presented quite a different character to those observed when the bile in the gall-bladder and the blood flowing from the vena cava inferior were formed into a circuit; the latter presented a steady character, they could be depended upon; whereas with the former a greater effect might be produced at first, and it would then cease, or perhaps go in the opposite direction.

The following fact may not perhaps be denied, viz. *that when the electrodes of a galvanometer are brought into contact with the bile flowing from the liver and with*

* If zinc and platinum be formed into an elementary voltaic circle with dilute sulphuric acid, the *current*, according to the usual mode of expression, goes from the zinc through the acid to the platinum; the platinum is therefore the *positive* electrode, and in contact with the *cation* hydrogen.

the venous blood flowing from the liver, we obtain evidence of the secreted product and the blood being in opposite electric states.

Can we refer these effects to any known actions? And

First. To the *heterogeneity of fluids.* Let us first ascertain what we mean by the phrase *heterogeneity of fluids,* otherwise it will serve as a convenient cloak for our ignorance. According to the researches of Becquerel, during the reaction of two liquids upon each other, that which performs the part of an *acid,* takes positive, that of an *alkali,* negative electricity. From the experiments we have just related, we find that the electrode in contact with the blood is *positive.* Are we then in a condition to say that the blood is *acid* to the bile, and that the effects are due to the *combination* of the blood with the bile? Where do they combine? Where are the acting points in the circuit? According to all chemical analyses the blood is supposed to contain a *free alkali,* and it is said that the bile contains acids, such as the *choleic,* or *bilic* acids, &c. If then we are to refer the effects to the heterogeneity of the fluids, we must consider the blood as acid, and not only that, but that immediately after the *separation* of the secreted product *from* the blood it immediately *recombines* with the blood.

Let us however just assume, according to the idea, first, we believe, entertained by Wollaston, that the effects are analogous to those which occur in the cell of a voltaic circuit, viz. to those of *decomposition,* and we shall now find that the results obtained with the galvanometer confirm this view. The electrode in contact with the *cation* (alkali) in a voltaic circuit is *positive*; if then we suppose that the blood contains the *cation* and the bile the *anion,* we should naturally expect that the results upon the galvanometer would be such as do occur. As this is the point to be proved, we shall now leave it and notice one or two other circumstances to which the effects may be referred; and

Secondly. To *catalytic actions,* or *the combining power of platinum.* We have strong experimental reasons for believing that when blood escapes from a wound it enters into *combination* with the oxygen of the atmosphere; when a plate of platinum therefore is in contact with the blood, actions similar to those which occur in the gas-battery take place. We have a right to suppose that similar actions would occur at the other electrode, namely, that in contact with the bile; still, it might not necessarily follow that the latter would counteract the effects of the former. Judging then from the *direction* of the current, the effects may be fairly supposed to be due to the actions which occur between the atmosphere and the blood, or, in other words, to *catalytic actions.*

There can be no doubt that the effects observed are partly due to *catalytic actions,* and we may even go further and say, that they *must* be so as a necessary consequence. We could not wish for a stronger confirmation of our views: just now we were obliged to suppose the blood to be *acid*—to contain an *anion*—to account for the effect; now we are obliged to suppose it to contain a *cation*—to be alkaline—to account

for the catalytic action, or if not in an alkaline state, still in such a state as to combine readily with the oxygen of the atmosphere. Now, this latter state we are necessarily driven to entertain, when supposing, which no physiologist will deny, that the blood, during secretion, undergoes a change similar to that of *decomposition**.

We shall now speak of,

Thirdly. *Thermo-electric actions.* BECQUEREL† and BRESCHET, as is well known, have shown that different parts of a living animal are of different temperatures; but it must be borne in mind that their experiments were intended to elucidate thermo-electric actions, and might not, therefore, be considered as comparable with the present. Although it would be considered rather a stretch of the imagination to suppose that the effects can be referred to thermo-electric actions, since no effect was obtained when the electrodes were inserted into the vena porta and hepatic veins, as in former experiments, or even in the experiments of MULLER, between the corresponding arteries and veins, still, it is for physiologists to show that the effects cannot be referred entirely to these actions. The following experiments may, therefore, be considered as worthy of being recorded.

A porcelain jar, 2 inches and a half in diameter and the same in depth, capable of holding about five ounces and a half of fluid, was used as the external cell; a portion of the small intestines of a rabbit, capable of holding half an ounce of fluid, was suspended by threads and formed the internal cell; the ends of the electrodes, to the extent of half an inch, were bent at a right angle and placed in each cell, the other extremities being connected with the galvanometer and mercurial cups as in the experiments on animals. Thus arranged, water at different temperatures was poured into each cell.

Experiment 1.	Temperature of atmosphere	71°
	Temperature of water in external cell	68
	Temperature of water in internal cell	120

Slight vibrations of the needle. Every endeavour to obtain a greater effect failed. The temperature of the fluid in each cell was then ascertained by means of a delicate thermometer.

Temperature of internal cell	105°
Temperature of external cell	80

Experiment 2.	Temperature of external cell	160°
	Temperature of internal cell	68

Vibrations as before; and it was then found that the

Temperature of external cell was	125°
Temperature of internal cell	98

* It is not necessary for us to point out in what manner, whether by parent-cells or secreting-cells.

† Traité de l'Électricité, tom. vii. p. 20.

Experiment 3.	Temperature of external cell	67°
	Temperature of internal cell	130
Vibrations; and		
	Temperature of external cell	81°
	Temperature of internal cell	110

In whatever manner the experiments were varied, whether by using water at greater or less differences of temperature, similar results were obtained. The vibrations were sharp and quick at the commencement, but soon terminated; in no instance could a decided effect upon the needle be obtained by making and breaking contact, and the effects were not in any way similar to those observed in the animal body. There is one remark, however, which might be made in reference to these experiments, viz. if we could keep the two fluids at constant temperatures at the point of contact, more decided effects might be expected.

Before we dismiss the subject of thermo-electric actions, we ought, perhaps, to relate some experiments in which a resistance—a liquid conductor—was added to the circuit, and see if the current would be capable of traversing it. We shall relate these experiments further on, and for the following reasons, assuming for a moment that the current would be arrested, we should not then be justified in coming to the conclusion that they are *therefore* due to thermo-electric actions; and our object is, as we have stated before, to ascertain if possible the *existence*, not the *force* of the current.

§ 2. *On the manifestation of Current Force during Urinary Secretion.*

Experiment 1.—Rabbit. Prussic acid dropped on the eye. One electrode in contact with the mucous surface of the ureter, the other inserted into the renal vein of the left kidney; no effect. One electrode inserted into the urinary bladder, the other into the renal vein of the right kidney; the latter slightly positive. There was great difficulty in catching the vibrations of the needle.

Experiment 2.—Rabbit. Prussic acid dropped on the eye. One electrode inserted into the bladder, the other into the left renal vein; the latter positive 5°: the latter electrode was then placed on the surface of the intestines; no effect.

After a short time similar circuits were formed between the bladder and the right renal vein; no effect.

Experiment 3.—Cat. Prussic acid, swallowed. Between the bladder and left renal vein; the latter positive 8°.

Experiment 4.—Cat. Prussic acid, swallowed. Between the bladder and both renal veins; the latter slightly positive.

Bladder much distended: urine acid to litmus.

Experiment 5.—Cat. Prussic acid, swallowed. Between left renal vein and blad-

der, the former positive 5°: the electrode in contact with the bladder was then placed on the surface of the intestines; blood slightly positive.

Experiment 6.—Rabbit. Prussic acid dropped on the eye. Between the bladder and left renal vein; the latter positive 3°: the electrode in contact with the vein was then placed on the surface of the intestines in the neighbourhood of the kidney; the electrode in contact with the bladder was now positive, slightly. The circuit between the vein and bladder was reformed; no effect. Between the right renal vein and bladder; no effect.

Bladder full: urine acid to litmus.

To what other conclusion can we arrive at, than that, *during urinary secretion, the blood and urine are in opposite electric states*? The effects are but small, certainly, amounting perhaps only to 3° or 4°; but such as they are they indicate the blood to be *positive*. Considering the small size of the organ, the nature of the secretion, *acid*, and the transient effects that are produced, we may feel some surprise at obtaining such satisfactory evidence.

In judging of the results upon the needle, we must take into consideration the *acting points* in the circuit; we have at least three acting points in the circuit; viz. at the point of secretion, and at the two electrodes. Although we are led to suppose that the current consequent upon the actions which occur at the point of secretion and those which occur at the electrode in contact with the blood assist each other, nevertheless the current consequent upon the actions which occur at the other electrode may be of such a nature as to counteract the effects of the two former, depending, in a great measure, upon the nature of the secretion. Hence we should be led to very erroneous conclusions if we judge merely from the effect upon the needle, either as to the *force* of the current, or its *origin*. We have also some reasons for supposing that the flowing of the blood through the organ would have some influence, acting by *convection** or carrying power. We might also add, that the very circumstance of a difference being observed, as to the amount of deviation of the needle in the different organs, would indicate that the effects cannot be due to one and the same cause, for instance, to thermo-electric effects.

§ 3. *On the manifestation of Current Force during Mammary Secretion.*

We have, unfortunately, only one experiment; the results however may be considered of some value.

A Cat. Prussic acid, swallowed. Unaware at the time that the cat was suckling, one of the mammary glands was divided whilst opening the walls of the abdomen. One electrode was placed in contact with the milk, the other in contact with the blood flowing from one of the mammary veins; blood positive 8°.

We should not be justified in coming to any definite conclusion from the result of one experiment, and therefore defer making any remarks.

* FARADAY, Experimental Researches, vol. i. p. 496.

§ 4. *On the manifestation of Current Force during Respiratory Actions.*

Whether the changes which take place in the lungs between the atmosphere and the blood, or whether the evolution of carbonic acid be considered as of the same nature as a secretion, may be a disputed point. The question, however, as to the *state* of the arterial blood is one of extreme interest, inasmuch as we have hitherto found the venous blood to be in a *positive* state to the secretions,—with the exception to the secretion in the stomach in rabbits, as related in former experiments; and the effects here may be reasonably considered as due to the predominant actions which occur at the electrode in contact with the mucous membrane of the stomach;—with this exception, we have found the *venous* blood *positive*. We have found the blood flowing from the portal vein to be *positive* to the bile, and in reference to the liver the portal vein may be considered as its artery; we can hardly consider the blood in the portal vein to form the *cation* to the bile, but look upon the effects as due to the blood being in contact with the *cathode*. We must bear in mind, that, *in order to obtain* CURRENT FORCE, *the* CIRCUIT FORM *must be given to the arrangement**; *i. e.* that the electrodes must be brought into contact, or by means of some conducting mass, with the ANION and CATION *originating* the power. The solution of the following question, viz. *What will be the effect if we apply one electrode in contact with the mucous membrane of the lungs, and the other in contact with the blood flowing from it*, i. e. *arterial blood*? is one of some importance.

Experiment 1.—Rabbit. Pithed. An opening was made in the lower part of the trachea and the chest laid open.

One electrode inserted into the right bronchus, the other into the left ventricle; the latter positive 4° or 5°, by making and breaking contact made to increase.

Between the right bronchus and right ventricle; the latter slightly positive.

The former circuit was reformed; no effect.

Experiment 2.—Rabbit. Pithed. In opening the chest the left subclavian vein was wounded. One electrode in contact with the right bronchus, the other with the blood from the vein; a very slight effect.

Between the right bronchus and left ventricle; the latter positive 2° or 3°, by making and breaking contact made to increase.

Experiment 3.—Rabbit. Prussic acid dropped on the eye. Between the left bronchus and the right ventricle; the latter positive 2° or 3°.

Between the left bronchus and left ventricle, a sudden effect upon the needle occurred, but soon became slight; blood positive.

Experiment 4.—Rabbit. Prussic acid dropped on the eye. Between the left bronchus and the right ventricle; the latter positive 3° or 4°.

Between the two ventricles; a slight effect occurred, sometimes in one direction, sometimes in the other.

Experiment 5.—Rabbit. Prussic acid dropped on the eye. Between the two ven-

* FARADAY, Experimental Researches, vol. ii. p. 51.

tricles; no effect. The electrode in contact with the right ventricle was inserted into the right bronchus; effect very slight; blood positive.

Experiment 6.—Cat. Prussic acid, swallowed. Between right bronchus and left ventricle; the latter positive 5°. Between the two ventricles; no effect.

Experiment 7.—Cat. Prussic acid, swallowed. Between left bronchus and the axillary vein, which was wounded in opening the chest; no effect. Between left bronchus and left ventricle; the latter positive 3° or 4°.

There was one circumstance worthy of notice; when the electrodes were inserted into the ventricles, blood did not necessarily escape from the wound upon being withdrawn.

We might relate several other circuits that were formed, viz. between blood on each side of the chest, or between blood in the chest and blood in the abdomen; but as no definite conclusion could be deduced from them, we think it unnecessary. Other experiments might also be related all tending to the same conclusion, viz. *that when the mucous membrane of the lungs and the blood flowing from the same part are formed into a circuit, the arterial blood is positive.*

When considering that the effects might be thought due to *catalytic actions*, we alluded to the changes which occur when *venous* blood is exposed to the atmosphere, to account for its being in a *positive* state. How can we apply the same reasoning to *arterial* blood? In every case it was not exposed to the air; but when exposed, should we be justified in concluding that it would again undergo the same changes which it had immediately undergone in the lungs?

In concluding our remarks in reference to the electric *state* of arterial blood, it is with some degree of pleasure that we can now look back upon our former failures, and which at that time were a source of extreme annoyance,—we allude to our endeavours to obtain evidence of *current force* by forming a circuit between the portal and hepatic veins. POUILLET and MULLER had already failed to obtain any effect by inserting the electrodes into corresponding arteries and veins*. Not only were we looking in the wrong quarter for our current, but we now find that the arterial and venous blood are *both* in the same electric state, and thus accounting for our failures.

In the following experiments a resistance was added to the circuit. A glass tube,

* We do not deny, but think it highly probable, that with *delicate* galvanometers some effect might occur. Assuming that a slight effect were obtained, it would then become a question whether the effects were not due to the changes which occur at the electrodes, rather than at the points of nutrition or secretion. The physical philosopher has an undoubted right to call upon the physiologist to point out the *anion* and *cation* in his circuit, or some adequate *cause* for the *current*. The fact is, the vagueness associated with the term *current* has misled physiologists. We are firmly convinced, that, without extreme care, a delicate galvanometer would only lead to confusion; there is no difficulty in obtaining an effect upon the needle; if anything, we obtain more than we want: the great point is to account for it when obtained, *i. e.* to show with what class of phenomena the effects may be referred.

nearly half an inch in diameter and 3 inches in length, was bent thus, U, and contained water; one limb was connected with one of the mercurial cups by a piece of copper wire of the same thickness as those connected with the galvanometer, and 3 inches in length; the other limb of the tube was connected with another mercurial cup by a similar piece of copper wire; each of these wires dipped, to the extent of a quarter of an inch, into the water. We thus had a resistance consisting of a column of water, nearly half an inch in diameter, 2 inches and a half in length, and 6 inches of copper wire. By this arrangement we could easily cause the current to travel through the galvanometer, with or without the resistance, at pleasure, by merely dipping the electrodes into one mercurial cup or the other, and without any loss of time.

Experiment 1.—Rabbit. Pithed. Between renal vein and bladder; *with* resistance, vibrations; *without*, 3° or 4°: *with*, vibrations; *without*, vibrations.

Between left bronchus and left ventricle; *with* resistance, no effect; *without*, 4°: *with*, 2° or 3°; *without*, no effect.

Between gall-bladder and blood from vena cava inferior; *with* resistance, 5°; *without*, 8°: *with*, 5°; *without*, 10°. The motion of the needle with the resistance was slow and steady.

Experiment 2.—Rabbit. Pithed. Between right bronchus and left ventricle; *without* resistance, 2° or 3°; *with*, vibrations; *without*, vibrations.

Between gall-bladder and blood from vena cava inferior; *without*, 10°; *with*, 3° or 4°; *without*, 10°.

Between renal vein and bladder; no effect either *with* or *without* the resistance.

We shall not attempt to deduce any conclusions from these experiments as to the *force* of the current, and leave it for the physical philosopher to decide whether the effects can be due to thermo-electric actions.

It might have been necessary to have made a few observations before we conclude our present paper, as to the difficulties we have to encounter in our inquiries; some very valuable remarks, however, have been already made by an eminent physical philosopher in reference to the development of electricity in the vegetable kingdom, and as they are so applicable to our present purpose we cannot do better than quote them:—"Il est démontré," says Becquerel, "que l'hétérogénéité des differents sucs qui se trouvent dans les tissus, est la cause première du dégagement de l'électricité, et que l'on doit y joindre encore les altérations qu'ils subissent au contact du platine et de l'air. Il est à regretter que les phénomènes observés ne puissent être mesurés; mais il y a impossibilité de le faire: essentiellement variables de leur nature, parce qu'ils sont modifiés à chaque instant par des agents extérieures, et d'autres causes que nous ne pouvons apprécier, leur existence seule peut être constatée. Au surplus, la physiologie parvient rarement à mesurer les effets qu'elle observe, tant ils sont fugitifs*."

Conclusion.—We are almost tempted to make a few remarks in reference to the

* Becquerel, Bibliothèque Universelle de Genève. Juin 1851.

nature of the *forces* concerned in the process of *secretion*; as it might, however, be considered premature to do so, we shall defer our observations to a future opportunity, and conclude by merely drawing attention to the resemblance first suggested, we believe, by WOLLASTON, between the *polar* decompositions which occur in the decomposing cell of a voltaic circle and the process of secretion in living animals.

The facts which the experiments related in the present paper tend to establish are as follows:—

First. That, during biliary secretion, the *bile* and *venous blood* flowing from the hepatic veins are in *opposite* electric states.

Secondly. During urinary secretion, the *urine* and *venous blood* flowing from the renal vein are in *opposite* electric states.

Thirdly. During mammary secretion, the *milk* and the *venous blood* flowing from the mammary veins are in *opposite* electric states. And,

Lastly. That when a circuit is formed between the *mucous membrane* of the lungs and *blood* (*arterial*) in the left ventricle of the heart, we obtain evidence of *current* force.

XVI. *On the Anatomy of the Stem of* Victoria regia.

By ARTHUR HENFREY, *F.L.S.* *Communicated by Professor* EDWARD FORBES, *F.R.S.*

Received February 19,—Read April 22, 1852.

THE interesting memoir on the anatomy of *Nuphar lutea*, published by M. TRÉCUL in the 'Annales des Sciences Naturelles' (Ser. 3, vol. iv.), having shown that the structure of the stem of that plant is decidedly of the Monocotyledonous type, it was with much pleasure I availed myself of an opportunity of examining the conditions in the remarkable plant of the same family which has been the object of so much attention lately. The specimen of *Victoria* which flowered in the Gardens of the Royal Botanic Society, was found floating, dead, upon the surface of the water a few weeks ago, and by the kindness of the Society's Curator, Mr. MARNOCK, I obtained one of the pieces, when it was sliced down through the middle to ascertain the cause of death.

It appeared to have decayed in the terminal bud; and as the remains of the leaves and roots upon the surface were in a somewhat decomposed condition, there was more difficulty in making out the external structure than would have been the case in a fresh healthy specimen, but I was enabled to ascertain the most important points with regard both to the external and internal anatomy.

The stem of the *Victoria*, as it grows in the tanks of our stoves, is an upright rhizome or rootstock, with the internodes undeveloped; the leaves which succeed very closely in a spiral course, leave projecting processes when they fall off, so that the external appearance of the stem acquires a striking resemblance to that of certain Palms, which are covered with spiral rows of the persistent bases of their petioles (Plate XIX. fig. 1). As in the Palms, there appear to be two or more spiral series running round the stem, like several threads to a screw, or like the spiral fibres in some spiral vessels of plants where several fibres occur lying side by side.

The place of the fallen leaves, however, is rendered much more evident by the cicatrices of the roots, or root-bundles, consisting of squarish flattened surfaces, situated at the underside of the leaf-scars (Plate XIX. fig. 1 *a*, *b*) upon a common process projecting from the stem, which gives origin to both; for in *Victoria*, as in *Nuphar*, the normal arrangement of the roots is in bundles springing from the lower side of the base of the leaf-stalks. The flat surfaces of the root-scars are divided into a number of tessellæ by raised lines (Plate XIX. fig. 1 *b*), each facet, as it may be called, being the scar of a single root, and presenting the projecting extremity of its central vascular bundle like an umbilicus in the middle. The leaves, or rather the petioles, and the

roots are articulated, and when they separate leave a clean fracture; the condition of the root-scar is, such as I have just described it, in all parts; that of the leaf-scars exhibits the open end of air-canals of large size, which traverse the petiole longitudinally; these are continued into the cortical part of the stem for a short distance, and then terminate abruptly in blind ends before reaching the central substance of the stem. The terminal portions of some of the vascular bundles supplying the leaves ramify very beautifully over these blind pouches forming the internal terminations of the air-canals of the petioles.

Midway between the rows of two successive spiral series of leaves are found rows, also spiral, of the scars of flower-stalks (Plate XIX. fig. 1 *c*), distinguished from those of the leaves by the absence of the root-scars beneath them, by their round section, their smaller size, the different arrangement of their air-canals, and, moreover, by the fact that they are not supported by a firm internal process, derived from the central substance of the stem. These are the principal points seen on the outside of the stem; it may be added, that the habit of growth is just what the arrangement of its structures would lead us to suppose; it grows by the continuous development of a terminal bud alone, which, like that of a Palm, throws out leaf after leaf in a spiral course, each leaf being furnished with a branch-like process of the central vascular substance, which remains as a projection, marked by the scar of the leaf and its bundle of roots after these have fallen off. The scars of the flower-stalks are remarkable for being so far distant from the axils of the leaves, which must be supposed to subtend them, and it appears to me that the flower-buds do not become developed until the leaves of the series above them, as well as of that below them, have been perfected.

There is no tap-root to the perfect plant; that which exists in the embryo never becomes developed, and its place is supplied by adventitious roots, as is regularly the case in Monocotyledons, to which class indeed the external characters of the stem of *Victoria* would lead us to refer it.

When we come to the examination of the internal structure of the stem, the Monocotyledonous character becomes still more apparent. There is no bark, no pith, no circular arrangement of the vascular structures, and nothing analogous to a cambium layer. Even in the simple vertical section of the stem (Plate XIX. fig. 2) we see the scattered, isolated condition of the vascular bundles (Plate XIX. fig. 2 *g*), the distinguishing mark of the Monocotyledonous stem, and when we look into the anatomy more closely the first impression is confirmed.

The outer casing of the stem consists of a thick layer of very spongy substance (Plate XIX. fig. 2 *d*; Plate XX. figs. 3, 4 *d*), composed wholly of firm cellular tissue, forming the boundaries of intercommunicating cavities, very much resembling in form and arrangement the cavities in the cellular tissue occurring between muscles, &c. in the higher animals; only it is stiff and resisting here, and does not collapse when the air is let out. Within this spongy layer is found a region of densish cellular tissue (Plate XIX. fig. 2 *e*; Plate XX. figs. 3, 4, 5 *e*), of a white opake colour to the naked

eye, which seems to be analogous to the layer of 'cortical substance' occurring in considerable thickness in the rhizomes of certain aquatic Monocotyledons, such as *Sparganium*, *Typha*, &c., and less extensively in all herbaceous plants of that class. This layer sends out broad flat laminæ which are directed both horizontally and vertically into the spongy substance, which they thus subdivide into compartments and greatly support and strengthen (Plate XIX. fig. 2 *f*; Plate XX. figs. 3, 4 *f*). As the firm walls of the cavernous spongy layer are of the same character and continuous with the white cortical layer, perhaps it would be most correct to consider the spongy substance as part of that layer, hollowed out by air-cavities to lighten the structure.

In those Monocotyledons, which, like *Sparganium*, have a broad cortical layer, there usually exists a firm 'fibrous layer,' composed of ducts consisting of the interlacing ends of the vascular bundles of the stem, which layer gives origin to the vascular bundles of the roots, and also forms the boundary between the 'cortical substance' and the vascular central mass, representing the wood in these plants. I could not detect a fibrous layer of this kind here, and perhaps its absence stands in some relation to the peculiar arrangement of the roots, for in the plants where it occurs, adventitious roots are given off in all parts of the stem, driving their vascular bundles from the fibrous layer everywhere present. In *Victoria* the roots occur only at the bases of the petioles, and they are there supplied by vascular bundles sent out expressly for them.

The central substance of the stem presents at first sight a confused mass of interlacing fibres imbedded in cellular tissue, which here exhibits no sign of division into regions corresponding to pith or medullary rays, Plate XIX. fig. 2 *g*; Plate XX. figs. 3, 4, 5 *g*. The character is quite Monocotyledonous, except that there is evidently frequent anastomosis of the interwoven fibres, which is not commonly found in the Monocotyledons. The outer part of this vascular region contains fibres of smaller diameter than those of the centre, many of which run horizontally round the stem (Plate XX. fig. 3); more internally, the fibres mostly run obliquely and sometimes transversely through the stem (Plate XX. fig. 4), and in the inner parts some of the fibres are nearly vertical, Plate XX. fig. 5. When examined by the naked eye the fibres appear opake, and are surrounded by a semi-transparent layer, which again is surrounded by the opake cellular parenchyma of this central layer; portions of the cellular tissue, near the outer part of this region, also present this semi-transparent appearance, which is caused by the absence of air-cavities in the tissue; the vessels containing air, and the general parenchyma, which is freely supplied with air-cavities, as in most water plants, appear opake under water, from the reflexion of light which the contained air causes.

At the place where each leaf and bundle of roots is borne, a branch-like process of the central vascular mass is given off (Plate XIX. fig. 2 *a*, *b*), in which run horizontally the vascular bundles for the leaf (*a*) and roots (*b*). Those for the former appear to be mostly derived from the central part of the vascular mass (Plate XX. fig. 4 *a*), those

for the roots chiefly from the more delicate fibres running horizontally around the outer part of the central region, Plate XX. figs. 3, 4 *b*. The bundles for the leaf run out as large fibres, independent of each other, and imbedded in firm cellular tissue continuous with that of the cortical layer; they form a group which presents a triangular section when cut across, Plate XX. figs. 7, 8 *a*. The vessels for the root-bundle are all collected into a somewhat cylindrical cord (Plate XX. figs. 7, 8 *b*), where they pass through the outer part of the cortical layer, which cord runs parallel to and just below the collection of fibres for the leaf; it subdivides quite close to the points of attachment of the roots.

It has been stated that plates of the firm cortical layer run through the spongy substance; these form buttresses, as it were, and cross-walls running between and connecting the processes which give origin to the leaves and roots, in the manner shown in the drawing (Plate XX. fig. 7), which is a *plane projection* of the cut surface of a portion of the cylindrical stem, showing the cut ends of the vessels of the leaves and roots (Plate XX. fig. 7, *a*, *b*), as also of those of flower-stalks (Plate XX. fig. 7, *c*); all of which are connected together by plates of firm tissue, the edges of which are shown in the section, Plate XX. fig. 7, *f f*.

The vascular bundles of the flower-stalks run out from the vascular region in the substance of these plates, having no proper branch-like process such as we find supporting the leaves. It is remarkable, as is seen in the section, that the flower-stalks lie nearer to the leaf above them than to those to the axils of which we must suppose them to belong.

The apex of the stem, with the delicate structures of the terminal bud, was so much decayed as to prevent satisfactory examination of the course of development of the vascular bundles; but so far as I could judge from the investigation of the sound portions of the stem, it is analogous to that of the Monocotyledonous rhizomes formed by the continuous development of a terminal bud, without the elongation of the internodes. The non-development of the internodes produces a prevalence of horizontal direction in the vascular bundles, very little vertical growth occurring to produce a perpendicular elongation of these structures. It is evident from the relative condition of development of the vascular bundles in the different parts of the stem, that the order of growth is the same as that in the Monocotyledons, and that the central bundles are the oldest, the outer and upper the youngest, and that the increase of thickness of the stem, which takes place only up to a certain point, is produced by expansion of rudimentary organs, chiefly in the outer part of the central vascular region.

The vascular system is exceedingly simple in its nature. There exists no analogue to wood or liber, the bundles are exclusively composed of vessels of large size, chiefly of spiral vessels with two or three parallel fibres, but also with reticulated and partially annular vessels, all unrollable. These vessels are surrounded by cellular tissue composed of longitudinal rows of small, oblong cells, with strong but thin

walls, which pass by gradations into the general parenchyma of the stem. The vascular bundles of the centre of the stem (Plate XX. fig. 13) are composed of very many such spiral vessels, or rather from their intercommunication and large size, ducts, collected into a cylindrical bundle. The vascular bundles of the petioles (Plate XX. fig. 8) present the ducts more scattered in the cross section, since the chief bundles ramify as they pass out; the vascular cord of the root-bundle (Plate XX. fig. 8 *b*) differs from that of the Monocotyledons, which has a central woody cylinder surrounded by ducts; for the firm oblong, parenchymatous cells of this cord have the ducts scattered pretty regularly through its substance, Plate XX. figs. 11, 12.

From the researches of M. Trécul, already referred to, it appears that the structure of the vascular bundles is similar in that plant, as is also the Monocotyledonous character of their arrangement.

In conclusion, it may be stated that so far as the general arrangement of the structure of the stem is concerned, *Victoria*, like *Nuphar*, would appear to afford evidence in favour of that view which regards the Nymphæaceæ as Monocotyledons. The main difference, in fact the only one, from the rhizomes of plants of that class, so far as I have examined them, lies in what I believe to be unimportant points, namely, the absence of the fibrous layer between the cortical and central substances, and the composition of the vascular bundles exclusively of ducts of the unrollable spiral fibres.

Explanation of the Plates.

PLATE XIX.

Fig. 1. View (natural size) of the side of a rhizome of *Victoria regia*, showing the spiral arrangement of the leaves, root-bundles and flowers. *a*, scar of leaf; *b*, scar of a root-bundle, each facet corresponding to a root; *c*, scar of a flower-stalk.

Fig. 2. Vertical section of the same stem, showing the central vascular region (*g*) surrounded by the cortical substance (*e*), which supports the spongy substance (*d*) by plates (*f, f*) of its tissue running out horizontally; *a, b*, process giving origin to a leaf and root-bundle; the vascular bundles of the leaf (*a*) are above those of the roots (*b*).

PLATE XX.

Fig. 3. Cross section of the stem (natural size), exhibiting the various regions; references as before; *b*, the vascular cord going to a root.

Fig. 4. Another cross section. *a*, vascular bundles going to a leaf; *b*, do. going to a root.

Fig. 5. Vertical section at the base of the stem with the spongy outer layer removed. *a*, *b*, vascular bundles supplying a leaf (*a*), and a root-bundle (*b*); other references as before.

Fig. 6. Vertical section of half the stem about the middle; references as before.

Fig. 7. A plane projection of a section of the cylindrical surface of the stem, showing the cut ends of the vascular bundles of the leaves (*a*), the roots (*b*), and the flower-stalks (*c*), and of the plates of cortical substance connecting them (*f*, *f*); the interspaces are filled up by spongy tissue like *d* in fig. 3.

Fig. 8. Magnified view of the cut end of a 'leaf and root' process with that of the vascular bundles of a flower below. *a*, vascular bundles of leaf; *b*, do. of root; *c*, do. of flower-stalk, with some of its air-canals running into the spongy tissue.

Fig 9. Cross section of the white cortical substance near the vascular bundles of the preceding figure. (Fig. 9–13 are highly magnified.)

Fig. 10. Cross section of vascular bundles of leaf, from *a* in do.

Fig. 11. Cross section of a portion of the vascular end *b*, supplying a root-bundle.

Fig. 12. Longitudinal section of do.

Fig. 13. Longitudinal section of a vascular bundle of the stem.

XVII. *On the Development of the Ductless Glands in the Chick.*

By Henry Gray, *Esq., F.R.S., Demonstrator of Anatomy at St. George's Hospital. Communicated by* William Bowman, *Esq., F.R.S.*

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HAVING been engaged for some period in investigating the evolution of the spleen, supra-renal and thyroid glands, and the several tissues of which each is composed, and having arrived at some conclusions which differ from those previously given, I have ventured to lay them before the Royal Society.

Development of the Spleen.

Before describing the results of my own investigations, I may mention that Arnold* states that the spleen arises like the pancreas from the duodenum, and exists at first as a common mass with that gland; whilst Bischoff† believes that it arises from a mass of blastema, at first common both to this organ and the pancreas, that forming the pancreas proceeding from the duodenum, and that of the spleen from the great curvature of the stomach.

The description which I now propose to offer, differs from either of those above mentioned.

About the seventy-second hour (Plate XXI. fig. 1), I found in the embryo of the chick that the vitelline sac had already sufficiently contracted to form two canals, of which the posterior was small, and only just observable; but the anterior one was much larger and longer, and took a somewhat tortuous course through the body of the embryo. No trace of either pancreas or spleen is yet to be observed, but a conical protrusion from the inferior part of this tube indicates the first rudiment of the liver.

At the ninetieth hour (fig. 2) the anterior of the two canals is longer and narrower than the posterior; it presents two slight dilatations, the first in the situation where the liver and pancreas are developed, the second, and larger, immediately in front of this, indicating the position of the future stomach. It is at the first-mentioned dilatation, at its upper part, and behind the stomach, that the pancreas is developed. This rudimentary gland, at this period, consists of a flask-shaped mass of dark granular blastema, connected by a broad peduncle with the wall of the intestinal tube, from which it is apparently a protrusion, being of a similar structure with it. That part connected with the intestine is narrow and tubular, its distal portion being

* F. Arnold, Salz. Med. Zeitung, 1831, T. W. p. 301.

† T. L. Bischoff, Entwick. der Saügthiere und des Menschen. Leipzig, 1842, p. 285.

spherical, and its surface slightly lobulated. There is no trace of any subdivision of this body into two portions, nor at its distal end can any trace of a spleen be as yet observed. Nearly the whole length of the above mentioned two canals have connected with them above a delicate fold of membrane, the "intestinal lamina," which serves to connect them with the vertebral column beneath. At the 114th hour (fig. 3) the anterior prolongation of the vitelline sac presents a double curvature, the concavity of one being directed towards the under, the other towards the upper surface of the embryo; it is in this latter that the rudimentary pancreas may now be clearly seen. This organ now appears as an elongate, dark granular, tubular mass, situated in that curved portion of the intestinal canal which is the rudiment of the duodenum; its direction is in the transverse axis of the embryo, one extremity being connected to the primitive intestinal tube, into which it may be seen to open by a distinct tubular prolongation, the other being separated from the spleen by a distinct granular blastema; its margins are not very distinct, and its contents and wall are darkly granular. The spleen makes its first appearance at this period in a fold of the "intestinal lamina," which below is continuous with the edge of the intestine, as far as the constricted portion of the vitellary sac, and above with the lower part of the rudimentary stomach. It is a small oval whitish mass, situated near to the distal end of the pancreas, but perfectly separate from this body. This separation is more evident at this period than at any other of its first stages of development; for a distinct granular membrane now divides them, whilst also the dark granular tinge of the pancreatic mass, and the lighter colour of the rudimentary spleen make this distinction more manifest. At a more advanced period of their development, the increased size of both organs causes them to approximate more closely, although not more intimately to one another, and it is this latter circumstance that has given rise to the great difference of opinion regarding the development of these parts. The organ, which has now a greyish tinge, lies parallel with the body, and tapers at both ends, the lower one being connected with the fold of the blastema, attached below to the intestine and vitellary membrane, the upper one continuous with the upper part of the same fold, which passes to the surface of the rudimentary stomach. On the fifth day (fig. 4) the pancreas and spleen may still be observed as two separate and independent masses of blastema, the former of which forms a distinct opake transverse tract of white matter, which stretches upwards and backwards from the rudimentary duodenal loop, the spleen existing as a reddish round mass occupying its opposite extremity. The rudimentary pancreas is elongated, and presents an indistinct external margin, which is somewhat convoluted. From the end, near the centre of that curvature of the canal which would correspond to the duodenum, an offset is given off, which consists of a dark narrow tube with very distinct margins, communicating with the dark margin of the canal itself. This I believe to be the rudiment of the pancreatic duct. The spleen at this period consists of a dark mass of blastema, distinct from all the surrounding parts;

when seen with the naked eye it is of an opake white, or slightly reddish colour; its shape is circular or reniform, and it is situated close to the end of the pancreas below and beneath the rudimentary stomach. The organ consists of part of a tract of blastema, which at the upper and lower extremities of the spleen are continued from it, in one direction, towards the lower end of the stomach, in the other, backwards towards the intestine. It occupies the free margin of the fold of the "intestinal lamina," which presents a direction similar to what was observed on the fourth day of incubation. On the sixth day (fig. 5), the anterior prolongation of the vitelline duct forms two very considerable curves, the one nearest to the stomach, the convexity of which is directed downwards, having become completely developed into the duodenal loop, and in this loop the pancreas is plainly visible, the rudimentary spleen occupying its distal extremity. The position of these parts is now somewhat altered, passing obliquely upwards and backwards, instead of transversely, as on the fifth day. This appears to depend upon the alteration in the form and position of the intestine, which, being curved more upwards, would necessarily give to these parts a more oblique direction. The spleen is at this period distinctly visible to the naked eye, it is seen below and beneath the stomach; its shape is reniform, being prolonged into two extremities, one at its upper, the other at its lower part; it is situated at the edge of the fold of the intestinal lamina, which stretches from the lower end of the stomach across the intestine to the vitelline duct. The lower prolonged extremity is the more distinct of the two, and is to be traced for some little distance along the margin and in the substance of this fold, being ultimately lost in it. The other extremity, or that connected to the stomach, is more rounded, and is connected through the interposition of the same fold of membrane to the lower part of the stomach itself. Its position is now somewhat altered, it being placed above and slightly behind the stomach; its upper edge being on a level with the rudimentary proventriculus. At the point where the spleen was connected with the pancreas, an apparent continuity of substance was found between them; but this was not really so, for the two organs were to be observed to be separate from one another, from the great difference in their texture, the substance of the rudimentary pancreas being darkly granular, whilst that of the spleen was of a lighter colour. On the seventh day (fig. 6) the spleen is of a patchy reddish tinge, from the presence of blood in its substance; and its position is now precisely the same as in the adult bird, occupying the space at the back part of the proventriculus, which exists now as a separate and distinct pouch; the surface of the organ is slightly lobulated, expecially towards its two extremities. It is now enclosed by a distinct membrane, and occupies the gastric surface of the distal end of the pancreas; its shape is pyriform, the rounded end being directed backwards and upwards, and the narrow end forwards, into the interval between the commencement of the duodenum and pancreas; these two extremities are still connected by folds of membrane with the stomach and mesentery; they proceed from the back part of the organ, and apparently

surround it. On the eighth day the organ has increased somewhat in size, being now about as large as a small millet-seed, forming a round reddish projecting mass from the surface of the membranous fold in which it was developed, and which is now becoming more delicate and indistinct, as if in process of absorption, although its attachments still remain the same. From this period the spleen enlarges gradually, being on the twelfth day about the size of a small pea; its colour becomes of a more vivid red, and its form somewhat circular; it is held in its position by the vessels which proceed to it, and by a fold of membrane, the edge of which is cord-like, that passes from its lower end to the inferior border of the stomach, the remaining portion of the membrane in which it was developed having become completely absorbed. With the exception of a continued increase in the growth of the organ, which takes place more rapidly after the vessels supplying it are formed, its form, position and attachments remain precisely the same as in the adult bird.

From the preceding observations it is seen that the spleen is developed from the surface of a fold of the "intestinal lamina," in the form of a small oval-shaped mass of blastema, quite independent either of the pancreas or stomach; that in process of time this organ approximates more closely, although not more intimately to both these organs; and as its vessels are formed, the membrane in which it was developed becomes absorbed, with the exception of a delicate fold, which in the adult bird serves to connect this organ with the stomach.

I shall in the next place proceed to consider the development of the different tissues of the spleen, the observations concerning which, I think, are very important, as they tend to prove, not only the glandular nature of the organ itself, but the great correspondence that the development of its tissues presents, as compared with the supra-renal and thyroid glands.

On the Development of the Tissues of the Spleen.

Of the Capsule.—The first indication of the external capsule of the spleen I observed on the ninth day, in the form of a thin and delicate transparent membrane, which completely surrounded the organ; its texture consisted of a fine granular membrane, in which were observed numerous minute nucleated fibres. About the twelfth day this membrane was more distinct, and more easily separable from the surface of the organ; its texture was soft, of a greyish colour, consisting of a finely delicate granular membrane, in which numerous small nuclei, with elongated fibres proceeding from either end, were observed. On the eleventh day the capsule was more distinct, less transparent, thicker in texture, and more adherent to the parts beneath; its texture being composed of numerous nuclei, contained in a granular membrane, with delicate fibres, some of which possessed a nucleus, but in others it had completely disappeared. On the twenty-first day, the capsule, although more distinct, was intimately connected with the parts beneath, and was composed of a dense mesh of delicate fibrillæ, in most of which no nuclei could be detected.

The Development of the Trabecular Tissue.

The development of the trabeculæ commences as early as the eighth day of incubation; previous to this period the chief mass of the substance of the spleen is made up of nuclei, containing in their interior one or more dark dotted granules; at this period, however, spindle-shaped pale granular fibres, containing in their centre an oval elongate nucleus, with a nucleolus, may be observed sparingly throughout the substance of the organ. On the ninth day, similar but more numerous fibres may be observed, either separately or in delicate cylindrical, or riband-like bundles. By the fourteenth day these fibres have increased very considerably in quantity, forming innumerable fine fibrillated bands, intersecting the entire substance of the organ, much as in the adult state.

On the Development of the Blood-Vessels and the Blood.

The blood-vessels which supply the spleen, the separate vessels of this organ, and the blood, have a completely separate, though concurrent development. They are all observed about the eighth day of incubation (fig. 7). The splenic artery, which is developed a little previous to the vein, may be now observed as a delicate white tract of blastema, running from the inferior and posterior angle of the spleen to the descending aorta; on arriving at the inner side of the organ, a prolongation of the same colour may be observed passing to the upper end of the stomach, just beneath the proventriculus, and one or two to the front of the same organ; it then runs along the inner margin of the spleen, and is ultimately lost in the substance of the pancreas and in the duodenal fold. No similar tract could however be found, after repeated examination, to pass into the substance of the spleen, although at the same period capillaries containing perfectly formed discs, but as yet having no proper coats, and apparently merely formed by the walls of cells agglomerated together, were seen to be arranged in a branching manner throughout the substance of the organ. On the ninth day this vessel presents a reddish tinge, and its distribution is the same as in the adult bird. Two small branches are now observed to be given off to the substance of the spleen, from the vessel previously described, as it runs along the inner side of the organ. The splenic vein makes its first appearance on the thirteenth day, in the form of a reddish white tract of blastema, which runs forward and joins with the left side of the mesenteric vein, which is formed on the twelfth day. On the fourteenth day, the splenic vein consists of three small branches, larger than the arteries, which, uniting together, empty themselves into the mesenteric vein.

The development of the blood-globules in the spleen, as well as the various changes they undergo in the substance of this organ, are points of the very highest import, from the great difference of opinion that at present exists regarding its use,—Gerlach and Schäffner believing that the spleen is the organ in which the blood-

discs are formed during extra-uterine life, whilst Kölliker, Beclard and Ecker, suppose that the blood-globules are destroyed there. The blood-discs in the spleen, as I have already mentioned, make their first appearance in this organ about the eighth day. They are oval or circular, varying somewhat in size, and consisting of an external envelope, pale, homogeneous and indistinct, having a nucleus on its wall, which is dark, highly refractive, irregularly circular, and in some cases of a granular texture. Some of these globules may be distinctly seen in an incipient stage of formation, and consist of a dark, and at first, a somewhat irregular granular nucleus, around which a delicate cell-wall may be observed. Although I have observed this in several cases, I do not presume the spleen to be the organ in which the development of the blood-globules takes place during intra-uterine life, nor have I observed that the development of the blood-discs continues to take place in it after its connection with the general vascular system is effected. With regard to the disintegration of the blood-globules, after the most repeated and careful examinations, I have failed in detecting anything that would lead me to suppose that this organ can, during its progress of development, perform the function Kölliker and others have assigned to it in its adult state.

On the Development of the Pulp.

The entire substance of the spleen at an early period of its development is almost precisely similar in structure with the supra-renal and thyroid glands, with which it may consequently be allied. As the evolution of the organ proceeds, part becomes developed into trabeculæ, part into blood-vessels and blood, whilst the greater portion remains to form the essential element of the organ, the pulp tissue. On the fifth day this substance presents the following elementary composition. It is composed of nuclei, varying in form and size. The greater majority are irregularly circular, their sides and edges being flattened at some points, so as to give them an angular form; some, however, are perfectly circular; they are pale, and their outer margins dark and well-defined, whilst in their interior may be observed, one, two, or more dark granules. These form a very considerable portion of the substance of the pulp, not only at this, but also at every other period of its development. A few nucleated spherical vesicles may also be observed, their outer margin exceedingly delicate; and on their wall may be seen a small irregular dark-edged nucleus; sometimes the nucleus is more circular, and contains a nucleolus, whilst the interior of the vesicle contains a few delicate pale granules. With the exception of a few small dark and highly refractive oil-granules, and a fine pale granular plasma, in which the above elements lie, they constitute the entire mass of the pulp tissue at this period. The next change observable takes place concurrent with the formation of the vessels which supply the organ, and which is soon followed by an increase of its size. Now, besides the elements already described as forming its structure, there may be observed nuclei, similar to those constituting the pulp at an early period, but

having a quantity of fine dark granules surrounding them in a circular form. There are also observed many nucleated vesicles, rather larger than the blood-corpuscles, the nuclei in which are circular, whilst the cavity of the vesicle contains also a few small pale granules. Some small masses of reddish brown granules may also be observed; they exist, however, very sparingly. From the time when the formation of the arteries supplying the organ is completed, up to that when the splenic vein is observed to be also constituted, these latter elements not only form a portion of the pulp, but are in fact its chief components. When, however, the splenic vein is nearly completed, a considerable change is observed to have occurred in the nucleated vesicles; those which had previously formed only a small portion of its substance, now exist as the chief element, and the majority contain a nucleus with irregular margins. Their form is chiefly circular, their outer wall in some cases very distinct, in others less so, from the cell being distended with dark granules. There is generally only a single nucleus, which has a dark outer margin, and contains either a nucleolus or two or three granules. In some the nucleus is of an irregular form, and more indistinct; the cavity of the cell in these cases containing a few granules, as the nuclei become more irregular and granular; these granules increase until at last the nucleus appears to be entirely broken up, when they all become crowded with small granules. Such is the structure the pulp tissue of the spleen presents, from the period when the splenic vein is formed, up to the time when incubation is completed.

Development of the Malpighian Vesicles.

The vesicles of the spleen are developed in a manner perfectly similar to those of the supra-renal and thyroid glands, with which they appear to bear a very close analogy; they are not developed, however, in the chick until the period of incubation is near to its completion. Between the twentieth and the twenty-first days there may be observed at the angles of division of the smaller blood-vessels, as well as upon the walls of the vessels themselves, rather large masses of nuclei and granules, arranged together in a circular form; these masses are not, however, at this period enclosed by any investing membrane, but are rather intimately connected with the walls of the vessels, as they are not removed by delicate manipulation, and only when a greater amount of force is used. A few days after incubation is completed these vesicles are observed to be partly surrounded by a faintly delicate homogeneous membrane, and in about a week the vesicles are distinctly formed, and present the same structure as in the adult bird; they are circular or oval, varying considerably in size, and consisting of an outer investing membrane, pale, homogeneous, or faintly granular in texture, and containing in their interior a mass of nuclei and numerous small dark granules.

Development of the Supra-renal Glands.

The development of the supra-renal glands being described differently by various physiologists, I shall first briefly detail the researches made by others upon this point.

Arnold (*op. cit.*) states that they are derived from the Wolffian bodies by means of a fissure, and that they have the same structure as these organs, a statement which has not, however, been confirmed by any other author. Valentin and Meckel believed that the supra-renal bodies exist first as a single mass, which is placed above and in front of the kidneys, and which afterwards divides itself into two lateral halves; whilst J. Muller and Bischoff, on the contrary, state that they have always observed them double, although no account of their origin has been given by them. Mr. T. Goodsir is the only author, as far as I am aware, who has attempted to prove the close affinity the ductless glands (excepting the spleen) have to each other, by tracing out their gradual evolution in the embryo. He states that all of them (excepting the spleen) arise in involuted portions of the membrana intermedia, and that at an early period of embryonic life they communicate with one another. He concludes from this apparent fact, their original identity of function, which he says is that of elaborating a nutrient material, an office which the germinal membrane itself performs during fœtal life.

The description that I shall now venture to offer, differs from any that has been previously given of the evolution of this gland. I trust however that I may be able to prove the close affinity that it has both with the spleen and also with the thyroid, not so much from any resemblance that it may present in its manner of evolution, but from the similitude it presents in its elementary parts, and in the development of those elements to their perfect form.

Between the sixth and seventh days of incubation, I observed that the Wolffian bodies had become contracted in their length, and more curved upon themselves than at an earlier period of their development, being somewhat reniform in shape, and attenuated at their upper extremities. Along the inner margin of these bodies was observed the ovaries, each of which consisted of a small oval mass of whitish blastema, occupying the central portion of each body, approximating, but not joining one another below, whilst they diverge from one another above. Between the Wolffian bodies may be observed the aorta, which bifurcates below into two lateral and a continuous branch. At this period the supra-renal glands, which, as will be seen, exist at a later period in the interval between the upper ends of the Wolffian bodies and the sides of the aorta, could not be observed. On the close of the seventh day (fig. 9), I observed in the interval between the sides of the descending aorta, and the upper and inner sides of the Wolffian bodies, a patchy reddish grey granular mass of blastema, without any distinct form or outline; whilst both the pointed but rounded margins of the primordial kidneys and the ovaries were plainly seen. On

tracing this granular mass upwards along the sides of the aorta, it was completely lost in the general formative mass which surrounds all these parts, and after many attempts I was unable to discover any connection between this body and the thyroid and the thymus above (as stated by Mr. GOODSIR), by means of the blastemal tract along the aorta; nor do I believe that any connection exists between them. The minute structure of the supra-renal gland at this period bears a very close resemblance to the spleen on the fifth day of incubation; it consists of nuclei about the size of the blood-discs, the majority of which are circular, pale, and contain in their interior, one, two, or even sometimes more dark granules,—nucleated spherical vesicles, their outer margin very delicate, the interior of the vesicle containing a few delicate granules. The only difference that exists between the minute structure of the spleen and the supra-renal gland, is the presence in the latter of numerous small dark granules, like fat granules, and which are in some cases accumulated in small circular masses. With this exception no difference can be observed in their minute structure at an early period. On the eighth day the supra-renal glands consist of two grayish white masses, which lie one on each side of the aorta, between this tube and the upper and inner extremity of the Wolffian bodies, and are perfectly separate. They are elongated and rounded at both ends, but they do not present any external circumscribed margin. They are situated, for part of their extent, higher than the upper extremity of the Wolffian bodies, the upper margins of which are distinct and well-defined, as contrasted with the uneven outline of the supra-renal gland. On the right side, the gland is placed completely above the corresponding ovary, but on the left a small part is hidden beneath the left ovary. The texture of the organ is opake, and darkly granular throughout. At this period a considerable advance is observed in the development of the tissues of the gland, for they even now present an incomplete vesicular arrangement; this however is not surprising, when it is considered that its chief function is probably performed during fœtal life. Its tissues have consequently a more rapid development than those of the spleen, an organ the function of which is exercised mainly in adult life. The imperfect vesicles of which this organ is partly formed, consists of a mass of nuclei, similar to those above described, amongst which are scattered a small quantity of fine dark granules; these masses of nuclei are arranged together in a circular form, without any investing membrane surrounding them; in some cases, however, a fine membrane could be observed in one part only of the circumference of the forming vesicle, whilst the remaining portion was entirely free. Some of the vesicles were formed of a mass of smaller vesicles, in which nuclei were observed. The chief majority however were composed of nuclei, fine dark granules, and small dark circular granular masses. On the ninth day the supra-renal glands are yellowish white, and occupy the same situation as on the previous day, but even at this period are not bounded by any external circumscribed margin; they are plainly separated from one another by the aorta, and although placed in

close contact with the Wolffian body, their different structures at once show them to be distinct organs; for the dark opake granular structure of the supra-renal glands contrast most strongly with the lighter and more transparent tissue of the Wolffian bodies. The vesicles of which the organ is at this period composed are circular, or elongate oval, and their contents far more transparent than those observed at a later period of development. They now present a delicate but well-defined external investing membrane, containing nuclei and a few dark granules. On the tenth day (fig. 10) the supra-renal glands are pyriform, the broad end being directed upwards, and the apex downwards; their margins are now also distinctly circumscribed, and their tissue, although dark and opake (when seen with a low magnifying power), is more transparent than the tissue of the gland at a more advanced period of its development. The vesicles of which they are composed at this period are not all circular, for some are elongated, and present two or sometimes three hemispherical bulgings on their wall, as if apparently formed of the junction of two or more vesicles; these contain several nuclei and numerous small dark highly refractive granules; the former are however at this period far more numerous than the latter. These vesicles vary both in size and form; they are grouped together in a mass, in which it is as yet impossible to detect any subdivision into cortical or medullary portions. On the fourteenth day (fig. 11) the supra-renal bodies have somewhat changed their position; they lie at the back part and inner side of the upper extremities of the Wolffian bodies, and in front, and at the inner side of the upper end of the kidneys; their form is of an elongated oval, and the one on the left side is like the corresponding ovary, slightly the larger of the two. At this period they are connected with the Wolffian bodies through the intervention of a delicate web of areolar tissue. The inferior vena cava is now observed passing upwards between these bodies, and closely adherent to their inner sides. On examining the organ with a low magnifying power, it is seen to consist of a mass of vesicles, which however are not equally distributed throughout the whole of its substance, being aggregated in much larger quantities at the upper, lower, and outer sides of the gland than at its inner side, where it is connected with the vena cava, and at its centre. These vesicles are large, and radiate from the circumference towards the centre of the gland, in some cases complete tubes of some length being formed by their junction, as indicated by hemispherical bulgings along their walls. They lie between the meshes of a close plexus of vessels, which run in straight lines from the centre towards the circumference of the gland. These vesicles, at this period, and up to the time when incubation is completed, consist of an external investing membrane, so delicate as to render its demonstration a matter of some difficulty; they are full, not of nuclei, as was observed during the first stages of development, but of dark and highly refractive granules, precisely like oil-globules, in such numbers as to completely distend the cavity of the vesicle, and also prevent the nuclei, a few of which still exist in small numbers, from being clearly detected. By the eighteenth day (fig. 12) the glands have enlarged considerably, and

are of a deep yellow colour; their position is now almost precisely the same as in the adult bird; numerous small vessels ramify upon their surface, and a large vein emerges from their inner sides, which empties itself into the vena cava. A complete division into cortical and medullary portions is now observed. The former consists of elongated tubes with exceedingly delicate walls, and radiating from the circumference towards the centre of the gland. These tubes, which at this period exist in greater number, are parallel with one another, and lie between a close mesh of delicate capillary vessels, having a similar direction. In some cases the simple vesicles placed end to end, but not as yet forming tubes, are disposed in a similar manner between the capillary vessels. The medullary portion of the gland consists of capillaries, which here join to form larger branches, either previous to their passage from the gland, or before passing into the cortical portion in the form of straight capillaries; numerous delicate fibres may also be observed joining them together, and forming a close mesh, in which are deposited numerous dotted corpuscles, and a quantity of fine dark granules. From this time, up to the completion of incubation, no change takes place in the structure of these glands. From the preceding observations it is seen that the supra-renal glands are developed by two separate masses of blastema, which are situated between the upper and inner extremities of the Wolffian bodies and the sides of the aorta, but which are totally independent (as far as concerns their development) of those bodies, or of each other. It is also seen that their minute structure at an early period closely corresponds with the structure of the spleen; and although the supra-renal glands attain their maximum degree of development at an earlier period than that organ, as regards the formation of its vesicles, still exactly the same process can be followed in the spleen as regards the development of the Malpighian vesicles; a fact which, I think, tends to prove the great similarity of the organs in question.

Development of the Thyroid Glands.

The evolution of the thyroid gland, like the spleen and supra-renal capsules, is involved in great obscurity.

The following are the principal accounts of its development that have yet been given.

According to Huschke* the thyroid gland is developed from the anterior branchial arches; no other observer however has verified his statement, and Rathke altogether denies it.

Arnold† states that it is developed from the membranous air-tube, in the situation where the larynx is formed, and that it is at first provided with an excretory duct.

Bischoff‡ states that they appear from a single formative mass, which is deposited

* Huschke, Isis, p. 621, 1826, p. 403, 1827.

† Arnold, Lehrbuch der Physiologie des Menschen. Zürich, 1842, vol. v. p. 1293.

‡ Th. L. Bischoff, Entwick. der Saüg. und des Mens. Leipzig, 1842.

on each side of the trachea. The latest researches on the development of the thyroid, appear to be those of Mr. GOODSIR, who states that it is, like the thymus and supra-renal gland, a development from the "membrana intermedia."

According to my observations, between the sixth and seventh days of incubation, the first trace of the thyroid gland becomes apparent. It consists of an exceedingly small spherical whitish mass of blastema, situated on each side of the root of the neck, close to the point where, at a later period, the carotid and subclavian arteries separate from one another. The outer margin of each mass is somewhat irregular and ill-defined, and is not apparently surrounded by any investing membrane; above each of them is seen the lower end of the rudimentary thymic tube, the outer margins of which are perfectly distinct from the thyroid. The carotid vessel runs beneath and on the outer side of the gland, to the wall of which it is somewhat adherent; and on the inner side runs the rudimentary tracheal tube, with which however it has no connection. The structure of the glands at this period approximate very closely to that of the spleen and supra-renal gland at the earliest stages of their evolution, consisting of granular matter, nuclei, and nucleated vesicles. The granular matter, which forms a large mass of the substance of the gland, is made up of innumerable minute pale granules, very similar to those observed in the structure of the spleen, but unlike those of the supra-renal glands, the particles composing which are darker and more refractive, not unlike minute fat granules. The nuclei are about the size of the red blood-discs; they are pale, perfectly circular, and contain in their interior from one to four or five small dark granules. These also form a very considerable portion of the structure of the organ. The vesicles are few in number, pale, and delicate, with nuclei on their walls, and containing in their interior a few fine dark granules. On the eighth day of incubation the organ has enlarged very slightly, occupying the same position as on the previous day; its colour is now reddish white, from the presence of blood in its structure; it is circular, and is bounded externally by a clearly defined marginal membrane, which now surrounds it. I have never been able, in tracing out the incipient development of these organs, to detect any connection between it and the branchial cleft, the trachea, or the thymus, although its position as regards these parts has probably given rise to the opinions, previously noticed, respecting the origin or connection of the thyroid from one or other of these structures. It has always appeared to me to arise as a separate mass of blastema, and unconnected, unless by simple apposition, with any other organ. From this period, up to the tenth day of incubation, the thyroid, with the exception of a slight enlargement, and a more distinct red colour, presents the same structure, and occupies a similar position as in the previous examinations. At this period they are found situated a little above the root of the neck, one on each side of the lower end of the trachea, just above its division into the bronchi, and at the point of origin of the carotid vessel. The organ is now more distinct from its reddish tinge; its form is circular, its outer margin clear and distinct; and its substance, when seen with a

low magnifying power, moderately transparent. Its minute structure also now presents a higher stage of development; for besides consisting, as at an earlier period, of granular matter, nuclei and vesicles, some of which are furnished with nuclei, there are observed numerous circular and tolerably transparent masses of nuclei and vesicles, some perfectly destitute of any investing membrane, but others presenting a fine membrane partially or completely investing them. In the interspaces between these vesicles numerous blood-discs may be observed, which are apparently not as yet contained in separate tubes.

On the twelfth day, the size of the thyroid about equals that of a millet-seed; it is bright red, and small vessels may now be traced passing into its substance. Its minute structure, however, undergoes no change until about the sixteenth day. At this period a delicate membrane, forming a complete capsule, can be removed from its external surface; it consists of numerous nucleated fibrillæ. Almost the entire mass of the organ is composed of large circular or oval vesicles, consisting of a mass of nuclei, enclosed in a faintly delicate, homogeneous, limitary membrane. On the twenty-first day the thyroid is rather smaller than the spleen, and rather larger than the supra-renal glands. It is of a reddish gray colour, and is surrounded by a complete fibrous capsule, which may be easily removed from its exterior. It consists of a mass of vesicles, which vary in size; they are chiefly circular, and their contents transparent. They consist of an external, homogeneous and transparent membrane, forming a closed cavity, which contains a mass of nuclei. In some cases, however, the vesicles, instead of being filled with nuclei, are lined with a layer of nucleated cells, and a cavity, although small, exists in their interior.

It may be seen, from the preceding observations, that the thyroid glands are developed in the form of two distinct separate masses of blastema, one at each side of the root of the neck, close to the point of separation of the carotid and subclavian vessels, and between the trachea and the bronchial clefts, but quite independent, as far as regards their evolution, of either of those parts. Their minute structure also, at an early period of their development, closely corresponds both with the spleen and supra-renal glands; and the tissues of which they are composed, at a later period, are formed in a manner precisely similar with the same parts in those organs, a fact which shows the analogy they bear to one another.

From the preceding observations, it will be seen that a close analogy exists between the glands already described; and the propriety of their classification, together with the thymus, under one group, as the "Ductless Glands," may be considered clearly proved. Now although most anatomists, excepting Mr. Goodsir in this country, adopt such a classification, and place the spleen under the head of the "ductless glands," many of our continental anatomists, among whom may be enumerated Ecker, in a very late and highly elaborate article, has attempted to prove that this organ cannot, either anatomically or physiologically, be enumerated with them. "For," says Ecker, "though the vesicles of the spleen have a similar function with the glands

of the blood-vessels, which is not improbable, these do not form the main element of the spleen. The function that I have explained to be particular to the spleen, takes place in the pulp, and in none of the glands of the blood-vessels are the characteristic elements of this found." That the spleen, however, may be classed with these glands is I think proved,—

1st. From the manner of its evolution, which is precisely similar both to the supra-renal and thyroid glands.

2ndly. From its structure, which at an early period almost exactly corresponds with that of the other glands in question; and

3rdly. From the development of its tissues following precisely the same law as that upon which the tissues both of the supra-renal and thyroid glands are formed.

It may be thought by some that I should have made this communication more perfect had I traced out the development of the highest of the ductless glands, the thymus. The elaborate investigations, however, of Mr. Simon upon this gland, have given all that can be desired upon this point, and the few observations I have myself made on the evolution of this organ in the chick accord so exactly with his statements, that I could add nothing but what was confirmatory of his observations.

Explanation of the Plates.

PLATE XXI.

Fig. 1. Represents the rudimentary intestinal canal from the embryo of the chick at the seventy-second hour. V. Vitelline sac. I. Intestine. L. Liver. S. Stomach.

Fig. 2. Represents the anterior prolongation of the intestinal canal at the ninetieth hour, with the liver and pancreas arising as protrusions from that tube. P. Pancreas.

Fig. 3. The same parts are represented as they are observed at the 114th hour. The first trace of the spleen is here shown as a small oval body developed in a fold of the intestinal laminæ, distinct from the pancreas. Sp. Spleen.

Fig. 4. Represents the intestinal canal, pancreas and spleen from an embryo chick between the fifth and sixth days. The spleen is now observed to have approximated close to the end of the pancreas.

Fig. 5. Represents the same parts from an embryo chick between the sixth and seventh days. The pancreas with its duct are observed occupying the fold of the duodenum; the spleen occupying its distal extremity.

Fig. 6. In this figure are represented the stomach, duodenum, pancreas and spleen from an embryo chick between the seventh and eighth days.

Fig. 7. The duodenal fold is here represented with the pancreas and its duct con-

tained in it. The spleen is seen near to its distal end with blood-vessels ramifying in its substance; no branches being derived from the vessel which runs along its inner side. From a chick at the eighth day.

PLATE XXII.

Fig. 8. The Wolffian bodies, with the diminutive and elongated ovaries lying upon them, are represented as observed in an embryo chick on the fifth day. W. Wolffian bodies. O. Ovaries.

Fig. 9. The same parts are represented as they are observed on the seventh day. The supra-renal glands are observed lying at their upper and inner margins, one on each side of the aorta and above the ovaries. S. Supra-renal glands.

Fig. 10. The same parts as observed on the tenth day. The ovaries now approach nearer to the supra-renal glands; the left one just overlapping the lower end of the corresponding gland.

Fig. 11. In this figure are represented the kidneys, Wolffian bodies, ovaries and right supra-renal gland from an embryo chick at the fourteenth day. The left supra-renal gland is covered over by the left ovary.

Fig. 12. The same parts as observed in the chick on the eighteenth day.

XVIII. *Researches on the Geometrical Properties of Elliptic Integrals.*

By the Rev. JAMES BOOTH, *LL.D., F.R.S. &c.*

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SECTION I.

1. IN placing before the Royal Society the following researches on the geometrical types of elliptic integrals, which nearly complete my investigations on this interesting subject, I may be permitted briefly to advert to what had already been effected in this department of geometrical research. LEGENDRE, to whom this important branch of mathematical science owes so much, devised a plane curve, whose rectification might be effected by an elliptic integral of the first order. Since that time many other geometers have followed his example, in contriving similar curves, to represent, either by their quadrature or rectification, elliptic functions. Of those who have been most successful in devising curves which should possess the required properties, may be mentioned M. GUDERMANN, M. VERHULST of Brussels, and M. SERRET of Paris. These geometers however have succeeded in deriving from those curves scarcely any of the properties of elliptic integrals, even the most elementary. This barrenness in results was doubtless owing to the very artificial character of the genesis of those curves, devised, as they were, solely to satisfy one condition only of the general problem*.

In 1841 a step was taken in the right direction. MM. CATALAN and GUDERMANN, in the journals of Liouville and Crelle, showed how the arcs of spherical conic sections might be represented by elliptic integrals of the third order and *circular* form. They did not, however, extend their investigations to the case of elliptic integrals of the third order and logarithmic form; nor even to that of the first order. These cases still remained, without any analogous geometrical representative, a blemish to the theory.

Some years ago, when engaged in the discussion of the problem of the rotation of a rigid body round a fixed point, by the help of an auxiliary ellipsoid, I had continually brought under my notice, in the course of my investigations, the sections of a sphere by a concentric cone, or as they now are generally named, spherical conic

* LEGENDRE a cherché à représenter en général, la fonction dig. (c, ϕ) par un arc de courbe; mais ses tentatives ne nous ont pas semblé heureuses, car il n'est parvenu à resoudre complétément le problème, qu'en employant une courbe transcendante, dans laquelle l'amplitude ϕ et l'arcs ont entre eux une relation géométrique encore plus difficile à saisir que dans la lemniscate.—VERHULST, *Traité des Fonctions Elliptiques*, p. 295.

sections. It accordingly became necessary that I should give especial attention to the nature of those curves. I succeeded in showing that the elliptic integral of the first order, which is merely a particular case of the circular form of elliptic integrals of the third order, represents a spherical conic section whose principal arcs have a certain relation to each other. Besides, I was so fortunate as to hit upon the true geometrical representative of an elliptic integral of the third order and logarithmic form. I discovered it to be the curve of intersection of a right elliptic cylinder by a paraboloid of revolution having its axis coincident with that of the cylinder. These researches were published in the early part of the present year*. There still remained, without investigation, the case when the parameter is negative and greater than 1. The geometrical representative of this peculiar form, I announced to be a curve, which I called the *Logarithmic hyperbola*. In the Theory of Elliptic Integrals, p. 159, I have said, "If a right cylinder standing on a plane hyperbola as a base, be substituted for the elliptic cylinder, the curve of intersection may be named the *logarithmic hyperbola*. It will have four infinite branches, whose asymptots will be the infinite arcs of two equal plane parabolas. This curve, and not the spherical ellipse, is the true analogue of the common hyperbola." No demonstration, however, of these properties was given in that treatise.

The main object of the following paper is to prove, that *Elliptic Integrals of every order, the parameter taking any value whatever between positive and negative infinity, represent the intersections of surfaces of the second order.*

To these curves may be given the appropriate name of *Hyperconic sections.*

These surfaces divide themselves into two classes, of which the sphere and the paraboloid of revolution are the respective types; from the one arise the circular functions, from the other the logarithmic and exponential. The circular integral of the third order is derived from the sphere, while the logarithmic function of the same order is founded on the paraboloid of revolution.

Although in the following pages I have, for the sake of simplicity, derived the properties of those curves, or of the integrals which represent them, from the intersections of these normal surfaces,—the sphere and the paraboloid,—with certain cylindrical surfaces; yet the intersections so produced may be considered as the intersections of these normal surfaces with various other surfaces of the second order. Let $U=0$ be the equation of the sphere or paraboloid, and $V=0$ the equation of the cylinder. The simultaneous equations $U=0$, $V=0$ give the equations of the curve of intersection. Let f be any abstract number whatever; then $U+fV=0$ is the equation of another surface of the second order passing through the curve of intersection. Let $U=0$ be the equation of a sphere, for example. Accordingly as we assign suitable values to the number f, we may make the equation $U+fV=0$ represent any central surface of the second order. But we cannot, by any substitution or

* The Theory of Elliptic Integrals, and the Properties of Surfaces of the Second Order, applied to the investigation of the motion of a body round a fixed point. London: G. Bell, 1851.

rational transformation, make the equation $U+fV=0$ represent a non-central surface instead of a central one, or *vice versâ*.

Although a remarkable relation exists between the areas and lengths of some of these hyperconics, such as the circle and the spherical ellipse, yet more distinctly to show the analogy which pervades all those curves, I have not had recourse in any case to the method of "elliptic quadratures," as it is termed*. We cannot admit such a violation of the law of geometrical continuity as to suppose, that while a function in one state represents a curve line, in another, immediately succeeding, it must express an area. Such can only be taken as a conventional explanation, until the real one, characterized by the simplicity of truth, shall present itself.

In the course of these investigations, it will be shown that the formulæ for the comparison of elliptic integrals, which are given by LEGENDRE and other writers on this subject, follow simply as geometrical inferences from the fundamental properties of those curves; and that the ordinary conic sections are merely particular cases of those more general curves above referred to, under the name of hyperconic sections.

It will doubtless appear not a little singular, that the principal properties of those functions, their classification, their transformations, the comparison of integrals of the third order, with conjugate or reciprocal parameters, were all investigated and developed before geometers had any idea of the true geometrical origin of those functions. It is as if the formulæ of trigonometry had been derived from an algebraical definition, before the geometrical conception of the circle had been admitted. As trigonometry may be defined, the development of the properties of circular arcs, whether described on a plane or on the surface of a sphere; so this higher trigonometry, or the theory of elliptic integrals, may best be interpreted as the development of the relations which exist between the arcs of hyperconic sections.

Indeed it may with truth be asserted, that nearly all the principal functions, on which the resources of analysis have chiefly been exhausted, whether they be circular, logarithmic, exponential or elliptic, arise out of the solution of this one general problem, to determine the length of an arc of a hyperconic section.

It may be said, we cannot by this method derive any properties of elliptic integrals which may not algebraically be deduced from the fundamental expressions appropriately assumed. But surely no one will assert that the properties of curve lines should be algebraically developed, without any reference to their geometrical types.

We might from algebraical expressions suitably chosen, derive every known property of curve lines, without having in any instance a conception of the geometrical types

* En considérant les fonctions elliptiques comme des secteurs, dont l'angle est précisément égal à l'amplitude ϕ, nous avons eu l'avantage de justifier la dénomination d'amplitude appliquée à l'angle ϕ; et même celle de *fonctions elliptiques*, en général, puisque les courbes algébraiques par lesquelles nous avons représentés ces transcendantes, se construisent avec facilité au moyen des rayons vecteurs d'une ou de deux ellipses données. —VERHULST, *Traité des Fonctions Elliptiques*, p. 295.

which they represent. The theory of elliptic integrals was developed by a method the inverse of that pursued in establishing the formulæ of common trigonometry. In the latter case, the geometrical type was given—the circle—to determine the algebraical relations of its arcs. In the theory of elliptic integrals, the relations of the arcs of unknown curves are given, to determine the curves themselves. This is briefly the object of the present paper.

The true geometrical basis of this theory would doubtless long since have been developed, had not geometers sought to discover the types of those functions among plane curves. They were beguiled into this course by observing, that in one case—that of the second order—the representative curve is obviously a plane ellipse. Hence they were led by a seeming analogy to search for the types of the other integrals among plane curves also.

The author hopes in a future communication to the Royal Society, the present having grown under his hands beyond the limits he anticipated, to extend his researches to elliptic integrals with imaginary parameters, and to show the true geometrical meaning of such expressions. It has long been known, that, by the aid of the imaginary transformation $\sin\phi=\sqrt{-1}\tan\psi$, we may pass from the logarithmic to the circular type, and conversely; but it has not, however, been observed that this transformation enables us to effect this transition, because it changes the algebraic expression for the arc of a parabola into that for a circular arc or area, and conversely. The striking analogies developed between the formulæ of the trigonometry of the circle and that of the parabola will be found very curious and instructive.

I have attempted thus to place on its true geometrical basis, a somewhat abstruse department of analysis, and to clear up the elementary notions from which it may, with the utmost simplicity, be developed. It is only in the maturity of a science, that the relations which bind together its cardinal ideas become simplified. An author, who has himself contributed much to the progress of mathematical science, well observes,—" qui il est bien rare qu'une théorie sorte sous sa forme la plus simple des mains de son premier auteur. Nous pensons qu'on sert peut-être plus encore la science en simplifiant, de la sorte, des théories déjà connues, qu'en l'enrichissant de théories nouvelles, et c'est là un sujet auquel on ne saurait s'appliquer avec trop de soin."—GERGONNE, *Annales des Mathématiques*, tom. xix. p. 338.

II. I have ventured to make some alterations in the established notation of elliptic integrals. I have written i for the modulus, instead of c; and j for its complement instead of b; so that $i^2+j^2=1$.

The symbol c, used by writers on this subject to designate the modulus, was adopted by analogy from the formula for the rectification of a plane elliptic arc by an integral of the second order. Although in the circular forms of the third order it still signifies a certain ellipticity, yet it has no longer the same signification in the usual form of the first order, or in the logarithmic form of the third.

Instead of the usual symbol, $\Delta=\sqrt{1-c^2\sin^2\phi}=\sqrt{1-i^2\sin^2\phi}$, $\sqrt{\text{I}}$ has been substituted, when i is the modulus. Should it become necessary to designate the amplitude, the expression may be written $\sqrt{\text{I}_\phi}$, or $\sqrt{{}_\phi\text{I}}$.

For the elliptic integrals of the first and second orders, which are usually written $F_c(\phi)$ and $E_c(\phi)$, I have substituted $\int\frac{d\phi}{\sqrt{\text{I}}}$ and $\int d\phi\sqrt{\text{I}}$. The surface of revolution may be named the *generating surface*, while the intersecting surface is always a cylindrical surface. The parameter, of which p is the general symbol, we shall suppose to vary from positive to negative infinity, and to pass through all intermediate states of magnitude.

The nature of the representative curve will depend on the value assigned to the parameter p in the expression $\text{K}\int\frac{d\phi}{[\pm p\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}$. The modulus we shall assume to be invariable and less than 1. In this progress from $+\infty$ to $-\infty$, the parameter passes through thirteen distinct values, each of which will cause a variation in the species or properties of the hyperconic section, the representative curve of the given elliptic integral.

In the following Table we may observe that the generating surface in passing from a sphere to a paraboloid, in its state of transition, becomes a plane.

It is somewhat remarkable, that the common form of the elliptic integral of the first order does not appear in the Table, although it is implicitly contained in cases II. and VIII.; for in the circular form of the third order, when the parameter is equal to the modulus i, we can reduce the third order to the first. The reason why the first form of elliptic integral does not appear in the Table is this; in the thirteen cases given, the origin is placed at the centre, or symmetrically with respect to the represented curve. When the elliptic integral of the first order is given in the usual form, without a parameter, it represents a spherical parabola, but the origin is non-symmetrical, that is, the origin is placed at a focus. See Theory of Elliptical Integrals, p. 33.

Instead of p, the general symbol for the parameter, we may substitute for it particular values, such as l, m, or n, as the case may require. The quantities l, m, n, i and j, are connected by the following equations:—

$$\left.\begin{array}{l} i^2+j^2=1,\quad lm=i^2,\quad \text{and } m-n+mn=i^2, \text{ in the circular form,} \\ i^2+j^2=1,\quad ln=i^2,\quad \text{and } m+n-mn=i^2, \text{ in the logarithmic form,} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (1.)$$

m and n may be called *conjugate parameters*; while l and m, or l and n may be termed *reciprocal parameters*.

These thirteen cases are exhibited in the following Table.

TABLE.

Case.	Sign.	Parameter.	Generating surface.	Cylindrical surface.	Hyperconic section.
I.	$+$	$p=n=\infty$	Sphere	Elliptic cylinder	Circular sections of elliptic cylinder.
II.	$+$	$p=n=i$, or $m=n$.	Sphere	Elliptic cylinder	Spherical parabola.
III.	$+$	$p=n>0$......	Sphere	Elliptic cylinder	Spherical ellipse.
IV.	$\pm$	$p=n=0$......	Plane	Elliptic cylinder	Plane ellipse.
V.	$-$	$p=m=1-\sqrt{1-i^2}$, or $m=n$.	Paraboloid indefinitely attenuated.	Circular cylinder.........	Circular logarithmic ellipse.
VI.	$-$	$p=m$, or $p=n<i^2$ ...	Paraboloid	Elliptic cylinder	Logarithmic ellipse.
VII.	$-$	$p=m=i^2$......	Plane	Elliptic cylinder	Plane ellipse.
VIII.	$-$	$p=m=i$......	Sphere	Elliptic cylinder	Spherical parabola.
IX.	$-$	$p=m>i^2$ $p=m<1$.	Sphere	Elliptic cylinder	Spherical ellipse.
X.	$-$	$p=l=1$......	Plane	Hyperbolic cylinder ...	Plane hyperbola.
XI.	$-$	$p=l>1$......	Paraboloid	Hyperbolic cylinder ...	Logarithmic hyperbola.
XII.	$-$	$p=l=1+\sqrt{1-i^2}$, or $m=n$.	Paraboloid	Hyperbolic cylinder.	Equiparametral logarithmic hyperbola.
XIII.	$-$	$p=l=\infty$	Paraboloid	Vertical plane	Parabola.

Cases I., IV., VII., X., XIII. give the formulæ for the rectification of the ordinary conic sections; the generating surface in these cases being a plane. When the generating surface is a sphere, we get the spherical hyperconic sections; when a paraboloid, the logarithmic hyperconic sections result.

SECTION II.—*On the Spherical Ellipse.*

III. A spherical ellipse may be defined as the curve of intersection of a cone of the second degree with a concentric sphere.

In the spherical ellipse there are two points analogous to the foci of the plane ellipse, such *that the sum of the arcs of the great circles drawn from those points to any point on the curve is constant.* Let α and β be the principal semiangles of the cone; 2α and 2β are therefore the *principal arcs* of the spherical ellipse. Let two right lines be drawn from the vertex of the cone in the plane of the angle 2α, making with the internal axe of the cone equal angles ϵ, such that

$$\cos\epsilon=\frac{\cos\alpha}{\cos\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

These lines are usually called *focals,* or the *focal lines* of the cone. The points in

which they meet the surface of the sphere are termed the *foci* of the spherical ellipse.

IV. *Every umbilical surface of the second order has two concentric circular sections, whose planes, in the case of cones, pass through the greater of the external axes. Perpendiculars drawn to the planes of those sections, passing through the vertex,—they may be called the* CYCLIC AXES *of the cone—make with the internal axis of the cone in the plane of* 2β*—the plane passing through the internal and the lesser external axis*—equal angles η, such that

$$\cos\eta=\frac{\sin\beta}{\sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Let a series of planes be drawn through the vertex, and perpendicular to the successive sides of the cone. This series of planes will envelope a second cone, which usually is called the *supplemental cone* to the former. The cones are so related, that the planes of the circular sections of the one are perpendicular to the focals of the other, and conversely.

V. The equation of the spherical ellipse may be found as follows, from simple geometrical considerations.

Let 2α and 2β be the greatest and least vertical angles of the cone; the origin of coordinates being placed at the common centre of the sphere and cone. Let the internal axis of the cone meet the surface of the sphere in the point Z, which may be taken as the pole. Let ρ be an arc of a great circle drawn from the point Z to any point Q on the curve. ψ being the angle which the plane of this circle makes with the plane of 2α, we shall have for the polar equation of the spherical ellipse,

$$\frac{1}{\tan^2\rho}=\frac{\cos^2\psi}{\tan^2\alpha}+\frac{\sin^2\psi}{\tan^2\beta}.$$

To show this, through the point Z let a tangent plane be drawn to the sphere. This plane will intersect the cone in an ellipse. This ellipse may be called the *plane base* of the cone, while the portion of the surface of the sphere within the cone may be termed the *spherical base* of the cone. The plane of the great circle passing through Z and Q will cut the plane base of the cone in the radius vector R; and if we write A and B for the semiaxes of this ellipse, whose plane touches the sphere, we shall have, for the common polar equation of this ellipse, the centre being the pole,

$$\frac{1}{R^2}=\frac{\cos^2\psi}{A^2}+\frac{\sin^2\psi}{B^2}.$$

Now the radius of the sphere being k, and ρ, α, β, the angles subtended at the centre by R, A, B, we shall clearly have

$$R=k\tan\rho, \qquad A=k\tan\alpha, \qquad B=k\tan\beta; \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

whence

$$\frac{1}{\tan^2\rho}=\frac{\cos^2\psi}{\tan^2\alpha}+\frac{\sin^2\psi}{\tan^2\beta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

We may write this equation in the form

$$\frac{1-\sin^2\rho}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}(1-\sin^2\alpha)+\frac{\sin^2\psi}{\sin^2\beta}(1-\sin^2\beta);$$

or reducing,

$$\frac{1}{\sin^2\rho}=\frac{\cos^2\psi}{\sin^2\alpha}+\frac{\sin^2\psi}{\sin^2\beta}. \quad \ldots \ldots \ldots \quad (6.)$$

This is the equation of the spherical ellipse under another form, which may be obtained independently, by orthogonally projecting the spherical ellipse on the plane of the external axes; or by taking the spherical ellipse as the symmetrical intersection of a right elliptic cylinder with the sphere.

VI. *If in the major principal arc* 2α *of the spherical ellipse, we assume two points equidistant from the centre, the distance* ε *being determined by the condition* $\cos\varepsilon=\frac{\cos\alpha}{\cos\beta}$, *as in* (2.), *the sum of the arcs of the great circles drawn from these points —the foci—to any point on the spherical ellipse is constant, and equal to the principal arc* 2α. For a proof of this well-known property, the reader is referred to the Theory of Elliptical Integrals, p. 12.

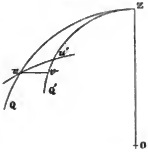
VII. *The product of the sines of the perpendicular arcs let fall from the foci of a spherical ellipse on the arc of a great circle touching it, is constant.*

Let ϖ and ϖ' be the perpendicular arcs let fall from the foci on the tangent arc of a great circle; we shall have

$$\sin\varpi \sin\varpi'=\sin(\alpha+\varepsilon)\sin(\alpha-\varepsilon). \quad \ldots \ldots \ldots \quad (7.)^*$$

VIII. To find an expression for the length of a curve described on the surface of a sphere, whose radius is 1.

Fig. 1.



Let u and u' be two consecutive points on the curve, ZQ, ZQ' the arcs of two great circles passing through them inclined to each other at the indefinitely small angle $d\psi$. Through u let a plane be drawn perpendicular to OZ, and meeting the great circle ZQ' in v.

Then ultimately uvu' may be taken as a right-angled triangle, whence $\overline{uu'}^2=\overline{uv}^2+\overline{u'v}^2$.

Now $uu'=d\sigma$, $uv=\sin\rho\, d\psi$, $u'v=d\rho$, whence

$$d\sigma=[d\rho^2+\sin^2\rho\, d\psi^2]^{\frac{1}{2}}. \quad \ldots \ldots \ldots \quad (8.)$$

Integrating this expression between the limits $\rho_{\prime}$ and $\rho_{\prime\prime}$, or ψ and 0, accordingly as we take ρ or ψ for the independent variable, we get

$$\sigma=\int_{\rho''}^{\rho'} d\rho\left[1+\sin^2\rho\left(\frac{d\psi}{d\rho}\right)^2\right]^{\frac{1}{2}}; \quad \text{or } \sigma=\int_0^{\psi} d\psi\left[\left(\frac{d\rho}{d\psi}\right)^2+\sin^2\rho\right]^{\frac{1}{2}}. \quad \ldots \ldots \quad (9.)$$

IX. To apply this expression to find the length of an arc of a spherical ellipse.

In this case it will be found simpler to integrate the differential expression for an

* Theory of Elliptical Integrals, &c., p. 13.

arc of a curve, taking ρ instead of ψ as the independent variable. We may derive from (6.) the following expressions,

$$\sin^2\psi=\frac{\sin^2\beta}{\sin^2\rho}\left\{\frac{\sin^2\alpha-\sin^2\rho}{\sin^2\alpha-\sin^2\beta}\right\},\ \cos^2\psi=\frac{\sin^2\alpha}{\sin^2\rho}\left\{\frac{\sin^2\rho-\sin^2\beta}{\sin^2\alpha-\sin^2\beta}\right\} \quad \ldots \quad (10.)$$

Differentiating the former with respect to ψ and ρ, and eliminating $\sin\psi$, $\cos\psi$; using for this purpose the relations established in (10.), we find

$$\frac{d\psi}{d\rho}=\frac{-\sin\alpha\sin\beta\cos\rho}{\sin\rho\sqrt{\sin^2\alpha-\sin^2\rho}\sqrt{\sin^2\rho-\sin^2\beta}} \quad \ldots \quad (11.)$$

Substituting this value of $\frac{d\psi}{d\rho}$ in the general expression for the arc; the resulting equation will become

$$\sigma=\int d\rho\left[\frac{\sin\rho\sqrt{\cos^2\rho-\cos^2\alpha\cos^2\beta}}{\sqrt{(\sin^2\alpha-\sin^2\rho)(\sin^2\rho-\sin^2\beta)}}\right], \quad \ldots \quad (12.)$$

an elliptic integral which may be reduced to the usual form by the following transformation: assume—

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\phi+\sin^2\beta\sin^2\phi}{\tan^2\alpha\cos^2\phi+\tan^2\beta\sin^2\phi}. \quad \ldots \quad (13.)$$

The limits of integration are 0 and $\frac{\pi}{2}$. Differentiating this expression, and introducing into (12.) the relations assumed in (13.), we obtain for the arc the following expression:—

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{d\phi}{\left[1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}\right)\sin^2\phi}}\right] \quad \ldots \quad (14.)$$

Let e be the eccentricity of the plane base of the cone, whose semiaxes are A and B, as in (V.),

$$e^2=\frac{A^2-B^2}{A^2}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}, \quad \text{as in (4.)},$$

(3.) gives $$\sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha},$$

and we derive from (2.) $$\sin^2\iota=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta};$$

or grouping these results together,

$$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=m$$

$$\sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2 \quad \ldots \quad (15.)$$

$$\sin^2\iota=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta}=n.$$

If we introduce these values into (14.), the transformed equation will become

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-\sin^2\eta\sin^2\varphi}}\right], \quad \ldots \ldots \quad (16.)$$

an elliptic integral of the third order and circular form, since e^2 is greater than $\sin^2\eta$, and less than 1.

This is case IX. in the Table, page 6.

This is one of the simplest forms to which the rectification of an arc of a spherical ellipse can be reduced. The parameter of the elliptic integral is the square of the eccentricity of the plane elliptic base, and the modulus is the sine of half the angle between the planes of the circular sections of the cone.

If we write m for e^2, i for $\sin\eta$, and express the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ in terms of m and i, the expression (16.) may be transformed into

$$\sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\left[\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}\right]. \quad \ldots \ldots \quad (17.)$$

It is easily shown that the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ of the elliptic integral in (16.) or its equal $\left(\frac{1-m}{m}\right)\sqrt{mn}$ is the square root of the *criterion of sphericity*,

$$\kappa=(1-m)\left(1-\frac{i^2}{m}\right).$$

For if we substitute in this expression for i, its value given in (1.) $m-n+mn=i^2$, we shall find

$$\sqrt{\kappa}=\frac{\tan\beta}{\tan\alpha}\sin\beta=\left(\frac{1-m}{m}\right)\sqrt{mn}. \quad \ldots \ldots \quad (18.)$$

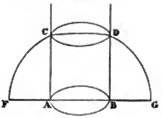
As $\sqrt{\kappa}$ is manifestly real, the elliptic integral is of the circular form.

X. We may, by the method of rectangular coordinates, derive an expression for the arc of a spherical ellipse.

In this case we shall consider the spherical ellipse as the curve of intersection of a right elliptic cylinder by a sphere having its centre on the axis of the cylinder.

Let $\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad$ and $x^2+y^2+z^2=k^2 \quad \ldots \quad (19.)$

Fig. 2.



be the equations of the cylinder and sphere, ABCD and FGCD, then $d\sigma$ being the element of an arc on the surface of a sphere whose radius is 1, $kd\sigma$ will be the element of the corresponding arc on the surface of the sphere whose radius is k.

Hence $$k\frac{d\sigma}{d\lambda}=\sqrt{\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2+\left(\frac{dz}{d\lambda}\right)^2}, \quad \ldots \ldots \quad (20.)$$

x, y and z being functions of the independent variable λ.

Assume
$$\left.\begin{aligned} x^2&=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda}, \quad y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda} \\ z^2&=\frac{a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda}. \end{aligned}\right\} \quad \ldots \quad (21.)$$

Differentiating these expressions,
$$\left(\frac{dx}{d\lambda}\right)^2=\frac{a^4b^4\sin^2\lambda}{[a^2\cos^2\lambda+b^2\sin^2\lambda]^3}, \quad \left(\frac{dy}{d\lambda}\right)^2=\frac{a^4b^4\cos^2\lambda}{[a^2\cos^2\lambda+b^2\sin^2\lambda]^3};$$

and as $xdx+ydy+zdz=0$, $\quad \ldots \quad (22.)$
$$\left(\frac{dz}{d\lambda}\right)^2=\frac{a^4b^4(a^2-b^2)^2\sin^2\lambda\cos^2\lambda}{[a^2\cos^2\lambda+b^2\sin^2\lambda]^3[a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda]}.$$

Substituting these expressions in (20.), we find
$$\left(\frac{d\sigma}{d\lambda}\right)^2=\frac{a^4b^4[a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda+(a^2-b^2)^2\sin^2\lambda\cos^2\lambda]}{k^2[a^2\cos^2\lambda+b^2\sin^2\lambda]^3[a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda]}. \quad \ldots \quad (23.)$$

The numerator of this expression may be resolved into the factors
$$[a^2\cos^2\lambda+b^2\sin^2\lambda][(k^2-a^2)\cos^2\lambda+(k^2-b^2)\sin^2\lambda],$$
and the equation may now be written
$$\frac{d\sigma}{d\lambda}=\frac{a^2b^2\sqrt{(k^2-a^2)\cos^2\lambda+(k^2-b^2)\sin^2\lambda}}{k[a^2\cos^2\lambda+b^2\sin^2\lambda]\sqrt{a^2(k^2-a^2)\cos^2\lambda+b^2(k^2-b^2)\sin^2\lambda}}. \quad \ldots \quad (24.)$$

Assume
$$\tan^2\psi=\left(\frac{k^2-b^2}{k^2-a^2}\right)\tan^2\lambda. \quad \ldots \quad (25.)$$

Hence
$$\frac{d\lambda}{d\psi}=\frac{\sqrt{(k^2-a^2)(k^2-b^2)}}{(k^2-a^2)\sin^2\psi+(k^2-b^2)\cos^2\psi}.$$

(24.) may now be transformed into
$$\frac{d\sigma}{d\psi}=\frac{d\sigma}{d\lambda}\frac{d\lambda}{d\psi}=\frac{a^2b^2\sqrt{(k^2-a^2)(k^2-b^2)}}{k[a^2(k^2-b^2)\cos^2\psi+b^2(k^2-a^2)\sin^2\psi]\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}}. \quad \ldots \quad (26.)$$

If we imagine a concentric cone to pass through the mutual intersection of the cylinder and the sphere, we shall have
$$\left.\begin{aligned} &a=k\sin\alpha, \quad b=k\sin\beta, \\ &\sin^2\eta=\frac{a^2-b^2}{a^2}, \quad e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=\frac{k^2(a^2-b^2)}{a^2(k^2-b^2)} \end{aligned}\right\} \quad \ldots \quad (27.)$$

Whence (26.) may be transformed into
$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\left[\frac{d\psi}{[1-e^2\sin^2\psi]\sqrt{1-\sin^2\eta\sin^2\psi}}\right], \quad \ldots \quad (28.)$$

an expression identically the same with (16.).

The angle ψ in this expression is identical with ϕ in (16.).

For
$$x^2+y^2=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda+b^2\sin^2\lambda}=\frac{a^4+b^4\tan^2\lambda}{a^2+b^2\tan^2\lambda};$$
eliminating $\tan\lambda$ by (25.),
$$x^2+y^2=\frac{a^4(k^2-b^2)\cos^2\psi+b^4(k^2-a^2)\sin^2\psi}{a^2(k^2-b^2)\cos^2\psi+b^2(k^2-a^2)\sin^2\psi}.$$

Now $a^2=k^2\sin\alpha$, $b^2=k^2\sin^2\beta$, $k^2-a^2=k^2\cos^2\alpha$, $k^2-b^2=k^2\cos^2\beta$, and $x^2+y^2=k^2\cos^2\rho$. Reducing, we get

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\psi+\sin^2\beta\sin^2\psi}{\tan^2\alpha\cos^2\psi+\tan^2\beta\sin^2\psi}. \qquad (29.)$$

Comparing this expression with (13.), we see that

$$\varphi=\psi. \qquad (30.)$$

XI. In the foregoing expressions (17.) and (28.) for the rectification of an arc of a spherical ellipse, the elliptic integrals are of the third order and circular form, with *negative* parameters. We shall now proceed to show that the same arc may be expressed by an elliptic integral of the third order and circular form, having a *positive* parameter.

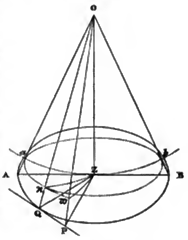
It is shown in most elementary treatises on the integral calculus, in its application to the rectification of plane curves, that if p the perpendicular let fall from a fixed point as pole on a tangent to the curve, makes the angle λ with a fixed right line drawn through the pole, t being the intercept of the tangent between the point of contact and the foot of the perpendicular, we shall have

$$\left.\begin{aligned}\pm s&=\int p\,d\lambda+\frac{dp}{d\lambda}\\ \text{and } t&=-\frac{dp}{d\lambda}\end{aligned}\right\} \qquad (31.)$$

The signs of s to be taken as the curve is concave or convex to the pole.

XII. To investigate an analogous formula for the rectification of a spherical curve, the intersection of a cone of any order with a concentric sphere.

Fig. 3.



Let a point Z be assumed on the surface of the sphere as pole, and through this point a tangent plane ZAQB, or (Θ), to the sphere being drawn, the cone whose vertex is at O, the centre of the sphere, and which passes through the given spherical curve, will cut this tangent plane (Θ) in a plane curve AQB, whose rectification may be effected, when possible, by (31.). Now a tangent plane OQP, or (T), may be conceived as drawn touching the cone, and cutting the tangent plane (Θ) in a right line QP or t, which will be a tangent to the plane curve in (Θ). It will also cut the sphere in an arc of a great circle ($\varkappa\varpi$) which will touch the spherical curve in $\varkappa$. Let the distance QO of the point of contact of the line t with the plane curve from the centre of the sphere be R. Through the centre of the sphere let a plane OZP, or (Π), be drawn at right angles to the straight line t. Now this plane, as it is perpendicular to t, must be perpendicular to the planes (Θ) and (T) which pass through t. As the plane (Π) is perpendicular to the plane (Θ), it must pass through (Z) the point of contact of this plane with the sphere, and cut the plane

of the curve AQB in a right line ZP, or p, which passes through the pole, the point of contact of (Θ) with the sphere. This line p being in (Π) must be perpendicular to t. The plane (Π) will also cut the sphere in an arc of a great circle $Z\varpi=\varpi$, perpendicular to $x\varpi$, the tangent arc to the spherical curve; for these arcs must be at right angles to each other, since the planes in which they lie, (Π) and (T), are at right angles. Let P be the distance OP of the point, in which the plane (Π) cuts the right line t, from the centre of the sphere; r the distance ZQ of the pole of the plane curve to the point in which t touches it, τ being the angle which t subtends at the centre of the sphere, and k its radius,

$$\left.\begin{array}{l} R^2=k^2+r^2,\ P^2=k^2+p^2,\ t^2=r^2-p^2=R^2-P^2 \\ p=k\sin\varpi,\ t=P\tan\tau \end{array}\right\} \quad (32.)$$

τ is the angle between OQ and OP.

Let ds be the element of an arc of the plane curve between any two consecutive positions of R, indefinitely near to each other; $k\mathrm{d}\sigma$ the corresponding element of the spherical curve between the same consecutive positions of R. Then the areas of the elementary triangles on the surface of the cone, between these consecutive positions of R, having their vertices at the centre of the sphere, and for bases the elements of the arcs of the plane and spherical curves respectively, are as their bases multiplied by their altitudes. Let S and S′ be these areas; then

$$S:S'::P\frac{\mathrm{d}s}{\mathrm{d}\lambda}:k^2\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}. \quad (a.)$$

But the areas of triangles are also as the products of their sides into the sines of the contained angles, *i.e.* in this case as the squares of the sides, or

$$S:S'::R^2:k^2, \quad (b.)$$

or

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}=\frac{P}{R^2}\frac{\mathrm{d}s}{\mathrm{d}\lambda}; \quad (c.)$$

putting for ds its value given in (31.),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}=\frac{P}{R^2}\left\{\frac{\mathrm{d}^2p}{\mathrm{d}\lambda^2}+p\right\}. \quad (d.)$$

Now $p=P\sin\varpi$, $P^2=R^2-t^2$, and $P^2=k^2+p^2$;

whence
$$P\frac{\mathrm{d}P}{\mathrm{d}\lambda}=p\frac{\mathrm{d}p}{\mathrm{d}\lambda},\ \text{and}\ t=-\frac{\mathrm{d}p}{\mathrm{d}\lambda}.$$

Substituting these values in (d.),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}=\sin\varpi+\frac{1}{R^2}\left\{P\frac{\mathrm{d}^2p}{\mathrm{d}\lambda^2}-\frac{\mathrm{d}P}{\mathrm{d}\lambda}\frac{\mathrm{d}p}{\mathrm{d}\lambda}\right\}. \quad (e.)$$

We now proceed to show that the last term of this equation is the differential of the arc, with respect to λ, subtended at the centre of the sphere.

This arc being τ, $\tan\tau=\frac{t}{P}$, $\cos\tau=\frac{P}{R}$.

$$\because \quad \frac{d\tau}{d\lambda}=\frac{1}{R^2}\left\{P\frac{dt}{d\lambda}-t\frac{dP}{d\lambda}\right\}, \quad \text{(f.)}$$

or as

$$t=-\frac{dp}{d\lambda}, \; \frac{d\tau}{d\lambda}=-\frac{1}{R^2}\left\{P\frac{d^2p}{d\lambda^2}-\frac{dp}{d\lambda}\frac{dP}{d\lambda}\right\}. \quad \text{(g.)}$$

Adding this equation to (e.), we get for the final result,

$$\left.\begin{array}{l} \pm\sigma=\int d\lambda \sin\varpi-\tau. \\ \text{If } t=\frac{dp}{d\lambda}, \text{ the formula becomes } \pm\sigma=\int d\lambda \sin\varpi+\tau. \end{array}\right\} \quad \text{(33.)}$$

Throughout these pages, to avoid circumlocution and needless repetitions, we shall designate as the *pro*-jected tangent, or briefly as the *protangent*, that portion of a tangent to a curve, whether it be a right line, a circle, or a parabola, between its point of contact, and a perpendicular from a fixed point let fall upon it, whether this perpendicular be a right line, or a circular, or a parabolic arc. This definition is the more necessary, as the protangent will continually occur in the following investigations. The term is not inappropriate, as the *pro*-tangent is the *projection* of the radius vector on the tangent.

XIII. To apply the formula (33.) to the rectification of the spherical ellipse.

Let, as before, A and B be the semiaxes of the plane elliptic base of the cone, r the central radius vector drawn to the point of contact of the tangent t, p the perpendicular from the centre on this tangent, t the intercept of the tangent to the plane ellipse between the point of contact and the foot of the perpendicular, λ the angle between p and A. Let α, β, ρ, ϖ, τ be the angles subtended at the centre of the sphere, whose radius is 1, by the lines A, B, r, p, t, we shall consequently have

$$A=k\tan\alpha, \; B=k\tan\beta, \; r=k\tan\rho, \; p=k\tan\varpi, \text{ and } t=\sqrt{k^2+p^2}\tan\tau. \quad \text{(34.)}$$

Now in the plane ellipse

$$p^2=A^2\cos^2\lambda+B^2\sin^2\lambda,$$

therefore in the spherical ellipse

$$\tan^2\varpi=\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda; \quad \text{(35.)}$$

whence

$$\sec^2\varpi=\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda.$$

Dividing the former by the latter,

$$\sin^2\varpi=\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda}. \quad \text{(36.)}$$

Introducing this value of $\sin\varpi$ into (32.), the general form for spherical rectification, the resulting equation will become

$$\sigma=\int d\lambda\left[\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda}\right]^{\frac{1}{2}}-\tau. \quad \text{(37.)}$$

XIV. To reduce this expression to the usual form of an elliptic integral.

Assume

$$\tan\chi=\cos\epsilon\tan\lambda. \quad \text{(38.)}$$

We must first show that this amplitude χ is equal to the amplitude φ in (13.), and therefore to ψ in (25.), as we proved in (X.).

In an ellipse, if ψ and λ are the angles which a central radius vector, and a perpendicular from the centre, on the tangent drawn through its extremity, make with the major axis, we know that $\tan\psi=\frac{B^2}{A^2}\tan\lambda=\frac{\tan^2\beta}{\tan^2\alpha}\tan\lambda$. Introducing this value of $\tan\psi$ into (6.) and reducing,

$$\cos^2\rho=\cos^2\alpha\cos^2\beta\left[\frac{\tan^2\alpha\cos^2\lambda+\tan^2\beta\sin^2\lambda}{\tan^2\alpha\cos^2\beta\cos^2\lambda+\tan^2\beta\cos^2\alpha\sin^2\lambda}\right].$$

Comparing this value of $\cos^2\rho$ with that assumed for $\cos^2\rho$ in (13.), namely,

$$\cos^2\rho=\frac{\sin^2\alpha\cos^2\varphi+\sin^2\beta\sin^2\varphi}{\tan^2\alpha\cos^2\varphi+\tan^2\beta\sin^2\varphi},$$

we get, after some reductions,

$$\tan\varphi=\cos\epsilon\tan\lambda. \quad \ldots \ldots \quad (39.)$$

But in (38.) we assumed $\tan\chi=\cos\epsilon\tan\lambda$. Hence the amplitudes φ, ψ and χ in (13.), (25.), and (38.) are equal. We may accordingly write φ instead of χ. Substituting the value of $\tan\lambda$, derived from the equation $\tan\varphi=\cos\epsilon\tan\lambda$, in (38.) the integral in (37.) becomes

$$\int\frac{\cos\alpha\cos\beta[\sin^2\alpha-(\sin^2\alpha-\sin^2\beta)\sin^2\varphi]d\varphi}{[\cos^2\alpha+(\sin^2\alpha-\sin^2\beta)\sin^2\varphi]\sqrt{\sin^2\alpha\cos^2\varphi+\sin^2\beta\sin^2\varphi}}.$$

Now $$\cos\epsilon=\frac{\cos\alpha}{\cos\beta},\ \tan^2\epsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}\ \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}. \quad \ldots \ldots \quad (40.)$$

Making the substitutions suggested by these relations and reducing, we get

$$\sigma=\frac{\cos\beta}{\cos\alpha\sin\alpha}\int\left[\frac{d\varphi}{[1+\tan^2\epsilon\sin^2\varphi]\sqrt{1-\sin^2\eta\sin^2\varphi}}\right]-\frac{\cos\alpha\cos\beta}{\sin\alpha}\int\frac{d\varphi}{\sqrt{1-\sin^2\eta\sin^2\varphi}}-\tau, \quad . \quad (41.)$$

an elliptic integral of the third order, with a *positive* parameter, and therefore of the *circular* form.

This is case IX. in the Table, page 316.

Writing n for $\tan^2\epsilon$, i for $\sin\eta$, and expressing $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$ in terms of n and i, (41.) becomes

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\varphi}{[1+n\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}\right]-\frac{i^2}{\sqrt{mn}}\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}-\tau. \quad \ldots \quad (42.)$$

XV. To express the *protangent* τ in terms of λ and φ. We found in XII.

$$\tan^2\tau=\frac{t^2}{P^2}=\frac{t^2p^2}{P^2p^2}=\frac{(A^2-B^2)^2\sin^2\lambda\cos^2\lambda}{[k^2+a^2\cos^2\lambda+b^2\sin^2\lambda][a^2\cos^2\lambda+b^2\sin^2\lambda]}.$$

Now $$A=k\tan\alpha,\ B=k\tan\beta,\ e^2=\frac{A^2-B^2}{A^2},\ \text{and}\ \sin^2\epsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\beta},$$

whence $$\tan\tau=\frac{e^2\sin\alpha\sin\lambda\cos\lambda}{\sqrt{1-e^2\sin^2\lambda}\sqrt{1-\sin^2\epsilon\sin^2\lambda}}. \quad \ldots \ldots \quad (43.)$$

To express $\tan\tau$ *in terms of the amplitude* ϕ.

Assume the relation established in (13.) or (25.) or (38.) or (39.), $\tan\phi=\cos\epsilon\,\tan\lambda$. Introducing this condition into (43.), we obtain

$$\tan\tau=\frac{e\tan\epsilon\sin\phi\cos\phi}{\sqrt{1-\sin^2\eta\sin^2\phi}}; \quad \ldots\ldots\ldots \quad (44.)$$

or as $\sqrt{m}=e,\quad \sqrt{n}=\tan\epsilon,\quad i=\sin\eta,$

the last equation becomes

$$\tan\tau=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}. \quad \ldots\ldots\ldots \quad (45.)$$

Hence (42.) may now be written

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\phi}{[1+n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right] \quad (46.)$$

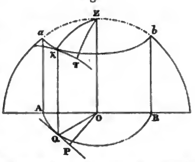
Now this formula and (17.) represent the same arc of the spherical ellipse; they may therefore be equated together. Accordingly

$$\left.\begin{aligned}\left(\frac{1+n}{n}\right)\int\left[\frac{d\phi}{[1+n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]-\left(\frac{1-m}{m}\right)\int\left[\frac{d\phi}{[1-m\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}\right]\\ =\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]\end{aligned}\right\} \quad \ldots \quad (47.)$$

This is the well-known theorem established by LEGENDRE, *Traité des Fonctions Elliptiques*, tom. i. p. 68, for the comparison of elliptic integrals of the circular form, with positive and negative parameters respectively. These circular forms arise from treating the element of the spherical conic either as the hypothenuse of an infinitesimal right-angled triangle, or as an element of a circular arc, having the same curvature. When we adopt the former principle, we obtain for the arc an elliptic integral of the third order, circular form and negative parameter. When we choose the latter, we get a circular form of the same order, with a positive parameter. Equating these expressions for the same arc of the curve, the resulting relation is LEGENDRE's theorem. We thus see how an elliptic integral with a *positive* parameter may be made to depend on another with a negative parameter less than 1 and greater than i^2.

XVI. We must not confound the angle λ in the preceding article with the angle λ in Art. (X.). Marking the latter λ by a trait thus, $\lambda_{\prime}$, to distinguish it from the former, we shall investigate the relation between them. Through ZO the axis of the cylinder, let a plane be drawn making the angle ψ with the plane ZOAa. Let this plane cut the spherical ellipse in the point x, and the plane ellipse the orthogonal projection of the latter in the point Q. Through x draw an arc of a great circle $x\tau$ touching the curve, and through Q draw a right line touching the plane ellipse. From Z let

Fig. 4.



fall the perpendicular arc $Z\varpi$ on the tangent arc of the circle, making the angle λ with the arc Za. From O let fall on the tangent to the plane ellipse at Q, the perpendicular OP making the angle $\lambda_{\prime}$ with OA.

Then $$\tan\lambda=\frac{\tan^2\alpha}{\tan^2\beta}\tan\psi, \text{ and } \tan\lambda_{\prime}=\frac{\sin^2\alpha}{\sin^2\beta}\tan\psi.$$

Hence we derive $\frac{\tan\lambda_{\prime}}{\tan\lambda}=\cos^2\epsilon$. Whence $\tan\lambda.\tan\lambda_{\prime}=\cos^2\epsilon\tan^2\lambda$.

But we have shown in (39.) that

$$\tan^2\phi=\cos^2\epsilon\tan^2\lambda,$$

whence $$\tan^2\phi=\tan\lambda\tan\lambda_{\prime}, \quad \dots\dots\dots\dots \quad (48.)$$

on *the tangent of the amplitude* ϕ *is a mean proportional between the tangents of the normal angles which a point of contact* $\varkappa$ *on the spherical ellipse and its projection* Q *on the plane ellipse the base of the cylinder produce.*

XVII. We may obtain, under another form, the rectification of the spherical ellipse.

Assume the equations of the right cylinder and generating sphere as given in (19.),

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ and } x^2+y^2+z^2=k^2.$$

Make $$x=a\sin\theta, \quad y=b\cos\theta; \quad \dots\dots\dots\dots \quad (49.)$$

hence $$z^2=k^2-a^2\sin^2\theta-b^2\cos^2\theta;$$

and therefore $$k\frac{d\sigma'}{d\theta}=\left[\frac{a^2(k^2-b^2)\cos^2\theta+b^2(k^2-a^2)\sin^2\theta}{(k^2-b^2)\cos^2\theta+(k^2-a^2)\sin^2\theta}\right]^{\frac{1}{2}}. \quad \dots\dots \quad (50.)$$

Now

$$a^2(k^2-b^2)=k^4\sin^2\alpha\cos^2\beta, \quad b^2(k^2-a^2)=k^4\sin^2\beta\cos^2\alpha, \quad k^2-b^2=k^2\cos^2\beta, \quad k^2-a^2=k^2\cos^2\alpha.$$

Substituting these values in (50.), and integrating,

$$\sigma'=\int d\theta\left[\frac{\tan^2\alpha\cos^2\theta+\tan^2\beta\sin^2\theta}{\sec^2\alpha\cos^2\theta+\sec^2\beta\sin^2\theta}\right]^{\frac{1}{2}}. \quad \dots\dots\dots \quad (51.)$$

If we now compare this formula with (37.) and make $\theta=\lambda$, we shall have

$$\sigma'-\sigma=\tau. \quad \dots\dots\dots\dots\dots\dots \quad (52.)$$

Hence we may represent the difference between two arcs of a spherical ellipse, measured from the vertices of the major and minor arcs of the curve, by the arc τ of a great circle which touches the curve.

XVIII. We may thus, by the help of the foregoing theorems, show that when any elliptic integral of the third order and circular form is given, whether the parameter be positive or negative, we may always obtain the elements of the spherical ellipse, of whose arc the given function is the representative.

Let the parameter be negative.

As $$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=m, \quad \text{and } \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2,$$

we shall have $$\tan^2\alpha=\frac{m-i^2}{i^2(1-m)},\quad \tan^2\beta=\frac{m-i^2}{i^2}. \quad \quad (53.)$$

In order that these values of $\tan\alpha$, $\tan\beta$ may be real, we must have $m>i^2$ and $m<1$.

Let the parameter be positive.

Now $$\tan^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}=n,\quad \text{and } \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=i^2,$$

hence $$\tan^2\alpha=\frac{n}{i^2},\quad \tan^2\beta=\frac{n}{i^2}\frac{(1-i^2)}{1+n}. \quad \quad (54.)$$

There is in this case no restriction on the magnitude of n.

XIX. To determine the value of the expression

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}\right],$$

when n is infinite.

As $$m-n+mn=i^2,\ \text{or } (1-m)(1+n)=1-i^2=j^2,$$

when n is infinite, $m=1$.

Resuming the expression given in (47.),

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}\right]-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right],$$

we find that when n is infinite, α is a right angle.

For $$n=\tan^2\varepsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}=\infty,\ \text{therefore } \alpha=\frac{\pi}{2}.$$

Now ψ being the angle between the spherical radius vector drawn to the extremity of the arc, and the major principal arc, we have

$$\tan\psi=\frac{\tan^2\beta}{\tan^2\alpha}\tan\lambda,\ \text{and } \tan\phi=\frac{\cos\alpha}{\cos\beta}\tan\lambda,\ \text{or } \tan\psi=\frac{\tan\beta}{\tan\alpha}\frac{\sin\beta}{\sin\alpha}\tan\phi.$$

Hence ψ is indefinitely less than ϕ, when n is infinite, or when α is a right angle. In this case therefore $\sigma=0$, and we get, when n is infinite, and ϕ not 0,

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\left[\frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}\right]=\frac{\pi}{2}. \quad \quad (55.)$$

We might have derived this theorem directly from (47.), by the transformation

$$\sqrt{n}\sin\phi=\tan\omega.$$

This is case I. in the Table, p. 316.

SECTION III.—*On the Spherical Parabola.*

XX. It remains now to exhibit a class of spherical conic sections whose rectification may be effected by elliptic integrals of the *first* order.

The curve which is the gnomonic projection of a plane parabola on the surface of a sphere, the focus being the pole, may be rectified by an elliptic integral of the first order.

Let a sphere be described touching the plane of the parabola at its focus. The spherical curve which is the intersection of the sphere with a cone, whose vertex is at its centre, and whose base is the parabola, may be called the *spherical parabola*.

To find the polar equation of this curve.

The polar equation of the parabola, the focus being the pole, is $r=\frac{2g}{1+\cos\omega}$, $4g$ being the parameter of the parabola. Let γ be the angle which g subtends at the centre of the sphere, and ρ the angle subtended by r, then

$$\tan\rho=\frac{2\tan\gamma}{1+\cos\omega}.$$

Let p be the perpendicular from the focus on a tangent to the parabola, μ the angle which this perpendicular makes with the axis of the parabola; $p=\frac{g}{\cos\mu}$. Whence in the spherical curve, as $p=k\tan\varpi$, $g=k\tan\gamma$,

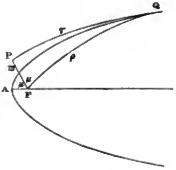
$$\tan\varpi=\frac{\tan\gamma}{\cos\mu}; \quad \ldots \ldots \quad (56.)$$

whence

$$\sin\varpi=\frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}. \quad \ldots \ldots \quad (57^*.)$$

Introduce this expression into the general form for spherical rectification, $\sigma=\int\sin\varpi d\mu+\tau$, given in (32.), we use the positive sign with τ, since $t=\frac{dp}{d\mu}$.

Fig. 5.



Now as τ, ϖ and μ are the sides and an angle of a right-angled spherical triangle, since $2\mu=\omega$, we get, by Napier's rules, $\tan\tau=\sin\varpi\tan\mu$, whence, by substitution,

$$\sigma=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left[\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right]. \quad (58.)$$

When the sphere becomes indefinitely great the spherical parabola approaches in its contour indefinitely near to the plane parabola. k being the radius of the sphere,

$$\sin\gamma=\tan\gamma=\frac{g}{k},$$

since γ in this case is indefinitely small, whence $\cos^2\gamma=1$. In this manner, since $s=k\sigma$,

* The expression for a perpendicular arc of a great circle let fall from the focus of a spherical ellipse on an arc of a great circle a tangent to this curve, is

$$\sin^2\varpi=\frac{2\sin^2\epsilon\cos^2\epsilon\cos^2\mu+(\sin^2\alpha-\sin^2\epsilon)\cos2\epsilon\pm\sin\epsilon\cos\epsilon\cos\mu\sqrt{\sin^22\alpha-\sin^22\epsilon\sin^2\mu}}{(1-\sin^22\epsilon\sin^2\mu)},$$

α being the principal major arc, ϵ the focal distance, and μ the angle which ϖ makes with α.

When the curve is the spherical parabola, $\alpha+\epsilon=\frac{\pi}{2}$, $\alpha-\epsilon=\gamma$, $2\epsilon=\frac{\pi}{2}-\gamma$, and the preceding expression, when we introduce these relations, will take the very simple form, $\sin\varpi=\frac{\sin\gamma}{\sqrt{1-\cos^2\gamma\sin^2\mu}}$, or $\sin\varpi=1$, as we take the sign $-$ or $+$. See Theory of Elliptic Integrals, p. 31.

(58.) may be transformed into

$$s=g\int\frac{d\mu}{\cos\mu}+g\frac{\sin\mu}{\cos^2\mu},$$

the well-known formula for the rectification of a plane parabola. When, on the other hand, the sphere becomes indefinitely small compared with the parabola, γ approximates to a right angle, and (58.) becomes

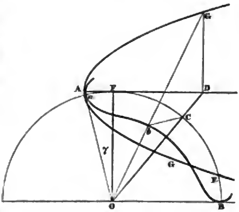
$$s=\mu+\tan^{-1}(\tan\mu)=2\mu,$$

as it should be, since 2μ is the angle which the radius vector ρ makes with the axis.

We shall find the notice of these extreme cases useful.

XXI. Although we have called this curve the spherical parabola, as indicating its mode of generation, it is in fact a closed curve, like all other curves which are the intersections of cones of the second degree with concentric spheres. It is a spherical ellipse, and we shall now proceed to determine its principal arcs.

Fig. 6.



Let ADG be a parabola, F its focus, O being the centre of the sphere which touches the plane of the parabola at F, and being also the vertex of the obtuse-angled cone, of which the parabola ADG is a section parallel to the side of the cone OB. Let the angle AOF or the arc Fa be γ, α and β being the principal semiangles of the cone,

$$2\alpha=\frac{\pi}{2}+\gamma=\text{AOB},$$

whence $$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma}.$$

To determine the angle β, or the arc Cb. Bisect the vertical angle AOB of the cone by the line OD, and draw DG an ordinate of the parabola. Then $\tan^2\beta=\left(\frac{\text{DG}}{\text{OD}}\right)^2$. As AOD is an isosceles triangle, $\text{AD}=\text{AO}=\frac{\text{OF}}{\cos\gamma}$; and

$$\text{OD}=\frac{\text{OF}}{\sin\alpha}=\frac{\text{OF}}{\sin\left(\frac{\pi}{4}+\frac{\gamma}{2}\right)}.$$

We have also, as DG is an ordinate of the parabola,

$$\overline{\text{DG}}^2=4\text{AF}\times\text{AD}=4\text{OF}.\tan\gamma\times\frac{\text{OF}}{\cos\gamma}=4\frac{\overline{\text{OF}}^2\sin\gamma}{\cos^2\gamma}.$$

Hence substituting, $$\tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}.$$

We may therefore announce the following important theorem:—

The spherical ellipse, whose principal arcs are given by the equations

$$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma},\quad \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}, \quad \ldots\ldots\ldots \quad (59.)$$

γ being any arbitrary angle, may be rectified by an elliptic function of the first order.

Write x for $\tan\alpha$, y for $\tan\beta$, and eliminate $\sin\gamma$ from the preceding equations,

$$\tan^2\alpha-\tan^2\beta=x^2-y^2=1, \quad \ldots \quad (59^*.)$$

the equation of an equilateral hyperbola. We thus obtain the following theorem:—

Any spherical conic section, the tangents of whose principal semiarcs can be the ordinates of an equilateral hyperbola, whose transverse semi-axis is 1, may be rectified by an elliptic integral of the first order.

XXII. When we take the complete function, and integrate between the limits 0 and $\frac{\pi}{2}$, we get, not the length of a quadrant of the spherical parabola, as we do when we take the centre as origin, but the length of two quadrants or half the ellipse. We derive also this other remarkable result, that when μ is a right angle, the spherical triangle whose sides are the radius vector, the perpendicular arc on the tangent, and the intercept of the tangent arc between the point of contact and the foot of the perpendicular, is a quadrantal equilateral triangle. For when $\mu=\frac{\pi}{2}$,

$$\rho=\frac{\pi}{2}, \quad \varpi=\frac{\pi}{2}, \quad \tau=\frac{\pi}{2}.$$

It may also easily be shown, that the arc of a great circle which touches the spherical parabola, intercepted between the perpendicular arcs let fall upon it from the foci, is in every position constant, and equal to a quadrant. See Theory of Elliptic Integrals, p. 35.

Hence the spherical parabola is the envelope of a quadrantal arc of a great circle, which always has its extremities on two fixed great circles of the sphere, the angle between the planes of these circles being $\frac{\pi}{2}+\gamma$.

Resuming the equations given in (59.), which express the tangents of the principal semiarcs of the spherical parabola in terms of $\sin\gamma$, namely,

$$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma}, \quad \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma},$$

writing i for $\cos\gamma$, and j for $\sin\gamma$, we get

$$\left.\begin{array}{ll} & \tan^2\iota=\frac{1-j}{1+j}, \quad e^2=\frac{1-j}{1+j}\sin^2\eta=\left(\frac{1-j}{1+j}\right)^2, \\ \text{whence} & \tan^2\iota=e^2=\sin\eta=\cos^2\beta. \\ \text{Now} & n=\tan^2\iota, \quad m=e^2; \ \text{hence } n=m=i. \end{array}\right\} \quad \ldots \quad (60.)$$

XXIII. We shall now proceed to the rectification of an arc of the spherical parabola, the centre being the pole. By this method we shall obtain certain geometrical results which have hitherto appeared as mere analytical expressions. In (14.) or (28.) we found for an arc of a spherical ellipse measured from the major principal arc, the following expression, the centre being the pole,

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\psi}{(1-e^2\sin^2\psi)\sqrt{1-\sin^2\eta\sin^2\psi}};$$

or substituting the values of the constants given by the preceding equations,

$$\sigma=\frac{2j}{1+j}\int\frac{d\psi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}. \quad (61.)$$

But when the focus is the pole, we found for the arc the following expression in (58.),

$$\sigma=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right].$$

Equating those values of σ, we get the resulting equation,

$$\frac{2j}{1+j}\int\frac{d\psi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right] \quad (62.)$$

XXIV. We shall now show that the amplitudes ψ and μ in the preceding formula are connected by the equation

$$\tan(\psi-\mu)=j\tan\mu, \quad (63.)$$

a relation established by Lagrange.

Let ϖ and ϖ' be the perpendicular arcs from the centre and focus of the spherical parabola on the tangent arc to the curve. Let λ and μ be the angles which these perpendicular arcs make with the major principal arc. The distance between the centre and focus of the spherical parabola, with the complements of those perpendiculars, constitute the sides of a spherical triangle. We shall therefore have

$$\sin^2\lambda=\sin^2\mu\frac{\sec^2\varpi}{\sec^2\varpi'}. \quad (64.)$$

Now $\sec^2\varpi=\sec^2\alpha\cos^2\lambda+\sec^2\beta\sin^2\lambda$, as in (35.); or writing for $\sec\alpha, \sec\beta$ their particular values in the spherical parabola, given in (59.),

$$\sec^2\varpi=\frac{2}{1-\sin\gamma}-\sin^2\lambda. \quad (65.)$$

Again, as $\quad \tan\varpi'=\frac{\tan\gamma}{\cos\mu},$

$$\sec^2\varpi'=\frac{\tan^2\gamma+\cos^2\mu}{\cos^2\mu};$$

reducing (64.), the result is

$$\tan^2\lambda=\frac{2(1+\sin\gamma)}{(\cot\mu-\sin\gamma\tan\mu)^2}. \quad (66.)$$

In the case of the spherical parabola,

$$\cos^2\iota=\frac{1+\sin\gamma}{2}, \text{ whence (66.) becomes}$$

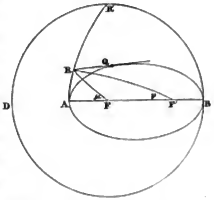
$$\cos\iota\tan\lambda=\frac{1+\sin\gamma}{\cot\mu-\sin\gamma\tan\mu}, \text{ or } \cos\iota\tan\lambda=\frac{\tan\mu+\sin\gamma\tan\mu}{1-\sin\gamma\tan\mu.\tan\mu}. \quad (67.)$$

The second member of this equation is manifestly the expression for the tangent of the sum of two arcs μ and ν, if we make $\tan\nu=\sin\gamma\tan\mu$.

Hence $$\cos\varepsilon\tan\lambda=\tan(\mu+\nu).$$
In (25.), or (38.) or (39.), we assumed $\tan\psi=\cos\varepsilon\tan\lambda$.
Hence $$\psi=\mu+\nu,\text{ or }\tan(\psi-\mu)=\tan\nu=\sin\gamma\tan\mu.$$

A geometrical interpretation of Lagrange's theorem $\tan(\psi-\mu)=\sin\gamma\tan\mu$ may be given by the aid of the spherical parabola.

Fig. 7.



Let $DR_{,}B$ be the great circle, the base of the hemisphere, whose pole is F. Let BQA be a spherical parabola, touching the great circle at B, and having one of its foci at F the pole of the hemisphere whose base is the circle $DR_{,}B$. Let RQ be an arc of a great circle, a tangent to the curve at Q. From F let fall upon it the perpendicular arc FR. The point R is in the great circle AR which touches the curve at its vertex A. The pole of this circle is the second focus $F_{,}$; for $AF_{,}=FB=\frac{\pi}{2}$. Let the arcs RF, $RF_{,}$ make the angles μ and ν with the transverse arc AB. Hence $AR=\nu$. In the spherical triangle FAR, right-angled at A, we have $\sin AF=\tan\nu\cot\mu$. Now as $AF=\gamma$, $\sin AF=\sin\gamma=j$; and if $\phi=\mu+\nu$, $\nu=\phi-\mu$, or reducing, $\tan(\phi-\mu)=j\tan\mu$; whence we infer that while the original amplitude is the angle μ at the focus F, the derived amplitude ϕ is the sum of the angles μ and ν at the foci F and $F_{,}$.

When the function is complete, or $\mu=\frac{\pi}{2}$, R will coincide with $R_{,}$ the pole of the great circle AB, whence ν is also $=\frac{\pi}{2}$, and as $\phi=\mu+\nu$, $\phi=\pi$. This shows, that when the function is complete, or the amplitude is a right angle, the amplitude of the derived function will be two right angles.

When the spherical parabola approximates to a great circle of the sphere, the second focus $F_{,}$ will approach to F the immoveable focus. The arc $RF_{,}$ will, therefore, approach to coincidence with the arc RF, or the angle ν will approximate to μ, so that $\phi=\mu+\nu=2\mu$ nearly.

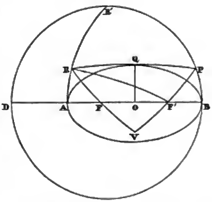
This is the geometrical explanation of the analytical fact observed in this theory, that when the modulus diminishes, or the spherical parabola approximates to a great circle of the sphere, the ratio of any two successive amplitudes approximates to that of two to one.

When the transverse arc of the spherical parabola is a right angle and a half, $\sin\gamma=\frac{1}{\sqrt{2}}$, and if C be its circumference, $C=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{1-\frac{1}{2}\sin^2\mu}}+\pi$. But two quadrants $2s$, or the loop of a lemniscate, are $=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{1-\frac{1}{2}\sin^2\mu}}$. Hence $2s=C-\pi$.

Or the loop of a lemniscate is equal to the difference between the circumference of the spherical parabola whose transverse arc is $\frac{3\pi}{2}$, and a semicircle.

When a quadrant of the spherical parabola is taken, or when the point of contact Q coincides with the extremity of the principal minor arc of the curve, we shall have $\phi=\frac{\pi}{2}$.

Fig. 8.



Since in this case $RQ=PQ$, $FV=F_{,}V$, therefore $\mu=OFV=OF_{,}V$, or $RF_{,}V=\mu+\nu$. As V is the pole of RP, and $F_{,}$ is the pole of AR, the point R is the pole of $VF_{,}$. Hence $RF_{,}V$ is a right angle, but $\mu+\nu=RF_{,}V$, whence $\phi=\frac{\pi}{2}$. As $\tan(\phi-\mu)=j\tan\mu$, when $\phi=\frac{\pi}{2}$, $\tan\mu=\frac{1}{\sqrt{j}}$ If in the expression $\tan\tau=\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}$ given in (58.), we substitute this value of $\tan\mu$, we shall get $\tan\tau=1$, or $\tau=\frac{\pi}{4}$.

Hence as two quadrants of a spherical parabola are together double of one, we shall have, writing the integral $\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}$ in the abbreviated form $\int\frac{d\mu}{\sqrt{I}}$,

$$j\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}+\frac{\pi}{2}=2j\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}+2.\frac{\pi}{4},\quad \text{or} \int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}=2\int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}. \quad . \quad . \quad (68.)$$

Now when i is nearly 1, $\int\frac{d\mu}{\sqrt{I}}=\log\left(\frac{1+\sin\mu}{\cos\mu}\right)$. Taking this expression between the limits $\mu=0$, and $\mu=\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}$, we shall have, since $\sin\mu=\frac{1}{\sqrt{1+j}}$, $\cos\mu=\frac{\sqrt{j}}{\sqrt{1+j}}$, and neglecting j and its powers when added to 1, j being very small,

$$\frac{1+\sin\mu}{\cos\mu}=\frac{2}{\sqrt{j}},\quad \text{whence} \int_0^{\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}}\frac{d\mu}{\sqrt{I}}=\log\left(\frac{2}{\sqrt{j}}\right).$$

Therefore (68.) gives
$$\int_0^{\frac{\pi}{2}}\frac{d\mu}{\sqrt{I}}=\log\left(\frac{4}{j}\right)^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69.)$$

XXV. To show that

$$\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}=\frac{1}{1+j}\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}},$$

the amplitudes ψ and μ being connected as before, by the equation $\tan(\psi-\mu)=j\tan\mu$.

Since
$$\tan\psi=\frac{1+\sin\gamma}{\cot\mu-\sin\gamma\tan\mu}=\frac{1+j}{\cot\mu-j\tan\mu}.$$

* . . . . . . "résultat fort remarquable, déjà signalé par Legendre; mais nous ignorons comment il y est parvenu."—Verhulst, *Traité Élémentaire des Fonctions Elliptiques*, p. 158.

Differentiating this expression with respect to ψ and μ,

$$\frac{(1+j)}{\sin^2\psi}\frac{d\psi}{d\mu}=\frac{\cos^2\mu+j\sin^2\mu}{\cos^2\mu\sin^2\mu}. \quad (70.)$$

We have also $$\tan^2\psi=\frac{(1+j)^2\sin^2\mu\cos^2\mu}{(\cos^2\mu-j\sin^2\mu)^2}. \quad (71.)$$

Whence, after some reductions, $$\sin^2\psi=\frac{(1+j)^2\sin^2\mu\cos^2\mu}{1-i^2\sin^4\mu}. \quad (72.)$$

Multiplying this expression by $\left(\frac{1-j}{1+j}\right)^2$, and reducing,

$$\frac{1}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=\frac{\sqrt{1-i^2\sin^2\mu}}{\cos^2\mu+j\sin^2\mu}, \quad (73.)$$

Multiplying together the left-hand members of the equations (70.), (72.) and (73.), and also the right-hand members together, we get, after some obvious reductions, and integrating,

$$\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=(1+j)\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}. \quad (74.)$$

This is the well-known relation between two elliptic integrals of the first order whose moduli are i and $\frac{1-j}{1+j}$, or in the common notation, whose moduli are c and $\frac{1-b}{1+b}$.

XXVI. Let τ be the arc whose tangent is $\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}$,

then $$\tan 2\tau=\frac{2j\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-i^2\sin^4\mu}; \quad (75.)$$

and combining (71.) and (73.), we shall find

$$\frac{\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=\frac{(1+j)\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}. \quad (76.)$$

Dividing (75.) by (76.), the result becomes $$\tan 2\tau=\frac{\frac{2j}{1+j}\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}. \quad (77.)$$

We are thus enabled to express τ, the portion of the tangent arc between the point of contact and the foot of the perpendicular arc on it, in terms of ψ instead of μ.

If we introduce this value of τ into (62.) and combine with it the relations established in (74.), the resulting equation will become

$$\left.\begin{aligned}2\int\frac{d\psi}{\left[1-\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\\ +\left(\frac{1+j}{2j}\right)\tan^{-1}\left[\frac{\frac{2j}{1+j}\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right]\end{aligned}\right\}. \quad (78.)$$

Adopting for the moment the ordinary notation of elliptic integrals,

$$m=-c=\frac{1-j}{1+j},\quad \text{whence } 1+c=\frac{2j}{1+j}.$$

Introducing this notation, the last formula will become

$$2\Pi_c(-c,\psi)=\mathrm{F}_c(\psi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\psi}{\sqrt{1-i^2\sin^2\psi}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79.)$$

In the Traité des Fonctions Elliptiques, tom. i. p. 68, we meet with the formula

$$\Pi_c(n,\psi)+\Pi_c\left(\frac{c^2}{n},\psi\right)=\mathrm{F}_c(\psi)+\frac{1}{\sqrt{a}}\tan^{-1}\left[\frac{\sqrt{a}\tan\psi}{\sqrt{1-c^2\sin^2\psi}}\right]. \quad . \quad . \quad . \quad . \quad (80.)$$

Now when $n=-c$, this formula becomes

$$2\Pi_c(-c,\psi)=\mathrm{F}_c(\psi)+\frac{1}{1+c}\tan^{-1}\left[\frac{(1+c)\tan\psi}{\sqrt{1-c^2\sin^2\psi}}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81.)$$

whence (79.) and (80.) are identical.

XXVII. Let us now proceed to rectify the spherical parabola by the formula for rectification given in (47.), the centre being the pole. For this purpose, resuming the formula for rectification established in (41.), and deducing the values of the parameter, modulus and coefficients in that expression from the given relations,

$$\tan^2\alpha=\frac{1+\sin\gamma}{1-\sin\gamma}=\frac{1+j}{1-j},\quad \tan^2\beta=\frac{2\sin\gamma}{1-\sin\gamma}=\frac{2j}{1-j}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (82.)$$

we get

$$\left.\begin{array}{ll} \text{The parameter, } \tan^2\varepsilon=\dfrac{1-j}{1+j} & \\ \text{The modulus, } \quad \sin\eta=\dfrac{1-j}{1+j} & \\ \text{The coefficient } \dfrac{\cos\beta}{\sin\alpha\cos\alpha}=\dfrac{2}{1+j}, & \text{the coefficient } \dfrac{\cos\alpha\cos\beta}{\sin\alpha}=\dfrac{1-j}{1+j} \\ \text{and} \quad e\tan\varepsilon=\dfrac{1-j}{1+j} & \end{array}\right\} \quad . \quad . \quad . \quad (83.)$$

Making these substitutions in (41.), the resulting equation will become

$$\left.\begin{array}{l} \sigma=\dfrac{2}{(1+j)}\displaystyle\int\frac{d\psi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}} \\ \quad -\dfrac{(1-j)}{(1+j)}\displaystyle\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}-\tan^{-1}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right] \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (84.)$$

But from (58.), the focus being the pole, we derive

$$\sigma=j\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}+\tan^{-1}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85.)$$

In (74.) we showed that

$$\int\frac{d\mu}{\sqrt{1-i^2\sin^2\mu}}=\frac{1}{1+j}\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}.$$

Introducing this relation into the last formula, and equating together the equivalent expressions for the arcs in (84.) and (85.), we get for the resulting equation,

$$\left.\begin{aligned}2\int\frac{d\psi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\\+(1+j)\overset{-1}{\tan}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right]+(1+j)\overset{-1}{\tan}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right]\end{aligned}\right\}\quad \text{(86.)}$$

We shall now proceed to show that the common formula for the comparison of elliptic integrals having the same modulus and amplitude but reciprocal parameters, is, in this particular case, identical with the geometrical theorem just established.

The formula is, in the ordinary notation,

$$2\Pi_c(c,\psi)=F_c(\psi)+\frac{1}{1+c}\overset{-1}{\tan}\left[\frac{(1+c)\tan\psi}{\sqrt{1-c^2\sin^2\psi}}\right]. \quad \text{(87.)}$$

We must accordingly show that, c being $\tan^2\iota$, and therefore $\frac{1}{1+c}=\frac{1+j}{2}$

$$\left.\begin{aligned}(1+j)\overset{-1}{\tan}\left[\frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right]+(1+j)\overset{-1}{\tan}\left[\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\right]\\=\frac{(1+j)}{2}\overset{-1}{\tan}\left[\frac{(1+\tan^2\iota)\tan\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\right]\end{aligned}\right\}\quad \text{(88.)}$$

If we write τ, τ' and Θ for these angles respectively, we have to show that

$$\Theta=2(\tau+\tau'). \quad \text{(89.)}$$

$\tau+\tau'$ is the arc of the great circle, which touches the spherical parabola, intercepted between the perpendicular arcs let fall from the centre and focus upon it.

We must, in the first place, by the help of LAGRANGE's equation between the amplitudes, established on geometrical principles in XXIV., reduce these angles to a single variable. μ is taken as the independent variable instead of ψ, as the trigonometrical function of ψ in terms of μ is in the first power only.

We have, therefore,

$$\left.\begin{aligned}\tan\Theta&=\frac{2\tan\psi}{(1+j)\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\\\tan\tau&=\frac{\left(\frac{1-j}{1+j}\right)\sin\psi\cos\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}\\\tan\tau'&=\frac{j\tan\mu}{\sqrt{1-i^2\sin^2\mu}}\end{aligned}\right\}\quad \text{(90.)}$$

The equation between the amplitudes ψ and μ,

$$\tan(\psi-\mu)=j\tan\mu, \text{ gives}$$

$$\tan\psi=\frac{(1+j)\sin\mu\cos\mu}{\cos^2\mu-j\sin^2\mu}. \quad \ldots \quad (91.)$$

Eliminating ψ by the help of this equation, from the value of $\tan\tau$ given in the preceding group,

$$\tan\tau=\frac{(1-j)\sin\mu\cos\mu}{\sqrt{1-i^2\sin^2\mu}}\times\frac{\cos^2\mu+j\sin^2\mu}{\cos^2\mu-j\sin^2\mu}.$$

Using this transformation and reducing,

$$\tan(\tau+\tau')=\tan\mu\sqrt{1-i^2\sin^2\mu}, \quad \ldots \quad (92.)$$

a simple expression for the length of the tangent arc to the spherical parabola between the perpendicular arcs let fall from the centre and focus upon it.

From the last equation we may derive

$$\tan2(\tau+\tau')=\frac{2\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}. \quad \ldots \quad (93.)$$

Using the preceding transformations, we may show that

$$\tan\Theta=\frac{2\sin\mu\cos\mu\sqrt{1-i^2\sin^2\mu}}{\cos^4\mu-j^2\sin^4\mu}.$$

Hence $$\Theta=2(\tau+\tau'). \quad \ldots \quad (94.)$$

Therefore (86.) becomes

$$2\int\frac{d\psi}{\left[1+\left(\frac{1-j}{1+j}\right)\sin^2\psi\right]\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}-\int\frac{d\psi}{\sqrt{1-\left(\frac{1-j}{1+j}\right)^2\sin^2\psi}}=(1+j)\frac{\Theta}{2}=(1+j)(\tau+\tau'). \quad (95.)$$

We have thus shown that in the particular case of the general formula for comparing elliptic functions of the third order with reciprocal parameters, when the parameter is *positive* and equal to the *modulus*, the circular arc in the formula of comparison (87.) is equal to twice the arc of the great circle touching the curve and intercepted between the perpendicular arcs let fall from the centre and focus upon it.

If we take the parameter with a *negative* sign, the circular arc τ in (62.) will represent the tangent arc between the point of contact and the foot of the focal perpendicular.

The spherical parabola, like any other spherical ellipse, may be considered as the intersection of an elliptic cylinder with a sphere whose centre is on the axis of the cylinder.

Let a and b be the semiaxes of the base of the cylinder, and k the radius of the sphere, α and β being the principal semiarcs of the spherical parabola,

$$\tan^2\alpha=\frac{a^2}{k^2-a^2}, \qquad \tan^2\beta=\frac{b^2}{k^2-b^2};$$

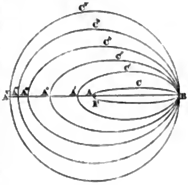
but in (59*.) we found $\tan^2\alpha-\tan^2\beta=1$; hence substituting,

$$k^2=a^2(1+i). \quad \ldots \quad (96.)$$

XXVIII. The foregoing investigations furnish us with the geometrical interpretation of the transformations of LAGRANGE. Let the successive amplitudes ϕ, ψ, χ of the derived functions, be connected by the equations

$$\tan(\phi-\mu)=j\tan\mu,\quad \tan(\psi-\phi)=j_{,}\tan\phi,\quad \tan(\chi-\psi)=j_{,,}\tan\psi. \quad . \quad . \quad . \quad . \quad (97.)$$

Fig. 9.



We may imagine a series of confocal parabolas having a common axis, described on a plane in contact with a sphere at their common focus. These parabolas will generate a series of confocal spherical parabolas on the surface of the sphere, BCA, BC'A', BC''A'', BC'''A''', which will all mutually touch at the vertex B remote from the common focus F. Let the distances between the common focus F and the vertices of the plane parabolas subtend at the centre of the sphere, angles $\gamma, \gamma', \gamma''$, &c., whose cosines i, $i_{,}$, $i_{,,}$, &c. are connected by the equations

$$i_{,}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}},\quad i_{,,}=\frac{1-\sqrt{1-i_{,}^2}}{1+\sqrt{1-i_{,}^2}},\quad i_{,,,}=\frac{1-\sqrt{1-i_{,,}^2}}{1+\sqrt{1-i_{,,}^2}}\ldots\text{\&c.}, \quad . \quad . \quad (98.)$$

it is plain that $\gamma=FA$, $\gamma'=FA'$, $\gamma''=FA''$, $\gamma'''=FA'''$, &c.

We may repeat this construction successively, until the parameter of the last of the applied tangent plane parabolas shall become so indefinitely small, compared with the radius of the sphere, that it may ultimately be taken to coincide with its projection. We shall in this way reduce, at least geometrically, the calculation of an elliptic integral of the first order to the rectification of an arc of a parabola, that is, to a logarithm, as in XX. If, on the contrary, the moduli i, $i_{,}$, $i_{,,}$, &c. proceed in a descending series, the angles γ, $\gamma_{,}$, $\gamma_{,,}$ continually increase, the magnitudes of the confocal applied parabolas increase, till at length their parameters become so large, compared with the radius of the sphere, that their central projections pass into great circles of the sphere. The evaluation of the elliptic integral will therefore ultimately be reduced to the rectification of a circular arc. These are the well-known results of the modular transformation of LAGRANGE.

The formulæ established in (58.) for the rectification of the spherical parabola, give

$$\sigma=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left[\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin\mu}}\right];$$

or writing i for $\cos\gamma$, j for $\sin\gamma$, and $\sqrt{I}$ for $\sqrt{1-i^2\sin^2\mu}$,

$$\sigma-\tau=j\int\frac{d\mu}{\sqrt{I}},$$

σ' and τ' being the corresponding quantities for the next derived spherical parabola,

$$\sigma'-\tau'=j_{,}\int\frac{d\phi}{\sqrt{I_{,}}}.$$

Now $j_{,}=\frac{2\sqrt{j}}{1+j}$, and $\int\frac{d\mu}{\sqrt{\mathrm{I}}}=\frac{1}{1+j}\int\frac{d\varphi}{\sqrt{\mathrm{I}_{,}}}$, as in (98.) and (74.),

whence $$2(\sigma-\tau)=\sqrt{j}(\sigma'-\tau'), \quad (99.)$$

Thus a simple ratio exists between the arcs, diminished by the protangents, of two consecutive confocal spherical parabolas.

When the functions are complete, μ is taken between 0 and $\frac{\pi}{2}$; φ therefore, as in article XXIV., must be taken between 0 and π; but when the amplitude is taken between 0 and π the function is doubled. Moreover, when the functions are complete, the point Q coincides with B; so that in this case the complete function represents, not one, but two quadrants of the spherical parabola, the focus being the pole. Hence as $\tau=\frac{\pi}{2}$, $\tau'=\pi$.

Whence putting C, C′, C″, C‴, &c. for the circumferences of the successive confocal spherical parabolas, derived by the preceding law, we may write

$$\left.\begin{aligned} C-\pi&=\sqrt{j}\,(C_{,}-\pi)\\ C_{,}-\pi&=\sqrt{j_{,}}\,(C_{,,}-\pi)\\ C_{,,}-\pi&=\sqrt{j_{,,}}\,(C_{,,,}-\pi)\\ C_{,,,}-\pi&=\sqrt{j_{,,,}}\,(C_{\mathrm{iv}}-\pi)\\ C_{\mathrm{iv}}-\pi&=\sqrt{j_{\mathrm{iv}}}\,(C_{\mathrm{v}}-\pi)\end{aligned}\right\} \quad (100.)$$

Multiplying successively by the square roots of $j\ j_{,}\ j_{,,}\ j_{,,,}$, &c., adding and stopping at the fifth derived parabola,

$$C-\pi=\sqrt{j\,j_{,}j_{,,}j_{,,,}j_{\mathrm{iv}}\ \&c.}\ (C_{\mathrm{v}}-\pi).$$

Let this coefficient be $\sqrt{Q}$, and we shall have $C-\pi=\sqrt{Q}(C_{\mathrm{v}}-\pi)$. . . . (101.)

Now we may extend this series, until the last of the derived spherical parabolas shall differ as little as we please from a great circle of the sphere. Let the circumference of this last derived spherical parabola be $C_{\prime\prime}$. Then $C_{\prime\prime}=2\pi$, and (101.) becomes

$$C=\pi(1+\sqrt{Q'}). \quad (102.)$$

Hence calculating the quantity Q', we may express the circumference of a spherical parabola by the circumference of a circle.

When all the spherical parabolas are nearly great circles of the sphere, $i=i_{,}=i_{,,}=i_{,,,}=0$, nearly; and $j=j_{,}=j_{,,}=j_{,,,}=1$, nearly. Whence $Q'=1$, nearly; or

$$C=2\pi \quad (103.)$$

When the spherical parabolas are indefinitely diminished,

$i=i_{,}=i_{,,}=1$, nearly, and $j=j_{,}=j_{,,}=j_{,,,}=0$, therefore $Q'=0$ nearly;

or $$C=\pi. \quad (104.)$$

Hence the circumferences of all spherical parabolas lie between two and four quadrants of a great circle of the sphere.

XXIX. Denoting the angles at the centre of the sphere, subtended by the halves of the semiparameters of the applied confocal parabolas, by γ, γ', γ'', &c., we have $\cos\gamma=i$, $\cos\gamma'=i_{,}$, $\cos\gamma''=i_{,,}$, $\cos\gamma'''=i_{,,,}$, and $\sin\gamma=j$, $\sin\gamma'=j_{,}$, $\sin\gamma''=j_{,,}$, $\sin\gamma'''=j_{,,,}$.

We may, using successively the equation $i_{,}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}$, determine in terms of j the successive values of i, $i_{,}$, $i_{,,}$, $i_{,,,}$ and of $j_{,}$, $j_{,,}$, $j_{,,,}$ &c., as follows:—

$$\left.\begin{array}{l} i_{,}=\frac{1-j}{1+j},\ i_{,,}=\left[\frac{1-j^{\frac{1}{2}}}{1+j^{\frac{1}{2}}}\right]^2,\ i_{,,,}=\left[\frac{(1+j)^{\frac{1}{2}}-2^{\frac{1}{2}}j^{\frac{1}{4}}}{(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}}\right]^2,\ i_{iv}=\left[\frac{1+j^{\frac{1}{2}}-2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}}{1+j^{\frac{1}{2}}+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}}\right]^2 \\ \qquad i_{v}=\left[\frac{(1+j)^{\frac{1}{4}}+2^{\frac{1}{4}}j^{\frac{1}{8}}-2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}}{(1+j)^{\frac{1}{4}}+2^{\frac{1}{4}}j^{\frac{1}{8}}+2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}}\right]^2 \ \&c. \end{array}\right\} \quad . \quad . \quad (105.)$$

Hence we may derive the successive values of $j_{,}$, $j_{,,}$, $j_{,,,}$ in terms of j.

$$\left.\begin{array}{l} \text{For}\quad j_{,}^2=\frac{2^2 j}{(1+j)^2},\quad j_{,,}^2=\frac{2^2 2\, j^{\frac{1}{2}}(1+j)}{(1+j^{\frac{1}{2}})^4},\quad j_{,,,}^2=\frac{2^2 2^1 2^{\frac{1}{2}} j^{\frac{1}{4}}(1+j)^{\frac{1}{2}}(1+j^{\frac{1}{2}})^2}{[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}]^4}, \\ j_{iv}^2=\frac{2^2 2^1 2^{\frac{1}{2}} 2^{\frac{1}{4}}(1+j^{\frac{1}{2}})(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}]^2}{[(1+j^{\frac{1}{2}})+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}]^4}, \\ j_{v}^2=\frac{(2^2 2^1 2^{\frac{1}{2}} 2^{\frac{1}{4}} 2^{\frac{1}{8}})(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}][(1+j^{\frac{1}{2}})+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}]^2}{[(1+j)^{\frac{1}{4}}+2^{\frac{1}{4}}j^{\frac{1}{8}}+2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}]^4} \end{array}\right\} \quad . \quad (106.)$$

We may express the coefficient Q, or the continued product of j, $j_{,}$, $j_{,,}$, $j_{,,,}$, &c., in terms of j, the complement of the original modulus. Including in our approximation the fifth derived modulus, we get

$$Q=\frac{(2)^1.(2)^{1+\frac{1}{2}}.(2)^{1+\frac{1}{2}+\frac{1}{4}}.(2)^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}.(2)^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}(j j^{\frac{1}{2}} j^{\frac{1}{4}} j^{\frac{1}{8}} j^{\frac{1}{16}})}{(1+j)^{\frac{1}{16}}(1+j^{\frac{1}{2}})^{\frac{1}{4}}[(1+j)^{\frac{1}{2}}+2^{\frac{1}{2}}j^{\frac{1}{4}}]^{\frac{1}{2}}[(1+j^{\frac{1}{2}})+2^{\frac{1}{2}}2^{\frac{1}{4}}(1+j)^{\frac{1}{4}}j^{\frac{1}{8}}][(1+j)^{\frac{1}{4}}+2^{\frac{1}{4}}j^{\frac{1}{8}}+2^{\frac{1}{2}}2^{\frac{1}{4}}2^{\frac{1}{8}}(1+j^{\frac{1}{2}})^{\frac{1}{2}}(1+j)^{\frac{1}{8}}j^{\frac{1}{16}}]^2} \quad (107.)$$

XXX. It may not be out of place here to show, although the investigation more properly belongs to another part of the subject, that the arc of a spherical parabola may be represented as the sum of two elliptic integrals of the third order, having imaginary parameters; or in other words, that every elliptic integral of the *first* order may be exhibited as the sum of two elliptic integrals of the third order, having *imaginary reciprocal* parameters.

Assume the expression given in (58.) for an arc of the spherical parabola, the focus being the pole, and μ the angle which the perpendicular arc from the focus, on the tangent arc of a great circle to the curve, makes with the principal transverse arc,

$$\sigma=\sin\gamma\int\frac{d\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}+\tan^{-1}\left\{\frac{\sin\gamma\tan\mu}{\sqrt{1-\cos^2\gamma\sin^2\mu}}\right\}.$$

Let $\cos\gamma=i$, $\sin\gamma=j$, and to preserve uniformity in the notation, write ϕ for μ. Then differentiating the preceding equation, it becomes after some reductions,

$$\frac{d\sigma}{d\phi}=\frac{j[1-i^2\sin^2\phi+\cos^2\phi+j^2\sin^2\phi]}{[\cos^2\phi-i^2\sin^2\phi\cos^2\phi+j^2\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a.)$$

Now the numerator is equivalent to $2j(1-i^2\sin^2\phi)$, and the denominator may be written in the form $1-2i^2\sin^2\phi+i^2\sin^4\phi$. But $i^2=i^2(i^2+j^2)$, hence this last expression may be put under the form $1-2i^2\sin^2\phi+i^4\sin^4\phi+i^2j^2\sin^4\phi$. This expression is the sum of two squares. Resolving this sum into its constituent factors, we get

$$\frac{d\sigma}{d\phi}=\frac{2j(1-i^2\sin^2\phi)}{[1-i(i+j\sqrt{-1})\sin^2\phi][1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad . \quad . \quad . \quad (b.)$$

Now this product may be resolved into the sum of two terms. Let

$$\frac{d\sigma}{d\phi}=\frac{P}{[1-i(i+j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}+\frac{Q}{[1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}. \quad \text{(c.)}$$

Or reducing these expressions to a common denominator,

$$\frac{d\sigma}{d\phi}=\frac{(P+Q)-(P+Q)i^2\sin^2\phi+\sqrt{-1}(P-Q)ij\sin^2\phi}{[1-i(i+j\sqrt{-1})\sin^2\phi][1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} \quad \text{(d.)}$$

Hence $$P+Q=2j,\ P-Q=0;\ \because P=j,\ Q=j. \quad \text{(e.)}$$

Integrating (c.), we get

$$\sigma=j\int\frac{d\phi}{[1-i(i+j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}+j\int\frac{d\phi}{[1-i(i-j\sqrt{-1})\sin^2\phi]\sqrt{1-i^2\sin^2\phi}} \quad (108.)$$

Now if we multiply together the imaginary parameters

$$(i^2+ij\sqrt{-1}) \text{ and } (i^2-ij\sqrt{-1}),$$

their product is i^2, or the parameters are reciprocal.

Since the parameters are each affected with a negative sign, and one is equal to i^2+ a certain quantity, while the other is equal to i^2- a certain quantity, the former parameter is of the *circular* form, while the other is of the *logarithmic* form.

It is very remarkable, that although the spherical parabola is a spherical conic, the imaginary parameters satisfy the criterion of conjugation which belongs to the logarithmic form, and not that which belongs to the circular form. Let $m=i(i-j\sqrt{-1})$, $n=i(i+j\sqrt{-1})$. These values of m and n satisfy the equation of logarithmic conjugation, $m+n-mn=i^2$, and not $m-n+mn=i^2$, the equation of circular conjugation.

On Spherical Conic Sections with Reciprocal Parameters.

XXXI. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation of an ellipse, the base of an elliptic cylinder. Let two spheres be described, having their centres at the centre of this elliptic base, and intersecting the cylinder in two spherical conic sections. These sections will have reciprocal parameters, if k, k', the radii of the spheres, are connected by the equation

$$(k^2-a^2)(k'^2-a^2)=a^4i^2, \quad (109.)$$

i^2 being, as before, equal to $\frac{a^2-b^2}{a^2}$.

When k and k' are equal, we get $k^2=a^2(1+i)$. This value of k agrees with that found for k in (96.), or, in other words, when the two spheres coincide, the section of the elliptic cylinder by the sphere is a spherical parabola. Hence also the spherical parabola always lies between two spherical conic sections with reciprocal parameters.

Let e^2 and e'^2 be the parameters of those sections of the cylinder made by the spheres. Then, as shown in (12.),

$$e^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha\cos^2\beta}=\frac{(a^2-b^2)k^2}{a^2(k^2-b^2)}=\frac{k^2i^2}{k^2-a^2+a^2i^2};$$

but the equation of condition (109.) gives

$$k^2-a^2=\frac{a^4i^2}{k'^2-a^2},\quad \text{hence } e^2=\frac{k^2(k'^2-a^2)}{a^2k'^2}.$$

In the same manner the spherical conic, whose radius is k', gives

$$e'^2=\frac{k'^2(k^2-a^2)}{a^2k^2};\quad \therefore\ e^2e'^2=\frac{(k^2-a^2)(k'^2-a^2)}{a^4}=i^2=lm, \quad . \quad . \quad . \quad . \quad (110.)$$

or e^2 and e'^2 are reciprocal parameters.

To compute in this case the value of the coefficient $\frac{\tan\beta}{\tan\alpha}\sin\beta$ in the expression given in (16.) for rectification,

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Since $$\tan^2\beta=\frac{b^2}{k^2-b^2},\quad \tan^2\alpha=\frac{a^2}{k^2-a^2},\quad \sin\beta=\frac{b}{k};$$

we obtain by substitution, $\frac{\tan^2\beta}{\tan^2\alpha}\sin^2\beta=\frac{b^4(k^2-a^2)}{a^2k^2(k^2-a^2+a^2i^2)};$

but the equation of condition (109.) gives

$$k^2-a^2=\frac{a^4i^2}{k'^2-a^2},\ \text{hence } \frac{\tan^2\beta}{\tan^2\alpha}\sin^2\beta=\frac{b^4(k^2-a^2)(k'^2-a^2)}{a^4i^2k^2k'^2}=\frac{b^4}{k^2k'^2}.$$

As this expression is symmetrical, we shall have for the spherical conic section, whose radius is k',

$$\frac{\tan\beta'}{\tan\alpha'}\sin\beta'=\frac{b^2}{kk'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111.)$$

Hence $$\frac{\tan\beta}{\tan\alpha}\sin\beta=\frac{\tan\beta'}{\tan\alpha'}\sin\beta', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112.)$$

or the coefficients of the elliptic integrals, which determine the arcs of two spherical conic sections, having reciprocal parameters, are equal.

Let $\varkappa$ be the criterion of sphericity; then as

$$\varkappa=(1-m)\left(1-\frac{i^2}{m}\right)=(1-e^2)(1-e'^2)=\frac{b^4}{k^2k'^2},$$

$$\varkappa=\varkappa'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113.)$$

XXXII. To determine the values of the angles λ and λ' which correspond to the same angle φ in the expressions for the arcs of spherical conic sections having reciprocal parameters.

Since $$\cos^2\varepsilon=\frac{\cos^2\alpha}{\cos^2\beta}=\frac{k^2-a^2}{k^2-b^2}=\frac{k^2-a^2}{k^2-a^2+a^2i^2}.$$

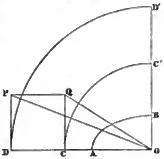
Introducing the equation of condition $(k^2-a^2)(k'^2-a^2)=a^4i^2$, we get $\cos\varepsilon=\frac{a}{k'}$; but $\tan\varphi=\cos\varepsilon\tan\lambda$, as in (39.); hence $\tan\lambda=\frac{k'}{a}\tan\varphi$, and $\tan\lambda'=\frac{k}{a}\tan\varphi$,

therefore $$k\tan\lambda=k'\tan\lambda', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114.)$$

or the tangent of the angle λ which the perpendicular arc from the centre of the

spherical conic, on the arc of a great circle touching it, makes with the principal major arc, is inversely as the radius of the sphere.

Fig. 10.



A simple geometrical construction will give the magnitude of those angles λ and λ'. Let the ellipse OAB be the base of the cylinder; OCC', ODD' being the bases of the hemispheres whose intersections with the cylinders give the spherical conic sections with reciprocal parameters. Erect the equal tangents DP, CQ, and join PO, QO. The angles AOP, AOQ are λ and λ'.

When $DP=CQ=0$, $\lambda=\lambda'=0$; when $DP=CQ=\infty$, $\lambda=\lambda'=\frac{\pi}{2}$. The condition (109.) shows that when $k=a$, $k'=\infty$. Now as $k'\tan\lambda'=a\tan\lambda$, is finite always, so long as λ is not absolutely $=\frac{\pi}{2}$; in order that its equal $k'\tan\lambda'$ may be finite also, we must have λ' always equal to 0, for every finite value of $\tan\lambda$.

XXXIII. The tangent of the principal arc of a spherical parabola is a mean proportional between the tangents of the principal arcs of two spherical conics with reciprocal parameters; the three curves being the sections of the same elliptic cylinder by three concentric spheres.

Since $\tan^2\alpha=\frac{a^2}{k^2-a^2}$, $\tan^2\alpha'=\frac{a^2}{k'^2-a^2}$, $\tan^2\alpha\tan\alpha'=\frac{a^4}{(k^2-a^2)(k'^2-a^2)}$.

Introducing the equation of condition $(k^2-a^2)(k'^2-a^2)=a^4i^2$ (109.), we get

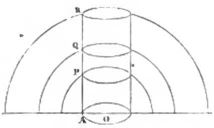
$$\tan\alpha\tan\alpha'=\frac{1}{i}. \qquad (115.)$$

Let k'' be the radius of the sphere whose intersection with the cylinder gives the spherical parabola; then $k''^2=a^2(1+i)$. See (96.)

Hence $k''^2-a^2=a^2i$; and $\tan^2\alpha''=\frac{a^2}{k''^2-a^2}=\frac{1}{i}$; therefore

$$\tan\alpha\tan\alpha'=\tan^2\alpha''. \qquad (116.)$$

Fig. 11.



The altitudes of the vertices of the three principal major arcs of the two spherical conics with reciprocal parameters, and of the spherical parabola, above the plane of the elliptic base of the cylinder, are in geometrical progression. Let AQ be the altitude of the vertex of the major arc of the spherical parabola. AP, AR the corresponding altitudes of the vertices of the major arcs of the spherical ellipses. Then $AP=\sqrt{k^2-a^2}$, $AR=\sqrt{k'^2-a^2}$, $AQ=\sqrt{k''^2-a^2}=a\sqrt{i}$. The equation of condition gives, as in (109.), $AP\times AR=\overline{AQ}^2$.

We shall give, further on, an expression for the sum of the arcs of two spherical conic sections having the same amplitude, but reciprocal parameters.

XXXIV. The projections of supplemental spherical ellipses on the plane of xy are confocal plane ellipses.

For $\sin\eta=\sin\epsilon'$, $\sin\eta'=\sin\epsilon$. Hence $\frac{a^2-b^2}{a^2}=\frac{a_1^2-b_1^2}{k^2-b_1^2}$, $\frac{a_1^2-b_1^2}{a_1^2}=\frac{a^2-b^2}{k^2-b^2}$.

This gives as the resulting value $k^2=a^2+b_1^2=a_1^2+b^2$, or $a^2-b^2=a_1^2-b_1^2$.

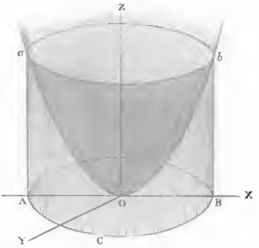
Two supplemental cones are cut by a plane at right angles to their common internal axe. The sections are concentric similar ellipses, having the major and the minor axes of the one, coinciding with the minor and major axes of the other.

For $\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}=e^2$, and $e_1^2=\frac{\tan^2\alpha'-\tan^2\beta'}{\tan^2\alpha'}=\frac{\cot^2\beta-\cot^2\alpha}{\cot^2\beta}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}$, or $e'=e$.

Section IV.—*On the Logarithmic Ellipse.*

XXXV. The logarithmic ellipse is the curve of symmetrical intersection of a paraboloid of revolution with an elliptic cylinder. This section of the cylinder by the paraboloid is analogous to the section of the cone by the concentric sphere in IX., for this cylinder may be viewed as a cone, having its vertex at the centre of the paraboloid, *i. e.* at an infinite distance.

Fig. 12.



Let the axes of the paraboloid and cylinder coincide with the axis of Z; the vertex of the paraboloid being supposed to touch the plane of xy at the origin O.

Let k be the semiparameter of the paraboloid Oab, and let a and b be the semiaxes of the base of the elliptic cylinder ACB; then the equations of these surfaces, and consequently of the curve in which they intersect, are

$$x^2+y^2=2kz, \text{ and } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad . \quad (117.)$$

Let $\mathrm{d}\Sigma$ be an element of the required curve, then

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}\theta}=\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118.)$$

x, y and z being dependent variables on a fourth independent variable θ.

Assume $\quad x=a\cos\theta,\ y=b\sin\theta$, then $a^2\cos^2\theta+b^2\sin^2\theta=2kz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (119.)$

Differentiating and substituting,

$$\left(\frac{\mathrm{d}\Sigma}{\mathrm{d}\theta}\right)^2=a^2\sin^2\theta+b^2\cos^2\theta+\frac{(a^2-b^2)^2}{k^2}\sin^2\theta\cos^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (120.)$$

To reduce this expression to a form suited for integration, it may be written,

$$k^2\left(\frac{\mathrm{d}\Sigma}{\mathrm{d}\theta}\right)^2=b^2k^2+(a^2-b^2)[k^2+a^2-b^2]\sin^2\theta-(a^2-b^2)^2\sin^4\theta. \quad . \quad . \quad . \quad . \quad (121.)$$

This expression may be reduced as follows:

Let $\quad P=b^2k^2, \quad Q=(a^2-b^2)[k^2+a^2-b^2], \quad R=-(a^2-b^2)^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (122.)$

and the preceding equation will become

$$k\Sigma=\int d\theta\sqrt{P+Q\sin^2\theta+R\sin^4\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123.)$$

Let this trinomial be put under the form of a product of two quadratic factors,

$$(A+B\sin^2\theta)(C-B\sin^2\theta)=AC+B(C-A)\sin^2\theta-B^2\sin^4\theta. \quad . \quad . \quad . \quad . \quad . \quad (124.)$$

Comparing this expression with the preceding in (121.), we get

$$AC=b^2k^2, \quad C-A=k^2+a^2-b^2, \quad B=a^2-b^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (125.)$$

To integrate (123.): assume $\quad \tan^2\varphi=\frac{A+B}{A}\tan^2\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126.)$

The limits of integration of the complete functions will continue as before. Making the substitutions indicated by the preceding transformations, the integral will now become

$$\frac{\sqrt{C(A+B)}}{AC}k\Sigma=\int\frac{d\varphi\left[1-\frac{B}{C}\left(\frac{A+C}{A+B}\right)\sin^2\varphi\right]}{\left[1-\frac{B}{A+B}\sin^2\varphi\right]^2\sqrt{1-\frac{B}{C}\left(\frac{A+C}{A+B}\right)\sin^2\varphi}}. \quad . \quad . \quad . \quad . \quad . \quad (127.)$$

Let $\frac{B}{A+B}=n, \ \frac{B}{C}\frac{(A+C)}{(A+B)}=i^2, \ \frac{A+C}{C}=\frac{i^2}{n}, \quad N=1-n\sin^2\varphi, \ I=1-i^2\sin^2\varphi, \quad . \quad . \quad (128.)$

and the preceding expression may be written

$$\frac{[2n-i^2-n^2]}{\sqrt{n(i^2-n)(1-n)}}\frac{\Sigma}{k}=(1-n)\int\frac{d\varphi I}{N^2\sqrt{I}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129.)$$

It will presently be shown that A and C must always have the same sign, whence $i^2>n$.

As $i^2=\frac{1+\frac{A}{C}}{1+\frac{A}{B}}$, and as C is always greater than B, $i^2<1$. From (125.) we may derive

$$\frac{a^2}{k^2}=\frac{(A+B)(C-B)}{(C-A-B)^2}, \qquad \frac{b^2}{k^2}=\frac{AC}{(C-A-B)^2}.$$

Now, that the values of a and b may be real, we must have $C>B$, while A and C must be of the same sign; but as B is essentially positive, C, and therefore A, must be positive.

Since $\qquad \frac{B}{A+B}=n, \quad$ and $\frac{A+C}{C}=\frac{i^2}{n}$, as in (128.)

we may eliminate A, B, C from the values of the semiaxes of the base of the elliptic cylinder, and express a, b and k, in terms of i and n. We may thus obtain

$$\frac{a^2}{k^2}=\frac{n(1-i^2)(i^2-n)}{[2n-i^2-n^2]^2}; \qquad \frac{b^2}{k^2}=\frac{n(i^2-n)(1-n)^2}{[2n-i^2-n^2]^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130.)$$

In order that these values of a and b may be real, we must have n positive, $i^2>n$, and $1>i^2$.

This is Case VI. in the Table, p. 316.

If we put c for the eccentricity of the plane elliptic base of the cylinder, we shall have after some obvious reductions, writing f for the complement of c,

$$(1-i^2)(1-c^2)=(1-n)^2, \text{ or } fj=1-n. \qquad (131.)$$

Now this simple equation between n, i and c enables us with great ease to determine the eccentricity c of the base of the elliptic cylinder, whose section with the paraboloid gives the logarithmic ellipse, when we know the parameter n, and the modulus i, of the given elliptic integral.

If we reduce this equation, it becomes $c^2j^2=2n-n^2-i^2$, the denominator of (130.).

XXXVI. To integrate the expression given in (127.), we must assume

$$\Phi_n=\frac{\sin\phi\cos\phi\sqrt{1-i^2\sin^2\phi}}{[1-n\sin^2\phi]}. \qquad (132.)$$

Differentiate this expression with respect to ϕ, and we shall have

$$\frac{d\Phi_n}{d\phi}=\frac{1-2(1+i^2)\sin^2\phi+3i^2\sin^4\phi}{[1-n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}+\frac{2n(\sin^2\phi-\sin^4\phi)(1-i^2\sin^2\phi)}{[1-n\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}. \qquad (a.)$$

Let $\qquad 1-n\sin^2\phi=\mathrm{N},\ 1-i^2\sin^2\phi=\mathrm{I}$, as before.

Separating the numerators of the preceding expression into their component parts, and attaching to each their respective denominators, we shall have

$$\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}=\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}, \text{ (b.) and } -\frac{2(1+i^2)\sin^2\phi}{\mathrm{N}\sqrt{\mathrm{I}}}=\frac{2(1+i^2)}{n}\frac{(1-n\sin^2\phi-1)}{\mathrm{N}\sqrt{\mathrm{I}}}=\frac{2(1+i^2)}{n\sqrt{\mathrm{I}}}-\frac{2(1+i^2)}{n\mathrm{N}\sqrt{\mathrm{I}}}. \qquad (c.)$$

The next term gives

$$\frac{3i^2\sin^4\phi}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{3i^2}{n}\frac{(1-n\sin^2\phi-1)\sin^2\phi}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{3i^2}{n}\frac{\sin^2\phi}{\sqrt{\mathrm{I}}}+\frac{3i^2\sin^2\phi}{n\mathrm{N}\sqrt{\mathrm{I}}}. \qquad (d.)$$

Now these two terms may be still further resolved; for

$$-\frac{3i^2}{n}\frac{\sin^2\phi}{\sqrt{\mathrm{I}}}=\frac{3}{n}\frac{(1-i^2\sin^2\phi-1)}{\sqrt{\mathrm{I}}}=\frac{3\sqrt{\mathrm{I}}}{n}-\frac{3}{n\sqrt{\mathrm{I}}}, \text{ and}$$

$$\frac{3i^2\sin^2\phi}{n\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{3i^2}{n^2}\frac{(1-n\sin^2\phi-1)}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{3i^2}{n^2\sqrt{\mathrm{I}}}+\frac{3i^2}{n^2\mathrm{N}\sqrt{\mathrm{I}}},$$

whence (d.) becomes

$$\frac{3i^2\sin^4\phi}{\mathrm{N}\sqrt{\mathrm{I}}}=\frac{3\sqrt{\mathrm{I}}}{n}-\frac{3}{n\sqrt{\mathrm{I}}}-\frac{3i^2}{n^2\sqrt{\mathrm{I}}}+\frac{3i^2}{n^2\mathrm{N}\sqrt{\mathrm{I}}}. \qquad (e.)$$

Combining the expressions in (b.), (c.), (d.) or (e.), the first term of the second member of (a.) may be written

$$\frac{[1-2(1+i^2)\sin^2\phi+3i^2\sin^4\phi]}{[1-n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}=\frac{3\sqrt{\mathrm{I}}}{n}+\left[\frac{2}{n}(1+i^2)-\frac{3i^2}{n}-\frac{3}{n}\right]\frac{1}{\sqrt{\mathrm{I}}}+\left[1-\frac{2}{n}(1+i^2)+\frac{3i^2}{n^2}\right]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}. \qquad (f.)$$

The second term, $\frac{2n(\sin^2\phi-\sin^4\phi)\sqrt{\mathrm{I}}}{(1-n\sin^2\phi)^2}$, of (a.) may be thus developed,

$$\frac{2n\sin^2\phi\sqrt{\mathrm{I}}}{\mathrm{N}^2}=-\frac{2n}{n}\frac{(1-n\sin^2\phi-1)\sqrt{\mathrm{I}}}{\mathrm{N}^2}=-\frac{2\mathrm{I}}{\mathrm{N}\sqrt{\mathrm{I}}}+\frac{2\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}; \qquad (g.)$$

and these two latter expressions may be written

$$-\frac{2\mathrm{I}}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{2(1-i^2\sin^2\phi)}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{2}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2i^2}{n}\frac{(1-n\sin^2\phi-1)}{\mathrm{N}\sqrt{\mathrm{I}}}=-\frac{2i^2}{n}\frac{1}{\sqrt{\mathrm{I}}}+\frac{2i^2}{n}\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2}{\mathrm{N}\sqrt{\mathrm{I}}};$$

whence (g.) becomes $$\frac{2n\sin^2\varphi\sqrt{\mathrm{I}}}{\mathrm{N}^2}=-\frac{2i^2}{n\sqrt{\mathrm{I}}}-2\left(1-\frac{i^2}{n}\right)\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}+\frac{2\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}. \quad \ldots \quad (h.)$$

The term $-\frac{2n\sin^4\varphi\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}$ may be written

$$-\frac{2n\mathrm{I}\sin^4\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-\frac{2\mathrm{I}}{n}\left[\frac{1-2n\sin^2\varphi+n^2\sin^4\varphi-2+2n\sin^2\varphi+1}{\mathrm{N}^2\sqrt{\mathrm{I}}}\right]=-\frac{2\mathrm{I}}{n.\sqrt{\mathrm{I}}}+\frac{4\mathrm{I}}{n.\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2\mathrm{I}}{n.\mathrm{N}^2\sqrt{\mathrm{I}}}. \quad (k.)$$

Now $$\frac{n\mathrm{I}}{n\sqrt{\mathrm{I}}}=\frac{2}{n}\sqrt{\mathrm{I}},$$

and $$\frac{4\mathrm{I}}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4(1-i^2\sin^2\varphi)}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4}{n\mathrm{N}\sqrt{\mathrm{I}}}+\frac{4i^2}{n^2}\frac{(1-n\sin^2\varphi-1)}{\mathrm{N}\sqrt{\mathrm{I}}},$$

whence $$\frac{4\mathrm{I}}{n\mathrm{N}\sqrt{\mathrm{I}}}=\frac{4i^2}{n^2\sqrt{\mathrm{I}}}-4\frac{(i^2-n)}{n^2}\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad \ldots \ldots \quad (m.)$$

Combining (k.) with (m.), we shall have

$$-\frac{2n\mathrm{I}\sin^4\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-\frac{2\sqrt{\mathrm{I}}}{n}+\frac{4i^2}{n^2\sqrt{\mathrm{I}}}-\frac{4}{n^2}(i^2-n)\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-\frac{2\mathrm{I}}{n\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad \ldots \quad (n.)$$

adding (n.) to (h.),

$$\frac{2n(\sin^2\varphi-\sin^4\varphi)\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-\frac{2\sqrt{\mathrm{I}}}{n}+\left(\frac{4i^2}{n^2}-\frac{2i^2}{n}\right)\frac{1}{\sqrt{\mathrm{I}}}+\left[\frac{2i^2}{n}-2+\frac{4}{n}-\frac{4i^2}{n^2}\right]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-2\left(\frac{1}{n}-1\right)\frac{1}{\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad (p.)$$

adding (f.) and (p.) together, we get as the final result,

$$\frac{d\Phi_n}{d\varphi}=\frac{\sqrt{\mathrm{I}}}{n}+\frac{1}{n}\left(\frac{i^2-n}{n}\right)\frac{1}{\sqrt{\mathrm{I}}}+\frac{1}{n^2}[2n-n^2-i^2]\frac{1}{\mathrm{N}\sqrt{\mathrm{I}}}-2\left(\frac{1-n}{n}\right)\frac{\mathrm{I}}{\mathrm{N}^2\sqrt{\mathrm{I}}}; \quad \ldots \quad (q.)$$

or multiplying by n, transposing and integrating,

$$2(1-n)\int\frac{\mathrm{I}d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}}=-n\Phi_n+\int d\varphi\sqrt{\mathrm{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\left[\frac{2n-n^2-i^2}{n}\right]\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad \ldots \quad (r.)$$

But we have shown in (129.) that

$$\frac{2[2n-i^2-n^2]}{\sqrt{n(i^2-n)(1-n)}}\frac{\Sigma}{k}=2(1-n)\int\frac{\mathrm{I}d\varphi}{\mathrm{N}^2\sqrt{\mathrm{I}}},$$

whence $$\frac{2[2n-i^2-n^2]}{\sqrt{n.(i^2-n)(1-n)}}\frac{\Sigma}{k}=-n\Phi_n+\int d\varphi\sqrt{\mathrm{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\left[\frac{2n-i^2-n^2}{n}\right]\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}. \quad (133.)$$

Hence, an arc of a logarithmic ellipse may be expressed by a line Φ_n, and in terms of elliptic integrals of the first, second and third orders; the latter being of the logarithmic form (127.) may be written in the form

$$\frac{\Sigma}{k}=\frac{b^2}{\sqrt{\mathrm{C}(\mathrm{A}+\mathrm{B})}}\int\frac{d\varphi[1-i^2\sin^2\varphi]}{[1-n\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}. \quad \ldots \ldots \quad (134.)$$

XXXVII. When the cylinder and the paraboloid are given, we may determine the parameter, modulus and constants of the functions which represent the curve of intersection of these surfaces, in the terms of the constants a, b and k.

The modulus parameter, coefficients and criterion of sphericity may be expressed, as linear products of constants, having simple relations with those of the given surfaces.

Resuming the equations given in (125.),

$$\mathrm{AC}=b^2k^2, \quad \mathrm{C}-\mathrm{A}=k^2+a^2-b^2, \quad \mathrm{B}=a^2-b^2,$$

we find $$(A+C)^2=(k^2+a^2-b^2)^2+4b^2k^2.$$

Assume $$4p^2=k^2+(a+b)^2,\quad 4q^2=k^2+(a-b)^2, \quad \ldots\ldots \quad (135.)$$

we shall then have the following equations:—

$$\left.\begin{array}{ll} A+C=4pq, & B=(a+b)(a-b) \\ A+B=(a+p-q)(a+q-p); & C-B=(p+q+a)(p+q-a) \\ A=(b+p-q)(b+q-p); & C=(p+q+b)(p+q-b) \\ ab=(p+q)(p-q), & k^2+a^2+b^2=2(p^2+q^2) \end{array}\right\} \quad \ldots \quad (136.)$$

Substituting these values in (129.) we obtain the resulting expressions

$$\left.\begin{array}{l} i^2=\dfrac{4(a+b)(a-b)pq}{(p+q+b)(p+q-b)(a+p-q)(a+q-p)} \\ n=\dfrac{(a+b)(a-b)}{(a+p-q)(a+q-p)},\quad m=\dfrac{(a+b)(a-b)}{(p+q+b)(p+q-b)} \end{array}\right\}. \quad \ldots\ldots \quad (137.)$$

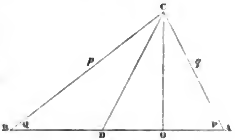
and if we denote by κ the criterion of sphericity,

$$\kappa=\frac{-b^4}{a^2(p+q)^2}\left(\frac{p+q+a}{p+q+b}\right)^2\left(\frac{p+q-a}{p+q-b}\right)^2, \quad \ldots\ldots \quad (138.)$$

we may express the parameters and modulus of the elliptic integral of the third order and logarithmic form by a geometrical construction of remarkable simplicity when the intersecting surfaces are given, or when a, b, and k are given.

Take $BA=a$, $BD=b$, and from O the point of bisection of AD, erect the perpendicular $OC=\frac{k}{2}$. Then (135.) gives $p=BG$, $q=AC$, and putting P and Q for the angles BAC and ABC, $a+b=2p\cos Q$, $a-b=2q\cos P$. As p, q, b are the sides of the triangle BCD, and the angle $BCD=P-Q$,

Fig. 13.



$$\cos^2\left(\frac{P-Q}{2}\right)=\frac{(b+p+q)(p+q-b)}{4pq};$$

again as a, p, q are the sides of the triangle ABC, and.

$$\cos^2\left(\frac{P+Q}{2}\right)=\frac{(a+p-q)(a+q-p)}{4pq}.$$

Substituting these values in (137.), we get

$$i^2=\frac{\cos P\cos Q}{\left[\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)\right]^2},\quad n=\frac{\cos P\cos Q}{\cos^2\left(\frac{P+Q}{2}\right)},\quad m=\frac{\cos P\cos Q}{\cos^2\left(\frac{P-Q}{2}\right)}, \quad \ldots \quad (139.)$$

and if c be the eccentricity of the elliptic base of the cylinder,

$$c^2=\frac{\sin 2P.\sin 2Q}{\sin^2(P+Q)}. \quad \ldots\ldots \quad (140.)$$

These are expressions remarkable for their simplicity.

We also find for the criterion of sphericity $\varkappa$,

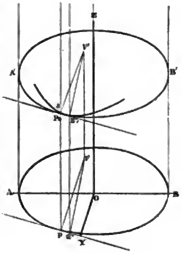
$$\varkappa=-\left[\frac{\sin^2\left(\frac{P-Q}{2}\right)}{\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)}\right]^2 . \quad . \quad . \quad . \quad (141.)$$

As $\frac{k}{2}$ is the altitude of a triangle whose sides are a, p, q,

$$a^2k^2=(a+p+q)(p+q-a)(a+q-p)(a+p-q).$$

XXXVIII. In the preceding investigations the element of the curve has been taken as a side of a limiting rectilinear polygon inscribed within it. We may however effect the rectification of the curve, starting from other elementary principles. Let APB be the plane base of the elliptic cylinder, and let a series of normal planes PP$'\nu'$ $\varpi\varpi'\nu'$ be drawn to the cylinder, indefinitely near to each other, and parallel to its axis. We may conceive of every element Pϖ of this plane ellipse between the normal planes as the projection of the corresponding element $s\varpi'$ of the logarithmic ellipse. Let τ be the inclination of the element dΣ of the logarithmic ellipse to the corresponding element ds of the plane ellipse. We shall have, dλ being the elementary angle between the planes PP$'\nu'$ and $\varpi\varpi'\nu'$,

Fig. 14.



$$\frac{d\Sigma}{d\lambda}=\sec\tau\frac{ds}{d\lambda}. \quad . \quad . \quad . \quad . \quad (142.)$$

Now (31.) gives $\frac{ds}{d\lambda}=p+\frac{d^2p}{d\lambda^2}$,

and therefore
$$\Sigma=\int\frac{p}{\cos\tau}d\lambda+\int\frac{d^2p}{d\lambda^2}\sec\tau.d\lambda. \quad . \quad . \quad . \quad . \quad (143.)$$

In the plane ellipse $p^2=a^2\cos^2\lambda+b^2\sin^2\lambda$, whence $\frac{d^2p}{d\lambda^2}=-\frac{(a^2-b^2)(a^2\cos^4\lambda-b^2\sin^4\lambda)}{(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}}$. (144.)

We have now to express $\cos\tau$ in terms of λ.

From (119.) combined with (120.) we may derive

$$\sec^2\tau=\frac{d\Sigma^2}{dx^2+dy^2}=\frac{b^2k^2+(a^2-b^2)[k^2+a^2-b^2]\sin^2\theta-(a^2-b^2)^2\sin^4\theta}{k^2(a^2\sin^2\theta+b\cos^2\theta)}. \quad . \quad . \quad (145.)$$

Eliminating $\frac{y}{x}$ between the equations $\tan\lambda=\frac{a^2}{b^2}\frac{y}{x}$, and $\frac{y}{x}=\frac{b}{a}\tan\theta$, we shall have

$$\tan\lambda=\frac{a}{b}\tan\theta. \quad . \quad . \quad . \quad . \quad (146.)$$

If we eliminate $\tan\theta$ by the help of this equation from (145.), we shall obtain

$$\cos^2\tau=\frac{k^2(a^2\cos^2\lambda+b^2\sin^2\lambda)}{a^2k^2+(a^2-b^2)[a^2-b^2-k^2]\sin^2\lambda-(a^2-b^2)^2\sin^4\lambda}. \quad . \quad . \quad . \quad (147.)$$

Substituting this value of $\cos\tau$ in (143.), and writing P′, Q′, R′ for the coefficients of powers of $\sin\lambda$, the resulting equation will become

$$k\Sigma=\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-(a^2-b^2)\int\frac{d\lambda(a^2\cos^4\lambda-b^2\sin^4\lambda)}{k(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}\cos\tau}. \quad . \quad . \quad (148.)$$

As the first of these integrals is precisely similar in form to the integral in (123.), we may in the same manner reduce the expression into factors. Accordingly let

$$P'+Q'\sin^2\lambda+R'\sin^4\lambda=(\alpha+\beta\sin^2\lambda)(\gamma-\beta\sin^2\lambda). \quad . \quad . \quad . \quad . \quad . \quad (149.)$$

Writing α, β, γ instead of A, B, C, and following step by step the investigation in Art. XXXV., we shall have, as in (126.) and (128.), ψ, m, and $i_{,}$ being the amplitude, parameter and modulus,

$$\tan^2\psi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda, \quad m=\frac{\beta}{\alpha+\beta}, \quad i_{,}^2=\frac{\beta}{\gamma}\left(\frac{\alpha+\gamma}{\alpha+\beta}\right). \quad . \quad . \quad . \quad . \quad (150.)$$

As $\qquad \alpha\gamma=a^2k^2, \quad \beta=a^2-b^2$, and $\gamma-\alpha=a^2-b^2-k^2, \quad . \quad . \quad . \quad . \quad (151.)$

we shall have the following relations between the constants $\alpha, \beta, \gamma, m, i_{,}$, and A, B, C, n, i, in (150.) and (128.),

$$\left.\begin{array}{l}\beta=B, \quad \alpha=C-B, \quad \gamma=A+B, \quad \alpha+\gamma=A+C, \\ \gamma-\beta=A, \quad \alpha+\beta=C, \quad \gamma-\alpha-\beta+C-A-B=0, \\ i_{,}^2=\frac{\beta(\alpha+\gamma)}{\gamma(\alpha+\beta)}=\frac{B(A+C)}{(A+B)C}=i^2, \text{ or } i_{,}=i, \quad m=\frac{\beta}{\alpha+\beta}=\frac{B}{C}.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (152.)$$

Hence the moduli are the same in the two forms of integration, and the parameters m and n will be found to be connected by the equation $m+n-mn=i^2$; . . (153.) m and n are, therefore, *conjugate* parameters, as they fulfil the condition assumed in (1.).

The amplitudes φ and ψ are equal.

In (126.) we assumed, $\tan^2\varphi=\frac{A+B}{A}\tan^2\theta$; and in (150.) $\tan^2\psi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda$, but $\tan\lambda=\frac{a}{b}\tan\theta$, as in (146.), whence $\tan^2\psi=\frac{a^2}{b^2}\frac{(\alpha+\beta)A}{(A+B)\alpha}\tan^2\varphi$.

In (152.) we have found $\alpha+\beta=C$, and $A+B=\gamma$,

whence $\qquad \tan^2\psi=\frac{a^2}{b^2}\frac{AC}{\alpha\gamma}\tan^2\varphi$. But $AC=b^2k^2$, and $\alpha\gamma=a^2k^2$,

as shown in (125.) and (151.), whence

$$\psi=\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154.)$$

We shall now proceed to find the value of the second integral in (148.).

From (147.) we may derive $\tan^2\tau=\frac{(a^2-b^2)^2\sin^2\lambda\cos^2\lambda}{k^2(a^2\cos^2\lambda+b^2\sin^2\lambda)}$. . . . . (155.)

Differentiating this expression, reducing, dividing by $\cos\tau$, and integrating, we finally obtain

$$k\int\frac{d\tau}{\cos^3\tau}=(a^2-b^2)\int\frac{d\lambda(a^2\cos^4\lambda-b^2\sin^4\lambda)}{\cos\tau\,(a^2\cos^2\lambda+b^2\sin^2\lambda)^{\frac{3}{2}}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156.)$$

(148.) may now be written

$$k\Sigma=\int d\lambda\sqrt{P+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\tau}{\cos^3\tau}. \quad (157.)$$

If we measure the arc of the logarithmic ellipse from the minor principal axe, or from the parabolic arc which is projected into b, instead of placing the origin at the vertex of the major axe as in (119.), we must put

$$x=a\sin\vartheta, \quad y=b\cos\vartheta; \quad (158.)$$

and following the steps indicated in that article, we shall obtain

$$kS=\int d\vartheta\sqrt{P+Q'\sin^2\vartheta+R'\sin^4\vartheta}. \quad (159.)$$

If we now make $\vartheta=\lambda$, and subtract the two latter equations, one from the other, the resulting equation will become

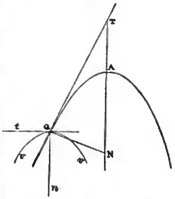
$$S-\Sigma=k\int\frac{d\tau}{\cos^3\tau}. \quad (160.)$$

But this integral is, we know, the expression for an arc of a common parabola, whose semi-parameter is k, measured from the vertex of the curve, to a point on it, where its tangent makes the angle τ with the ordinate.

Thus the difference between two elliptic arcs measured from the vertices of the curve, which in the plane ellipse may, as we know, be expressed by a right line; and in the spherical ellipse by an arc of a circle, as shown in Art. XV.; will in the logarithmic ellipse be expressed by an arc of a parabola. As a parabolic arc can be rectified only by a logarithm, we may hence see the propriety of the term *logarithmic*, by which this function is designated.

XXXIX. If from the vertex A of a paraboloid, an arc of a parabola be drawn, at right angles to a parabolic section of the paraboloid, it will meet this parabolic section at its vertex. Let the arc AQ be drawn at right angles to the parabolic section Qv of the paraboloid, the point Q is the vertex of the parabola Qv.

Fig. 15.



Draw QT and Qt tangents to the arcs QA and Qv. Then QT and Qt are at right angles. As QT is a tangent to a principal section passing through the axis of the paraboloid, it will meet this axis in a point T; and as Qt is a tangent to the surface of the paraboloid, it will be perpendicular to the normal to the surface QN. Now as Qt is perpendicular to QT and to QN, it is perpendicular to the plane QTN which passes through them, and therefore to every line in this plane, and therefore to the axis AN, or to any line parallel to it, as the diameter Qn. Hence, as the tangent Qt to the parabola Qv is perpendicular to the diameter Qn, Q is the vertex of the parabola.

Hence in the logarithmic ellipse, one extremity of the protangent arc is always the vertex of the parabola which touches the logarithmic ellipse at its other extremity.

This is a very important theorem, as the protangents are arcs of equal parabolas, all measured from the vertices of the parabolas. Hence also the length of the protangent arc depends solely on its normal angle.

As an arc of a circle may be expressed by the notation $s=\sin^{-1}\left(\frac{y}{k}\right)$, y being the ordinate and k the radius, so in like manner an arc of a parabola may be designated by the form $s=\tan^{-1}\left(\frac{y}{k}\right)$; y being the ordinate and k the semiparameter. To distinguish the parabolic arc from the circular arc, the former may be written $s=\tau\alpha\nu^{-1}\left(\frac{y}{k}\right)$. Again, as we say, in the case of the circle, the angle ω and the arc $k\omega$, ω being the angle contained between the normals to the curve at the extremities of the arc: so in the parabola, we may write ω for the angle between the normals, and $(k.\omega)$ for the corresponding parabolic arc. In the case of the parabola the arc is always supposed to be measured from the vertex; in the circle the arc may be measured from any point, as every point is a vertex.

XL. Resuming the equation (157.), $k\Sigma=\int d\lambda\sqrt{P'+Q'\sin^2\lambda+R'\sin^4\lambda}-k^2\int\frac{d\tau}{\cos^3\tau}$. We shall now proceed to develope the first integral of the second side of this equation. As the integral is precisely the same in form as (123.), and the amplitude $\psi=\varphi$, as also the modulus $i_{\prime}=i$, we may substitute α, β, γ for A, B, C, m for n, Φ_m for Φ_n, retaining the modulus and amplitude, which continue unchanged, as we have established in (152.) and (154.); or substituting for α, β, γ their values in m and i, we get

$$\left.\begin{aligned}\frac{2[i^2+m^2-2m]}{\sqrt{m(i^2-m)(1-m)}}\,\frac{\Sigma}{k}=-m\Phi_m-\frac{[i^2+m^2-2m]}{m}\int\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}\\ +\frac{[i^2-m]}{m}\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}+\int d\varphi\sqrt{1-i^2\sin^2\varphi}-\frac{[i^2+m^2-2m]2}{\sqrt{m(i^2-m)(1-m)}}\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\}. \quad . \quad (161.)$$

If we eliminate i from the coefficients of (133.) and (161.), putting M for $(1-m\sin^2\varphi)$, and N for $(1-n\sin^2\varphi)$, as also $\sqrt{\mathrm{I}}$ for $\sqrt{1-i^2\sin^2\varphi}$; (133.) will become

$$\frac{2(n-m)}{\sqrt{mn}}\,\frac{\Sigma}{k}=-n\Phi_n+\frac{(1-n)(n-m)}{n}\int\frac{d\varphi}{N\sqrt{\mathrm{I}}}+\frac{m}{n}(1-n)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\int d\varphi\sqrt{\mathrm{I}}, \quad . \quad . \quad (162.)$$

and (161.) will be transformed into

$$\frac{2(n-m)}{\sqrt{mn}}\,\frac{\Sigma}{k}=-m\Phi_m-\frac{(1-m)(n-m)}{m}\int\frac{d\varphi}{M\sqrt{\mathrm{I}}}+\frac{n}{m}(1-m)\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\int d\varphi\sqrt{\mathrm{I}}-\frac{2(n-m)}{\sqrt{mn}}\int\frac{d\tau}{\cos^3\tau}. \quad (163.)$$

If we compare together (162.) and (163.), which are expressions for the same arc of the logarithmic ellipse, and make the obvious reductions, putting for Φ_n and Φ_m their values $\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{N}$ and $\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{M}$, we shall get the following resulting equation of comparison,

$$\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{N\sqrt{\mathrm{I}}}+\left(\frac{1-m}{m}\right)\int\frac{d\varphi}{M\sqrt{\mathrm{I}}}=\frac{i^2}{mn}\int\frac{d\varphi}{\sqrt{\mathrm{I}}}-\frac{2}{\sqrt{mn}}\int\frac{d\tau}{\cos^3\tau}+\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{MN}. \quad . \quad (164.)$$

From (155.) we may deduce $\sin\tau = \frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}$; . . . . (165.)

we shall therefore have $\tan\tau\sec\tau = \frac{\sqrt{mn}\sin\phi\cos\phi\sqrt{I}}{MN}$. . . . . (166.)

It may easily be shown that $\tan\tau\sec\tau$ represents the portion of a tangent to a parabola intercepted between the point of contact and the perpendicular from the focus.

Hence $\tan\tau\sec\tau = 2\int\frac{d\tau}{\cos^3\tau} - \int\frac{d\tau}{\cos\tau}$. . . . . (167.)

Combining (164.), (166.) and (167.), and using the ordinary notation of elliptic integrals,

$$\left(\frac{1-n}{n}\right)\Pi_c(n,\phi) + \left(\frac{1-m}{m}\right)\Pi_c(m,\phi) = \frac{c^2}{mn}F_c(\phi) - \frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau},$$

or as
$$\frac{d\tau}{\cos\tau} = \frac{d\sin\tau}{1-\sin^2\tau};\quad \frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau} = \frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\phi}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]d\phi}{1-\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]^2}, \quad \ldots\ldots \quad (168.)$$

we have therefore

$$\left(\frac{1-n}{n}\right)\Pi_c(n,\phi) + \left(\frac{1-m}{m}\right)\Pi_c(m,\phi) = \frac{c^2}{mn}F_c(\phi) - \frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\phi}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-c^2\sin^2\phi}}\right]d\phi}{1-\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-c^2\sin^2\phi}}\right]^2}. \quad \ldots \quad (169.)$$

This is the expression given by LEGENDRE, *Traité des Fonctions Elliptiques*, tom. i. p. 68. Written in the notation adopted in this paper, the formula would be

$$\left(\frac{1-n}{n}\right)\int\frac{d\phi}{N\sqrt{I}} + \left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}} = \frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}} - \frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}. \quad \ldots\ldots \quad (170.)$$

XLI. We may express a and b, the semiaxes of the elliptic base of the cylinder, in terms of m and n, the conjugate parameters of the elliptic integrals in the preceding equations. From the equation of condition $m+n-mn=i^2$, and (130.) we may eliminate i^2, and get

$$\frac{a^2}{k^2} = \frac{mn(1-m)}{(n-m)^2};\quad \frac{b^2}{k^2} = \frac{mn(1-n)}{(n-m)^2}. \quad \ldots\ldots\ldots\ldots \quad (171.)$$

Therefore
$$\frac{b}{a} = \sqrt{\frac{1-n}{1-m}} = \frac{\sqrt{(1-n)(1-m)}}{(1-m)} = \frac{\sqrt{1-i^2}}{1-m} = \frac{j}{1-m}.$$

Hence the ratio of the axes of the elliptic base of the cylinder is a function of the modulus and parameter.

The ratio of the corresponding quantities in the case of the spherical ellipse may be derived from the equation

$$\frac{a^2-b^2}{a^2} = i^2;\ \text{or}\ \frac{b}{a} = \sqrt{1-i^2} = j.$$

This ratio is therefore independent of the parameter. There is then an important difference in the two cases. In the one case, the ratio of the axes is independent of the parameter, and will continue invariable, while the parameter passes through every

stage of magnitude. But in the logarithmic ellipse the vertical cylinder will change its base with the change of the parameter. We shall see the importance of this remark presently.

These ratios are—

In the sphere $\frac{b}{a}=j$. In the paraboloid $\frac{b}{a}=\frac{j}{1-m}$. . . . . (172.)

XLII. Resuming equation (157.) and developing it by a process similar to that applied to (127.), we get

$$\Sigma=\frac{\alpha\gamma}{k\sqrt{\gamma(\alpha+\beta)}}\int\frac{[1-i^2\sin^2\varphi]d\varphi}{\left[1+\frac{\beta}{\alpha+\beta}\sin^2\varphi\right]^2\sqrt{1-i^2\sin^2\varphi}}-k\int\frac{d\tau}{\cos^3\tau}. \quad . \; . \; . \; . \; . \; . \quad (173.)$$

Now (151.) and (152.) give

$$\frac{\beta}{\alpha+\beta}=m,\ \alpha\gamma=a^2k^2,\ \sqrt{\gamma(\alpha+\beta)}=\frac{k^2\sqrt{mn}}{(n-m)},\ \text{and}\ a^2=\frac{k^2mn(1-m)}{(n-m)^2}.$$

Making these substitutions, we get

$$\Sigma=a\sqrt{(1-m)}\int\frac{[1-i^2\sin^2\varphi]d\varphi}{[1-m\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}-k\int\frac{d\tau}{\cos^3\tau}, \quad . \; . \; . \; . \; . \; . \quad (174.)$$

Now let $m=0$, then (165.) gives $\tau=0$, and we shall have

$$\Sigma=a\int d\varphi\sqrt{1-i^2\sin^2\varphi}.$$

This is the common expression for the rectification of a plane ellipse, whose greater semiaxis is a, and eccentricity i. This is case IV. of the Table, p. 316.

We cannot arrive at this limiting expression by making $e^2=m=0$ in (53.); for this supposition would render $i=0$, which, throughout these investigations, is assumed to be invariable.

XLIII. If, as in the case of the spherical parabola, we make $n=m$, or $n=1-\sqrt{1-i^2}$, the values of $\frac{a}{k}$ and $\frac{b}{k}$ become infinite. What, then, is the meaning of the elliptic integral of the logarithmic form of the third order, when $n=m$, or $n=1-\sqrt{1-i^2}$? In the circular form of the third order, when $m=n$, $n=i$, and the spherical ellipse becomes the spherical parabola, which, as we know, may be rectified by an elliptic integral of the first order. Not only do the ratios $\frac{a}{k}$, $\frac{b}{k}$ become infinite, but they become equal, for $\frac{b^2}{a^2}=\frac{1-n}{1-m}=1$, when $m=n$. What, then, does the integral in this case signify? It does not become imaginary or change its species.

Resuming the equation established in (133.),

$$\frac{2[2n-i^2-n^2]}{\sqrt{n(1-n)(i^2-n)}}\frac{\Sigma}{k}=-n\Phi_n+\frac{[2n-i^2-n^2]}{n}\int\frac{d\varphi}{N\sqrt{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\sqrt{I}.$$

If we now introduce the relation given in (130.) $\frac{a}{k}=\frac{\sqrt{n(i^2-n)(1-i^2)}}{2n-i^2-n^2}$, we shall have by substitution

$$\frac{2\sqrt{1-i^2}}{\sqrt{1-n}}\frac{\Sigma}{a}=-n\Phi_n+\left(\frac{2n-i^2-n^2}{n}\right)\int\frac{d\varphi}{N\sqrt{I}}+\left(\frac{i^2-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\sqrt{I}. \quad . \quad . \quad . \quad . \quad (175.)$$

If we now suppose $m=n$, or $n=1-\sqrt{1-i^2}$, or $2n-i^2-n^2=0$, the last equation will become

$$2\sqrt{j}\,\frac{\Sigma}{a}=j\int\frac{d\varphi}{\sqrt{I}}+\int d\varphi\sqrt{I}-n\Phi_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176.)$$

In this case

$$\Phi=\frac{\tan\varphi\sqrt{I}}{1+j\tan^2\varphi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177.)$$

This is the expression for the length of an arc of a logarithmic ellipse, the intersection of a cylinder, now become circular, with a paraboloid whose semiparameter $k=0$; therefore the dimensions of the paraboloid being indefinitely diminished in magnitude, this intersection of a finite circular cylinder by a paraboloid indefinitely attenuated, must take place at an infinite altitude. We naturally should suppose that the section of a cylinder which indefinitely approaches in its limit to a circular cylinder, by a paraboloid of revolution, would be a circle; yet the fact is not so. The intersection of these surfaces, instead of being a circle, is a logarithmic ellipse, whose rectification may be effected by an elliptic integral of the second order, as we shall now proceed to show.

In the first place let us conceive the paraboloid as of definite magnitude, and the cylinder to be elliptical; its semiaxes as before being a and b. Then as a and b are the ordinates of a parabola, at the points where the elliptic cylinder meets the paraboloid, at its greatest and least distances from the axis of the surfaces, we shall have

$$a^2=2kz', \quad b^2=2kz''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178.)$$

Hence $a^2-b^2=2k(z'-z'')$. Let $z'-z''=h$, then h is the thickness or height of that portion of the cylinder within which the logarithmic ellipse is contained.

Now (171.) gives $\qquad a^2-b^2=\frac{k^2mn}{n-m}; \quad \therefore 2h=\frac{kmn}{n-m},$

and we have also $\qquad a=\frac{k\sqrt{mn(1-m)}}{n-m}, \quad$ hence $h=\frac{a}{2}\,\frac{\sqrt{mn}}{\sqrt{1-m}}.$

Now when $n=m$, $a=b$, $k=0$, while we get for h

$$h=\frac{a}{2}\,\frac{n}{\sqrt{1-n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179.)$$

We thus arrive at this most remarkable result, that though the cylinder changes from elliptic to circular, while the parameter of the paraboloid approximates to its limiting value 0, yet the thickness of the zone, that is h, does not also indefinitely diminish, but assumes the limiting value given above.

Now if we cut this circular cylinder, the radius of whose base is a, by a plane

making with the plane of the circular section, or with the plane of xy, an angle whose tangent is $\frac{h}{a}$, the semiaxes A and B of this plane section will manifestly be

$$B=a, \text{ and } A=\sqrt{a^2+h^2} \text{ or } A=\frac{a(2-n)}{2\sqrt{1-n}}. \quad \ldots \ldots \quad (180.)$$

If we denote the eccentricity of this plane ellipse by

$i_{,}$, $i_{,}=\frac{n}{2-n}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}$, or writing j for $\sqrt{1-i^2}$, $i_{,}=\frac{1-j}{1+j}$. Hence $j=\frac{1-i_{,}}{1+i_{,}}$. . . (181.)

It is shown in every treatise on elliptic integrals, (see HYMER's Integral Calculus, p. 220,) that if c and $c_{,}$ are two moduli connected by the equation

$$c_{,}=\frac{1-\sqrt{1-c^2}}{1+\sqrt{1-c^2}}=\frac{1-b}{1+b}, \quad \ldots \ldots \quad (182.)$$

and φ and ψ two angles related, as in (63.), so that

$$\tan(\psi-\varphi)=b\tan\varphi, \quad \ldots \ldots \quad (183.)$$

we shall have $\quad (1+c_{,})E_{c}(\varphi)=E_{c_{,}}(\psi)+c_{,}\sin\psi-\frac{1}{2}b_{,}^2F_{c_{,}}(\psi).$

Now $\quad 1+c_{,}=\frac{2}{1+b}, \qquad b_{,}^2=1-c_{,}^2=\frac{4b}{(1+b)^2}$, hence

$$E_{c}(\varphi)=\frac{(1+b)}{2}E_{c_{,}}(\psi)+\frac{(1-b)}{2}\sin\psi-\frac{b}{1+b}F_{c_{,}}(\psi), \quad \ldots \ldots \quad (184.)$$

and, using the common notation for the present, (74.) gives

$bF_{c}(\varphi)=\frac{b}{1+b}F_{c_{,}}(\psi)$. Adding these equations, we get

$$E_{c}(\varphi)+bF_{c}(\varphi)=\frac{(1+b)}{2}E_{c_{,}}(\psi)+\frac{(1-b)}{2}\sin\psi, \quad \ldots \ldots \quad (185.)$$

or, using the notation adopted in this paper,

$$\int d\varphi\sqrt{\mathrm{I}}+j\int\frac{d\varphi}{\sqrt{\mathrm{I}}}=\frac{(1+j)}{2}\int d\psi\sqrt{\mathrm{I}_{,}}+\frac{n}{2}\sin\psi, \quad \ldots \ldots \quad (186.)$$

since $\quad n=1-b=1-j.$

Substituting the value of the first member of this equation in (176.), the resulting equation will become

$$2\sqrt{j}\frac{\Sigma}{a}=\frac{(1+j)}{2}\int d\psi\sqrt{\mathrm{I}_{,}}+\frac{n}{2}\sin\psi-n\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi}. \quad \ldots \ldots \quad (187.)$$

Having put for Φ_n its value in this case, namely,

$$\Phi_n=\frac{\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi},$$

we must now combine the last two members of this equation. Adding, they become

$$\frac{n}{2}\left\{\sin\psi-\frac{2\sin\varphi\cos\varphi\sqrt{\mathrm{I}}}{\cos^2\varphi+j\sin^2\varphi}\right\} \quad \ldots \ldots \quad (188.)$$

From this expression we must eliminate the functions of φ.

Now (73.) gives
$$\sqrt{\mathrm{I}}=\frac{\cos^2\varphi+j\sin^2\varphi}{\sqrt{\mathrm{I}'}}, \qquad (189.)$$
writing φ for μ.

Substituting this value of $\sqrt{\mathrm{I}}$ in the preceding expression, for which we put t, we get
$$t=\frac{n}{2}\left\{\sin\psi-\frac{2\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}_{,}}}\right\}. \qquad (190.)$$

From this equation we must eliminate $\sin\varphi$, $\cos\varphi$.

If we solve the preceding equation (189.) we shall obtain the resulting expressions
$$\left.\begin{aligned}2\sin^2\varphi&=1-\sqrt{\mathrm{I}_{,}}\cos\psi+i_{,}\sin^2\psi\\ 2\cos^2\varphi&=1+\sqrt{\mathrm{I}_{,}}\cos\psi-i_{,}\sin^2\psi\end{aligned}\right\} \qquad (191.)$$

Multiplying these equations together, and recollecting that $\mathrm{I}_{,}=1-i_{,}^2\sin^2\psi$, we find
$$4\cos^2\varphi\sin^2\varphi=\sin^2\psi[\mathrm{I}_{,}+2\sqrt{\mathrm{I}_{,}}i_{,}\cos\psi+i_{,}^2\cos^2\psi]. \qquad (192.)$$

Now the second member of this equation is a perfect square,

whence
$$2\sin\varphi\cos\varphi=\sin\psi[\sqrt{\mathrm{I}_{,}}+i_{,}\cos\psi]. \qquad (193.)$$

Substituting this value of $2\sin\varphi\cos\varphi$ in (190.), we get
$$t=\frac{n}{2}\sin\psi\left[1-\frac{\sqrt{\mathrm{I}_{,}}+i_{,}\cos\psi}{\sqrt{\mathrm{I}_{,}}}\right]=-\frac{n}{2}\frac{i_{,}\sin\psi\cos\psi}{\sqrt{\mathrm{I}_{,}}}. \qquad (194.)$$

As $n=1-j$, and $i_{,}=\frac{1-j}{1+j}$, $n=\frac{2i_{,}}{1+i_{,}}$, equation (187.) may now be written
$$2\Sigma=\frac{a}{2}\left(\frac{1+j}{\sqrt{j}}\right)\int d\psi\sqrt{\mathrm{I}_{,}}-\frac{ai_{,}^2\sin\psi\cos\psi}{\sqrt{j}(1+i_{,})\sqrt{\mathrm{I}_{,}}}. \qquad (195.)$$

Now as
$$\mathrm{A}=\frac{a}{2}\frac{(2-n)}{\sqrt{1-n}}=\frac{a}{2}\frac{(1+j)}{\sqrt{j}} \text{ and } 1+i_{,}=\frac{2}{1+j},$$

we get ultimately
$$2\Sigma=\mathrm{A}\int d\psi\sqrt{\mathrm{I}_{,}}-\mathrm{A}\frac{i_{,}^2\sin\psi\cos\psi}{\sqrt{1-i_{,}^2\sin^2\psi}}. \qquad (196.)$$

The second term of the last member of this equation is evidently the common expression for a portion of a tangent to a plane ellipse between the point of contact and the foot of a perpendicular on it from the centre; while $\mathrm{A}\int d\psi\sqrt{\mathrm{I}_{,}}$, or $\mathrm{A}\int d\psi\sqrt{1-i_{,}^2\sin^2\psi}$, is the expression for the arc of a plane ellipse whose semitransverse axis is A, and eccentricity $i_{,}$.

When the function is complete, $\varphi=\frac{\pi}{2}$ and $\psi=\pi$. See (183.)

Hence as $\int_0^\pi d\psi\sqrt{\mathrm{I}_{,}}=2\int_0^{\frac{\pi}{2}}d\psi\sqrt{\mathrm{I}_{,}}$,
$$\Sigma=\mathrm{A}\int_0^{\frac{\pi}{2}}d\psi\sqrt{\mathrm{I}_{,}}. \qquad (197.)$$

Σ therefore, in this case, is equal to a quadrant of the plane ellipse whose principal semiaxis A, and eccentricity $i_{\prime}$ are given by the equations

$$A=\sqrt{a^2+h^2}, \text{ and } i_{\prime}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (198.*)$$

To distinguish this variety of the curve, we may call it the *circular logarithmic ellipse*, as it is a section of a circular cylinder. Accordingly, in the two forms of the third order, when the conjugate parameters are equal, or $m=n$, the representative curves of those forms become the spherical parabola, and the circular logarithmic ellipse.

This is Case V. in the Table, p. 316. The results of the preceding investigation will reappear in the demonstration of the theorem, that quadrants of the spherical or logarithmic ellipse may be expressed by the help of integrals of the first and second orders.

XLIV. It is not difficult to show that this particular case of the logarithmic form, when the parameters m and n are equal, represents the curve of intersection of a circular cylinder, by a paraboloid whose principal sections are unequal.

Let
$$x^2+y^2=a^2, \text{ and } \frac{x^2}{k}+\frac{y^2}{k'}=2z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (199.)$$

be the equations of the circular cylinder and of the paraboloid.

Assume
$$x=a\cos\theta, \quad y=a\sin\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (200.)$$

Then
$$2z=a^2\left\{\frac{\cos^2\theta}{k}+\frac{\sin^2\theta}{k'}\right\},$$

and
$$\frac{dx}{d\theta}=-a\sin\theta, \ \frac{dy}{d\theta}=a\cos\theta, \ \frac{dz}{d\theta}=a^2\left(\frac{1}{k'}-\frac{1}{k}\right)\sin\theta\cos\theta. \quad . \quad . \quad . \quad . \quad . \qquad (201.)$$

Hence
$$\frac{d\Sigma}{d\theta}=a\left[1+a^2\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^2\theta\cos^2\theta\right]^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (202.)$$

Now we may reduce this expression by two different methods to the form of an elliptic integral.

By the first method, eliminating $\cos^2\theta$, this expression becomes

$$\frac{d\Sigma^2}{d\theta^2}=a^2+a^4\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^2\theta-a^4\left(\frac{1}{k'}-\frac{1}{k}\right)^2\sin^4\theta. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (203.)$$

We may, as in (124.), reduce this expression to the form of a product of two quadratic factors,

$$(A+B\sin^2\theta)(C-B\sin^2\theta)=AC+B(C-A)\sin^2\theta-B^2\sin^4\theta. \quad . \quad . \quad . \qquad (204.)$$

Comparing this expression with the preceding,

$$AC=a^2, \ B=a^2\left(\frac{1}{k'}-\frac{1}{k}\right), \ C-A=a^2\left(\frac{1}{k'}-\frac{1}{k}\right) \text{ or } C=A+B, \text{ and } AC=A^2+AB=a^2. \quad (205.)$$

* Professor STOKES of Cambridge has pointed out to me, that this curve, like the plane ellipse, when the cylinder is developed on a plane, becomes a curve of sines.

Let us now, as in (126.), assume $\tan^2\theta=\frac{A}{A+B}\tan^2\varphi$; . . . . (206.)

and following the steps there indicated, we shall have

$$\Sigma=A\int\frac{d\varphi\left[1-\frac{B(2A+B)}{(A+B)^2}\sin^2\varphi\right]}{\left[1-\frac{B}{A+B}\sin^2\varphi\right]^2\sqrt{1-\frac{B(2A+B)}{(A+B)^2}\sin^2\varphi}}, \qquad (207.)$$

an expression of the same form as (127.).

Let $\qquad \frac{B}{A+B}=n,\ \frac{B(2A+B)}{(A+B)^2}=i^2$; . . . . (208.)

therefore $\qquad 1-n=\frac{A}{A+B}$, and $1-i^2=\frac{A^2}{(A+B)^2}$

Hence $\qquad 1-n=\sqrt{1-i^2}$, or $n=m$ $\Big\}$ . . . . (209.)

If we develope this integral by the method indicated in XXXVI., the coefficient $\frac{2n-i^2-n^2}{n}$ of the integral $\int\frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-i^2\sin^2\varphi}}$, in the result will be 0, and the reduced integral will become, as

$$\frac{B}{A+B}=n,\ B=\frac{nA}{1-n},\ \text{and } B=a^2\left(\frac{1}{k'}-\frac{1}{k}\right). \qquad (210.)$$

$$\Sigma=\frac{nA}{2(1-n)}\left[\frac{1}{n}\int d\varphi\sqrt{I}+\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}-\Phi\right]. \qquad (211.)$$

Let z' and z'' be the altitudes of the points above the plane of xy, in which the principal sections of the paraboloid meet the circular cylinder. Then $z''-z'$ is the height or thickness of the zone of the cylinder on which the curve is traced.

Now $\qquad a^2=2kz',\ a^2=2k'z''$, whence $z''-z'=\frac{a^2}{2}\left(\frac{1}{k'}-\frac{1}{k}\right)$.

Let this altitude or thickness of the zone be put h, and we shall have

$$\Sigma=h\left[\frac{1}{n}\int d\varphi\sqrt{I}+\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{\sqrt{I}}-\Phi\right]. \qquad (212.)$$

Hence the arc of this species of logarithmic ellipse may be expressed by integrals of the first and second orders.

It is not a little remarkable that whether the integrals of the third order be circular or logarithmic, or, looking to their geometrical origin, spherical or parabolic, when the conjugate parameters are equal, or $m=n$, we may express the arcs of the hyperconic sections thus represented, in terms of integrals of the first and second orders only; the integral of the third order being in this case eliminated.

If we now resume equation (202.) and make

$$2\theta=\frac{\pi}{2}+\psi, \qquad (213.)$$

$\sin 2\theta = 2\sin\theta\cos\theta = \cos\psi$, and $2d\theta = d\psi$. Therefore (202.) will become

$$\frac{4d\Sigma^2}{d\psi^2} = a^2 + \frac{a^4}{4}\left(\frac{1}{k'} - \frac{1}{k}\right)^2 \cos^2\psi, \quad \ldots \ldots \quad (214.)$$

hence as $h = \frac{a^2}{2}\left(\frac{1}{k'} - \frac{1}{k}\right)$, we shall have $2\Sigma = \sqrt{a^2+h^2}\int d\psi\sqrt{1 - \frac{h}{\sqrt{a^2+h^2}}\sin^2\psi}$. . (215.)

This is the common form for the rectification of a plane ellipse, whose principal semi-axes are $\sqrt{a^2+h^2}$ and a. Let $i_{,}$ be the eccentricity of this plane ellipse,

$$i_{,} = \frac{h}{\sqrt{a^2+h^2}} = \frac{B}{2A+B} = \frac{n}{2-n} = \frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}, \quad \ldots \ldots \quad (216.)$$

and the relation between φ and ψ is given by the equations

$$2\theta = \frac{\pi}{2} + \psi, \quad \tan^2\theta = \frac{A}{A+B}\tan^2\varphi, \text{ or } \tan\theta = \sqrt{1-n}\tan\varphi.$$

Hence
$$\frac{1+\sin\psi}{1-\sin\psi} = (1-n)\tan^2\varphi. \quad \ldots \ldots \quad (217.)$$

When $\psi = 0$, $\tan\varphi = \frac{1}{\sqrt{j}}$; when $\psi = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$; when $\psi = -\frac{\pi}{2}$, $\varphi = 0$. Hence ψ is measured from the perpendicular on the tangent to the ellipse, at the point which divides the elliptic quadrant into two segments whose difference is equal to $a-b$, as will be shown further on: while φ is measured from the semitransverse axe a. Thus while ψ varies from $-\frac{\pi}{2}$ (that is from the position at right angles to this perpendicular, and *below* it,) to 0, that is to the perpendicular itself, φ varies from 0 to $\tan^{-1}\frac{1}{\sqrt{j}}$; and while ψ varies from 0 to $\frac{\pi}{2}$, φ varies from $\tan^{-1}\frac{1}{\sqrt{j}}$ to $\frac{\pi}{2}$. Thus while ψ passes over two right angles, φ passes over one right angle.

We may now equate the two expressions (211.) and (215.),

$$\int d\psi\sqrt{1-i_{,}^2\sin^2\psi} = \frac{2h}{\sqrt{a^2+h^2}}\left[\frac{1}{n}\int d\varphi\sqrt{I} + \frac{(1-n)}{n}\int\frac{d\varphi}{\sqrt{I}} - \Phi\right], \quad \ldots \quad (218.)$$

or we may express an elliptic integral of the first order by means of two elliptic integrals of the second order. Thus we obtain the geometrical origin of this well-known theorem.

When the functions are complete, since

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi\sqrt{1-i_{,}^2\sin^2\psi} = 2\int_0^{\frac{\pi}{2}} d\psi\sqrt{1-i_{,}^2\sin^2\psi}, \quad \text{we get}$$

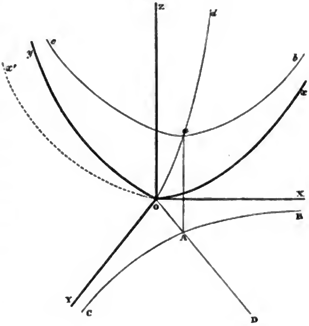
$$\int_0^{\frac{\pi}{2}} d\psi\sqrt{1-i_{,}^2\sin^2\psi} = i_{,}\left[\frac{1}{n}\int_0^{\frac{\pi}{2}} d\varphi\sqrt{I} + \left(\frac{1-n}{n}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right], \quad \ldots \ldots \quad (219.)$$

which agrees with (186.).

SECTION V.—*On the Logarithmic Hyperbola.*

XLV. The Logarithmic hyperbola may be defined as the curve of symmetrical intersection of a paraboloid of revolution with a right cylinder standing on a plane hyperbola as a base.

Fig. 16.



Let $Oxx_{\prime}$ be a paraboloid of revolution, whose vertex is at O, and whose axis is OZ. Let ACB be an hyperbola in the plane of xy, whose vertex is at A, whose asymptots are the lines OX, OY, and whose axis is the right line OAD. Let the planes ZOX, ZOD, ZOY cut the paraboloid in the plane parabolas Ox, Od, Oy, and let cab be the curve on the surface of the paraboloid whose orthogonal projection on the plane of xy is the plane hyperbola ABC. Then acb is the logarithmic hyperbola.

As OX is an asymptot to the hyperbolic arc AB, it is manifest that the parabolic arc Ox is a curvilinear asymptot to the arc ab of the logarithmic hyperbola.

Let
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ and } x^2+y^2=2kz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220.)$$
be the equations of the hyperbolic cylinder and of the paraboloid of revolution, and consequently of the curve in which they intersect. Let Υ be an arc of this curve,

then
$$\Upsilon=\int d\lambda\left[\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2+\left(\frac{dz}{d\lambda}\right)^2\right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221.)$$
x, y, z being functions of a fourth independent variable λ.

Assume
$$x^2=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda},\ y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}*. \quad . \quad . \quad . \quad (222.)$$
It is manifest that these assumptions are compatible with the first of equation (220.), and the second of that group gives
$$x^2+y^2=\frac{a^4\cos^2\lambda+b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}=2kz.$$

---

* We might, by the help of the imaginary transformation $\sin\theta=\sqrt{-1}\tan\theta'$, pass at once from the elliptic cylinder to the hyperbolic cylinder. Let $\tan\theta'=u$, and the resulting equation will be of the form
$$\frac{d\Upsilon}{du}=\frac{\alpha+\beta u^2+\gamma u^4}{\sqrt{A+Bu^2+Cu^4+Du^6}},$$
an expression which, on trial, it would be found very difficult to reduce. The difficulty is eluded by making the transformation pointed out and adopted in the text.

Differentiating (222.), we get

$$\left(\frac{dx}{d\lambda}\right)^2=\frac{a^4b^4\sin^2\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^3},\quad \left(\frac{dy}{d\lambda}\right)^2=\frac{a^4b^4\cos^2\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^3},\quad \left(\frac{dz}{d\lambda}\right)^2=\frac{(a^2+b^2)^2a^4b^4\sin^2\lambda\cos^2\lambda}{k^2(a^2\cos^2\lambda-b^2\sin^2\lambda)^4}. \quad (223.)$$

Hence $$\frac{k}{a^2b^2}\frac{d\Upsilon}{d\lambda}=\frac{[a^2k^2+(a^2+b^2)(a^2+b^2-k^2)\sin^2\lambda-(a^2+b^2)^2\sin^4\lambda]^{\frac{1}{2}}}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^2}. \quad (224.)$$

Let this radical be put$=\sqrt{V}$.

Assume $V=(A+B\sin^2\lambda)(C-B\sin^2\lambda)=AC+B(C-A)\sin^2\lambda-B^2\sin^4\lambda$, . . (225.)

hence $AC=a^2k^2,\quad B=a^2+b^2,\quad C-A=a^2+b^2-k^2$. . . . . (226.)

Let us now assume $\sin\varphi$ such, that

$$\sin^2\lambda=\frac{AC\sin^2\varphi}{AB+BC\cos^2\varphi}, \quad (227.)$$

then $$A+B\sin^2\lambda=\frac{A(A+C)}{A+C\cos^2\varphi},\quad C-B\sin^2\lambda=\frac{C(A+C)\cos^2\varphi}{A+C\cos^2\varphi},$$

and $$a^2\cos^2\lambda-b^2\sin^2\lambda=a^2-\frac{(a^2+b^2)AC\sin^2\varphi}{B(A+C\cos^2\varphi)};$$

or as $$a^2+b^2=B,\quad AC=a^2k^2,\quad C+k^2=A+B,$$

we get $$a^2\cos^2\lambda-b^2\sin^2\lambda=\frac{a^2(A+C)}{A+C\cos^2\varphi}\left[1-\frac{A+B}{A+C}\sin^2\varphi\right].$$

Hence $$\frac{k}{a^2b^2}\frac{d\Upsilon}{d\lambda}=\frac{\sqrt{AC}.[A+C\cos^2\varphi]\cos\varphi}{a^4(A+C)[1-l\sin^2\varphi]^2}. \quad (228.)$$

Making $$l=\frac{A+B}{A+C}. \quad (229.)$$

Differentiating the equation $\sin^2\lambda=\dfrac{AC\sin^2\varphi}{AB+BC\cos^2\varphi}$, ... (230.)

we get $$\frac{d\lambda}{d\varphi}=\frac{ak\sqrt{A+C}\cos\varphi}{\sqrt{B}[A+C\cos^2\varphi]\sqrt{1-\frac{C}{B}\left(\frac{A+B}{A+C}\right)\sin^2\varphi}}, \quad (231.)$$

or as $$\frac{d\Upsilon}{d\varphi}=\frac{d\Upsilon}{d\lambda}\frac{d\lambda}{d\varphi},\quad \text{making } i^2=\frac{C(A+B)}{B(A+C)}, \quad (232.)$$

we get, finally, $$\frac{\Upsilon}{k}=\frac{b^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}. \quad (233.)$$

XLVI. We may develope another formula for the rectification of an arc of the logarithmic hyperbola.

Assuming the principles established in Sect. XXXVIII., we may put

$$\Upsilon=-\int p\sec\upsilon d\lambda-\int\frac{d^2p}{d\lambda^2}\sec\upsilon d\lambda. \quad (234.)$$

In this formula p is the perpendicular from the axis of the hyperbolic cylinder let fall on a tangent plane to it, passing through the element of the curve; and υ is the

angle which a tangent to this element makes with the plane of the base. v in this equation is analogous to τ in the last section.

In the above expression, the negative sign is used as the curve is *convex* towards the origin.

Now $p^2=a^2\cos^2\lambda-b^2\sin^2\lambda$, and $\tan v=\dfrac{\frac{dz}{d\lambda}}{\sqrt{\left(\frac{dx}{d\lambda}\right)^2+\left(\frac{dy}{d\lambda}\right)^2}}$. We must substitute for these differentials, their values given in (223.), and introduce the value of φ assumed in (227.), whence

$$\sec^2 v=\frac{(A+C)^2AC\cos^2\varphi}{k^2[A+C\cos^2\varphi]^2(a^2\cos^2\lambda-b^2\sin^2\lambda)}; \quad (235.)$$

$$\therefore\ p\sec v=\frac{\sqrt{AC}(A+C)\cos\varphi}{k[A+C\cos^2\varphi]}. \quad (236.)$$

But (231.) gives

$$\frac{d\lambda}{d\varphi}=\frac{\sqrt{A+C}.ak\cos\varphi}{\sqrt{B}[A+C\cos^2\varphi]\sqrt{1-i^2\sin^2\varphi}},$$

whence

$$p\sec v d\lambda=\frac{a^2k\cos^2\varphi d\varphi}{\sqrt{B(A+C)}\left[1-\frac{C}{A+C}\sin^2\varphi\right]^2\sqrt{1-i^2\sin^2\varphi}}. \quad (237.)$$

We must now determine the value of the second integral in (234.), namely,

$$\int\frac{d^2p}{d\lambda^2}\sec v d\lambda,$$

since $p^2=a^2\cos^2\lambda-b^2\sin^2\lambda$, $\quad \dfrac{d^2p}{d\lambda^2}\sec v d\lambda=-\dfrac{(a^2+b^2)[a^2\cos^4\lambda+b^2\sin^4\lambda]\sec v d\lambda}{(a^2\cos^2\lambda-b^2\sin^2)^{\frac{3}{2}}}$. $\quad$ (238.)

Now we may derive from (223.) $\tan v=\dfrac{(a^2+b^2)\sin\lambda\cos\lambda}{k(a^2\cos^2\lambda-b^2\sin^2\lambda)^{\frac{1}{2}}}$. $\quad$ (239.)

Differentiating this expression, then multiplying by $\sec v$, and integrating, we obtain

$$k\int\frac{dv}{\cos^3 v}=(a^2+b^2)\int\frac{[a^2\cos^4\lambda+b^2\sin^4\lambda]\sec v d\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^{\frac{3}{2}}}. \quad (240.)$$

Comparing this expression with (238.), and introducing into (234.) the values found in (237.) and (240.), we obtain

$$\frac{T}{k}=\int\frac{dv}{\cos^3 v}-\frac{a^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\varphi d\varphi}{[1-m\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}, \quad (241.)$$

making

$$m=\frac{C}{A+C}, \quad (242.)$$

since

$$l=\frac{A+B}{A+C},\ \text{and}\ i^2=\frac{C}{B}\left(\frac{A+B}{A+C}\right),\ \text{assume}\ n=\frac{i^2}{l}=\frac{C}{B}. \quad (243.)$$

and we shall have m and n connected by the equation of condition, defined in (1.),

$$m+n-mn=i^2.$$

The three parameters l, m, n, and the modulus i are connected by the equations

$$ln=i^2,\quad m+n-mn=i^2. \quad (244.)$$

l and n are *reciprocal* parameters, the reader will recollect, while m and n are *conjugate* parameters.

XLVII. It was shown in (226.), that $C-A=a^2+b^2-k^2$, $B=a^2+b^2$, $k^2=A+B-C$, and $a^2k^2=AC$, whence

$$\frac{a^2}{k^2}=\frac{AC}{(A+B-C)^2}, \quad \frac{b^2}{k^2}=\frac{(A+B)(B-C)}{(A+B-C)^2}. \quad . \quad . \quad . \quad . \quad (245.)$$

In order that these values of a and b may be real, we must have $B>C$, and A of the same sign with C, both positive, otherwise $\sqrt{V}$ in (225.) will be imaginary. As $l=\frac{A+B}{A+C}$, $l>1$; here the parameter l is greater than 1, while m and n are each less than 1.

We may express the semiaxes of the hyperbola, the base of the hyperbolic cylinder, in terms of the modulus i and the parameter l; for by the equations immediately preceding we may eliminate A, B and C in (243.). We thus find

$$\frac{a^2}{k^2}=\frac{li^2(l-1)(1-i^2)}{[l^2+i^2-2li^2]^2}, \quad \frac{b^2}{k^2}=\frac{l(l-1)(l-i^2)^2}{[l^2+i^2-2li^2]^2}; \quad . \quad . \quad . \quad . \quad (246.)$$

therefore

$$\frac{B}{k^2}=\frac{a^2+b^2}{k^2}=\frac{l(l-1)}{l^2+i^2-2li^2}. \quad . \quad . \quad . \quad . \quad (247.)$$

We may express the semiaxes in terms of the conjugate parameters m and n,

$$\frac{a^2}{k^2}=\frac{n^2m(1-m)}{[n+m-2mn]^2}, \quad \frac{b^2}{k^2}=\frac{m(1-n)(n+m-mn)}{(n+m-2mn)^2}; \quad . \quad . \quad . \quad . \quad (248.)$$

hence

$$\frac{B}{k^2}=\frac{a^2+b^2}{k^2}=\frac{m}{(m+n-2mn)} \text{ and } \sqrt{B(A+C)}=\frac{\sqrt{mn}}{(m+n-2mn)}; \quad . \quad . \quad . \quad . \quad (249.)$$

or we may express a and b more simply in terms of l and m. Eliminating n and i^2, we get

$$\frac{a^2}{k^2}=\frac{m(1-m)}{(l-m)^2}, \quad \frac{b^2}{k^2}=\frac{l(l-1)}{(l-m)^2}. \quad . \quad . \quad . \quad . \quad (250.)$$

Let c_i be the eccentricity of the hyperbolic base of the cylinder, we shall easily discover the following equation between c_i, i and l, analogous to (131.),

$$(c_i^2-1)i^2j^2=(l-i^2)^2. \quad . \quad . \quad . \quad . \quad (251.)$$

Hence when i and l are given, c_i may easily be found.

XLVIII. If we equate together the values found for Υ, the arc of the logarithmic hyperbola, in (233.) and (241.), we shall have

$$b^2\int\frac{\cos^2\phi d\phi}{[1-l\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}+a^2\int\frac{\cos^2\phi d\phi}{[1-m\sin^2\phi]^2\sqrt{1-i^2\sin^2\phi}}=\sqrt{B(A+C)}\int\frac{dv}{\cos^3 v}. \quad . \quad (252.)$$

For brevity, put

$$L=1-l\sin^2\phi, \; M=1-m\sin^2\phi, \; N=1-n\sin^2\phi, \; I=1-i^2\sin^2\phi. \quad . \quad . \quad (253.)$$

The preceding equation may now be written

$$b^2\int\frac{\cos^2\phi d\phi}{L^2\sqrt{I}}+a^2\int\frac{\cos^2\phi d\phi}{M^2\sqrt{I}}=\sqrt{B(A+C)}\int\frac{dv}{\cos^3 v}; \quad . \quad . \quad . \quad . \quad (254.)$$

or, if we substitute for the coefficients of this equation their values given in (246.), we shall have

$$(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=\frac{[l^2+i^2-2li^2]\sqrt{l-i^2}}{\sqrt{l(l-1)}}\int\frac{dv}{\cos^3 v}. \quad . \quad . \quad . \quad (255.)$$

Let $\delta=l^2+i^2-2li^2$, $\delta'=m^2+i^2-2mi^2$, $\Phi_l=\frac{\sin\varphi\cos\varphi\sqrt{I}}{L}$, $\Phi_m=\frac{\sin\varphi\cos\varphi\sqrt{I}}{M}$. (256.)

Now the process given in XXXVI. will enable us to develope the integrals

$$\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}} \text{ and } \int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}, \text{ as follows:—}$$

$$2(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}=l(l-i^2)\Phi_l-(l-i^2)\int d\varphi\sqrt{I}+\frac{(l-i^2)^2}{l}\int\frac{d\varphi}{\sqrt{I}}+\frac{\delta}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}, \quad . \quad . \quad (257.)$$

and

$$2i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=-\frac{m(1-i^2)i^2}{(i^2-m)}\Phi_m+\frac{i^2(1-i^2)}{i^2-m}\int d\varphi\sqrt{I}+\frac{i^2(1-i^2)}{m}\int\frac{d\varphi}{\sqrt{I}}-\frac{\delta' i^2(1-i^2)}{m(i^2-m)}\int\frac{d\varphi}{M\sqrt{I}}. \; (258.)$$

The equations of condition $ln=i^2$, and $m+n-mn=i^2$, give

$$\frac{i^2(1-i^2)}{i^2-m}=l-i^2, \text{ and } \frac{(l-i^2)^2}{l}+\frac{i^2(1-i^2)}{m}=\frac{(l-i^2)\delta}{l(l-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (259.)$$

We have also, since $m=\frac{i^2(l-1)}{l-i^2}$, $l(l-i^2)\Phi_l-\frac{mi^2(1-i^2)}{(i^2-m)}\Phi_m=\frac{\delta\sin\varphi\cos\varphi\sqrt{I}}{LM}$. . (260.)

Making these substitutions, adding together (257.) and (258.), the coefficient of $\int d\varphi\sqrt{I}$ vanishes, and we shall have

$$2(l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+2i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=\frac{\delta\sin\varphi\cos\varphi\sqrt{I}}{LM}+\frac{\delta(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}+\frac{\delta}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}-\frac{\delta'(l-i^2)}{m}\int\frac{d\varphi}{M\sqrt{I}},$$

but (255.) gives $\quad (l-i^2)^2\int\frac{\cos^2\varphi d\varphi}{L^2\sqrt{I}}+i^2(1-i^2)\int\frac{\cos^2\varphi d\varphi}{M^2\sqrt{I}}=\delta\sqrt{\frac{(l-i^2)}{l(l-1)}}\int\frac{dv}{\cos^3 v}.$

Combining this equation with the preceding,

$$\frac{\delta}{l}(l-i^2)\int\frac{d\varphi}{L\sqrt{I}}-\frac{\delta'(l-i^2)}{m}\int\frac{d\varphi}{M\sqrt{I}}+\frac{\delta(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}+\frac{\delta\sin\varphi\cos\varphi\sqrt{I}}{LM}=2\delta\sqrt{\frac{l-1}{l(l-1)}}\int\frac{dv}{\cos^3 v}. \quad (261.)$$

Now $\quad \delta'=m^2+i^2-2mi^2=\frac{i^2(1-i^2)\delta}{(l-i^2)^2}$, and as $m=\frac{i^2(l-1)}{l-i^2}$, $\frac{\delta'(l-i^2)}{m}=\frac{(1-i^2)\delta}{(l-1)}$.

In the last equation, substituting this value of δ', and then dividing by δ, we get

$$\frac{\sin\varphi\cos\varphi\sqrt{I}}{LM}+\frac{(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{I}}+\frac{(l-i^2)}{l}\int\frac{d\varphi}{L\sqrt{I}}-\frac{(1-i^2)}{(l-1)}\int\frac{d\varphi}{M\sqrt{I}}=2\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{dv}{\cos^3 v}. \quad . \quad . \quad (262.)$$

Now $\quad 2\int\frac{dv}{\cos^3 v}=\tan v\sec v+\int\frac{dv}{\cos v}$ and $\cos^2 v=\frac{LM}{\cos^2\varphi}$, ... (263.)

as may be shown by combining (226.) with (235.).

Hence $\quad \sin v=\sqrt{\frac{l(l-1)}{l-i^2}}\tan\varphi\sqrt{I}$, ... (264.)

and therefore $\tan\upsilon\sec\upsilon=\sqrt{\frac{l(l-1)}{l-i^2}}\frac{\sin\varphi\cos\varphi\sqrt{\text{I}}}{\text{LM}}$. . . . . (265.)

Substituting this value in the preceding equation, we get

$$\left(\frac{l-i^2}{l}\right)\int\frac{d\varphi}{\text{L}\sqrt{\text{I}}}-\frac{(1-i^2)}{(l-1)}\int\frac{d\varphi}{\text{M}\sqrt{\text{I}}}+\frac{(l-i^2)}{l(l-1)}\int\frac{d\varphi}{\sqrt{\text{I}}}=\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{d\upsilon}{\cos\upsilon}. \quad (266.)$$

In (170.) we showed that, m and n being conjugate parameters connected by the equation $m+n-mn=i^2$,

$$\left(\frac{1-n}{n}\right)\int\frac{d\varphi}{\text{N}\sqrt{\text{I}}}+\frac{(1-m)}{m}\int\frac{d\varphi}{\text{M}\sqrt{\text{I}}}=\frac{i^2}{mn}\int\frac{d\varphi}{\sqrt{\text{I}}}-\frac{1}{\sqrt{mn}}\int\frac{d\tau}{\cos\tau}.$$

Now $\left(\frac{1-n}{n}\right)=\frac{l}{i^2}\left(\frac{l-i^2}{l}\right)$, $\left(\frac{1-m}{m}\right)=\frac{l}{i^2}\left(\frac{1-i^2}{l-1}\right)$, $\frac{i^2}{mn}=\frac{l}{i^2}\left(\frac{l-i^2}{l-1}\right)$ and $\frac{1}{\sqrt{mn}}=\frac{l}{i^2}\sqrt{\frac{l-i^2}{l(l-1)}}$.

Substituting these values in the preceding equation, and dividing by $\frac{l}{i^2}$, we get

$$\left(\frac{l-i^2}{l}\right)\int\frac{d\varphi}{\text{N}\sqrt{\text{I}}}+\left(\frac{1-i^2}{l-1}\right)\int\frac{d\varphi}{\text{M}\sqrt{\text{I}}}=\left(\frac{l-i^2}{l-1}\right)\int\frac{d\varphi}{\sqrt{\text{I}}}-\sqrt{\frac{l-1}{l(l-1)}}\int\frac{d\tau}{\cos\tau}. \quad (267.)$$

If we add this equation to (266.), the coefficient of the integral $\int\frac{d\varphi}{\text{M}\sqrt{\text{I}}}$ will vanish, and the resulting equation will become

$$\int\frac{d\varphi}{\text{L}\sqrt{\text{I}}}+\int\frac{d\varphi}{\text{N}\sqrt{\text{I}}}=\int\frac{d\varphi}{\sqrt{\text{I}}}+\frac{\sqrt{l}}{\sqrt{(l-1)(l-i^2)}}\left[\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}\right]. \quad (268.)$$

We shall now proceed to show that $\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}$ may be put under the form $\int\frac{d\upsilon'}{\cos\upsilon'}$, if we make the assumption $\sin\upsilon'=\frac{\sqrt{x'}\tan\varphi}{\sqrt{\text{I}}}$, ... (269.)

x' being equal to $(1-n)\left(\frac{i^2}{n}-1\right)=\frac{(l-i^2)(l-1)}{l}$.

Now $$\cos^2\upsilon=\frac{(1-m\sin^2\varphi)(1-l\sin^2\varphi)}{\cos^2\varphi}, \text{ as in (263.).}$$

Hence $$\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{d\upsilon}{\cos\upsilon}=\int\frac{d\varphi}{\sqrt{\text{I}}}\left[\frac{l[1-i^2\sin^2\varphi-i^2\sin^2\varphi\cos^2\varphi]}{\text{LM}}\right]; \quad (270.)$$

but we derive from (165.) and (166.) the value

$$\sqrt{\frac{l-i^2}{l(l-1)}}\int\frac{d\tau}{\cos\tau}=\int\frac{d\varphi}{\sqrt{\text{I}}}\frac{[n\cos^2\varphi-n\sin^2\varphi+ni^2\sin^4\varphi]}{\text{MN}}, \quad (271.)$$

or subtracting,

$$\sqrt{\frac{l-i^2}{l(l-1)}}\left[\int\frac{d\upsilon}{\cos\upsilon}-\int\frac{d\tau}{\cos\tau}\right]=\int\frac{\sqrt{\text{I}}}{\text{M}}\left[\frac{1}{\text{L}}+\frac{n\sin^2\varphi}{\text{N}}\right]d\varphi-\int\frac{\cos^2\varphi}{\text{M}\sqrt{\text{I}}}\left[\frac{i^2\sin^2\varphi}{\text{L}}+\frac{n}{\text{N}}\right]d\varphi. \quad (272.)$$

These two latter integrals may be combined into the single integral,

$$\int\frac{[1-i^2\sin^2\varphi-n\cos^2\varphi][1-i^2\sin^4\varphi]d\varphi}{\text{LMN}\sqrt{\text{I}}}. \quad (273.)$$

Now as $m+n-mn=i^2$, the first factor of the numerator becomes $(1-n)(1-m\sin^2\varphi)$ $=(1-n)\mathrm{M}$, and therefore

$$\sqrt{\frac{l-i^2}{l(l-1)}}\left[\int\frac{dv}{\cos v}-\int\frac{d\tau}{\cos\tau}\right]=\left(\frac{l-i^2}{l}\right)\int\frac{[1-i^2\sin^4\varphi]}{\mathrm{LN}\sqrt{\mathrm{I}}}. \quad \ldots \quad (274.)$$

Substituting the second member of this equation for the last in (268.), we find

$$\int\frac{d\varphi}{\mathrm{L}\sqrt{\mathrm{I}}}+\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}-\int\frac{d\varphi}{\sqrt{\mathrm{I}}}=\int\frac{[1-i^2\sin^4\varphi]}{\mathrm{LN}\sqrt{\mathrm{I}}}. \quad \ldots \quad (275.)$$

Now, since we have assumed in (269.)

$$\sin v'=\frac{\sqrt{\varkappa'}\tan\varphi}{\sqrt{\mathrm{I}}}, \quad \cos^2 v'=\frac{\mathrm{LN}}{\mathrm{I}\cos^2\varphi}, \quad \text{hence } \frac{dv'}{\cos v'}=\frac{\sqrt{\varkappa'}[1-i^2\sin^4\varphi]d\varphi}{\mathrm{LN}\sqrt{\mathrm{I}}}; \quad \ldots \quad (276.)$$

and consequently
$$\int\frac{d\varphi}{\mathrm{L}\sqrt{\mathrm{I}}}+\int\frac{d\varphi}{\mathrm{N}\sqrt{\mathrm{I}}}=\int\frac{d\varphi}{\sqrt{\mathrm{I}}}+\frac{1}{\sqrt{\varkappa'}}\int\frac{dv'}{\cos v'}. \quad \ldots \quad (277.)$$

This formula is usually written

$$\int\frac{d\varphi}{[1-n\sin^2\varphi]\sqrt{1-c^2\sin^2\varphi}}+\int\frac{d\varphi}{\left[1-\frac{c^2}{n}\sin^2\varphi\right]\sqrt{1-c^2\sin^2\varphi}}=\mathrm{F}_c(\varphi)+\frac{1}{\sqrt{\alpha}}\int\frac{\frac{d}{d\varphi}\left(\frac{\sqrt{\alpha}\tan\varphi}{\Delta}\right)d\varphi}{1-\left(\frac{\sqrt{\alpha}\tan\varphi}{\Delta}\right)^2}. \quad (278.)$$

We have thus shown that from the comparison of two expressions for the same arc of the logarithmic hyperbola, we may derive the well-known equation which connects two elliptic integrals of the third order, and of the logarithmic form, whose parameters are reciprocal*.

Hence also it follows that if v, τ, and v' are three normal angles, which normals to a parabola make with the axis, and if their angles are connected by the equations

$$\left.\begin{aligned}\cos^2 v &= \frac{\mathrm{ML}}{\cos^2\varphi}, & \sin v &= \sqrt{\frac{m}{n}}\tan\varphi\sqrt{\mathrm{I}},\\ \cos^2\tau &= \frac{\mathrm{MN}}{\mathrm{I}}, & \sin\tau &= \frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{\mathrm{I}}},\\ \cos^2 v' &= \frac{\mathrm{LN}}{\mathrm{I}\cos^2\varphi}, & \sin v' &= \sqrt{\frac{m}{n}}(1-n)\frac{\tan\varphi}{\sqrt{\mathrm{I}}},\end{aligned}\right\} \quad \ldots \quad (279.)$$

we shall have

$$\int\frac{dv}{\cos v}=\int\frac{dv'}{\cos v'}+\int\frac{d\tau}{\cos\tau}. \quad \ldots \quad (280.)$$

* We might by the aid of the imaginary transformation $\sin\varphi=\sqrt{-1}\tan\psi$ have passed from this theorem, connecting integrals with reciprocal parameters, to the corresponding theorem in the circular form. It seems better to give an independent proof of this theorem by the method of differentiating under the sign of integration, as we shall do further on. Although these theorems have algebraically the same form, their geometrical significations are entirely different. In the logarithmic form, the theorem results from the comparison of two expressions for the *same* arc of the logarithmic hyperbola. But in the circular form, the theorem represents the sum of the arcs of two different spherical conic sections described on the same cylinder by two concentric spheres, or on the same sphere by two cylinders having their axes coincident.

Section VI.

XLIX. *The difference between an arc of a logarithmic hyperbola, and the corresponding arc of the tangent parabola, may be expressed by the arcs of a plane, a spherical and a logarithmic ellipse.*

Resuming the equation (241.), $\int\frac{d\upsilon}{\cos^3\upsilon}-\frac{\Upsilon}{k}=\frac{a^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\phi d\phi}{M^2\sqrt{I}}$,

and combining (248.) with (249.), we may easily show that

$$\frac{a^2}{\sqrt{B(A+C)}}=\frac{n(1-m)\sqrt{mn}}{m+n-2nm}; \quad \dots \dots \quad (281.)$$

and from (258.) we deduce that

$$2n(1-m)\int\frac{\cos^2\phi d\phi}{M^2\sqrt{I}}=\frac{n}{m}(1-m)\int\frac{d\phi}{\sqrt{I}}+\int d\phi\sqrt{I}-m\Phi_m-\left(\frac{1-m}{m}\right)(m+n-2mn)\int\frac{d\phi}{M\sqrt{I}}.$$

Let $$G=\frac{n}{m}(1-m)\int\frac{d\phi}{\sqrt{I}}+\int d\phi\sqrt{I}-m\Phi_m. \quad \dots \dots \quad (282.)$$

Substituting this value of $\int\frac{\cos^2\phi d\phi}{M^2\sqrt{I}}$ in the preceding equation we get, after some obvious reductions,

$$2\int\frac{d\upsilon}{\cos^3\upsilon}-\frac{2\Upsilon}{k}=\frac{\sqrt{mn}}{m+n-2mn}G-\frac{n(1-m)}{\sqrt{mn}}\int\frac{d\phi}{M\sqrt{I}}.$$

Now $a_{\prime}$ and $b_{\prime}$ being the semiaxes of the base of an elliptic cylinder whose curve of section with the paraboloid is a logarithmic ellipse, let, as in (171.),

$$\frac{a_{\prime}^2}{k^2}=\frac{mn(1-m)}{(n-m)^2}, \quad \frac{b_{\prime}^2}{k^2}=\frac{mn(1-n)}{(n-m)^2}; \quad \dots \dots \quad (283.)$$

and if we put Σ for an arc of this logarithmic ellipse, we shall have, as in (163.),

$$\frac{2\Sigma}{k}=\frac{\sqrt{mn}}{n-m}G-\frac{n(1-m)}{\sqrt{mn}}\int\frac{d\phi}{M\sqrt{I}}-2\int\frac{d\tau}{\cos^3\tau}.$$

Subtracting this equation from the preceding, and replacing G by its value in (282.), we finally obtain

$$\Upsilon=k\int\frac{d\upsilon}{\cos^3\upsilon}-k\int\frac{d\tau}{\cos^3\tau}-\Sigma+\frac{\sqrt{mn}(1-n)mk}{(n-m)(m+n-2nm)}G. \quad \dots \quad (284.)$$

We may express the arc Υ by the help of one parabolic arc only, if we introduce the equation established in (160.), $\quad S=\Sigma+k\int\frac{d\tau}{\cos^3\tau}$, $\quad$ hence

$$\Upsilon=k\int\frac{d\upsilon}{\cos^3\upsilon}-S+\frac{\sqrt{mn}(1-n)mk}{(n-m)(n+m-2mn)}\left[\frac{n}{m}(1-m)\int\frac{d\phi}{\sqrt{I}}+\int d\phi\sqrt{I}-m\Phi\right]. \quad . \quad (285.)$$

When $\sin\phi=\frac{1}{\sqrt{l}}$, $\upsilon=\frac{\pi}{2}$, and the arc of the logarithmic hyperbola becomes infinite, the arc of the parabola also becomes infinite, and an asymptot to the logarithmic hyper-

bola; the difference, however, between these infinite quantities is finite, and equal to $\frac{\sqrt{mn}(1-n)mk}{(n-m)(n+m-2mn)}G-S$, integrated between the limits $\varphi=0$, and $\varphi=\sin^{-1}l^{-\frac{1}{2}}$.

It is needless here to dwell on the analogy which this property bears to the finite difference between the infinite arc of the common hyperbola and its asymptot. When $n=m$, the above expression becomes illusory. We shall, however, in the next article find a remarkable value for the arc of the logarithmic hyperbola, when $m=n$.

We may express the above formula somewhat more simply.

As in (248.) $\frac{b}{k}=\frac{i\sqrt{m(1-n)}}{m+n-2mn}$, and $\frac{b_{\prime}}{k}=\frac{\sqrt{mn(1-n)}}{n-m}$ $\frac{bb_{\prime}}{k^2}=\frac{i}{\sqrt{m}}\frac{\sqrt{mn}(1-n)m}{(n-m)(n+m-2mn)}$.

The equation given in (285.) now becomes

$$\Upsilon=k\int\frac{dv}{\cos^3 v}-S+\frac{\sqrt{m}}{i}\frac{bb_{\prime}}{k}G. \quad \quad (286.)$$

The ratio between the axes of the original hyperbolic cylinder, and of the derived elliptic cylinder, may easily be determined; for

$$\frac{b^2}{a^2}=\frac{i^2(1-m)}{n^2(1-n)}, \text{ (a.)} \qquad \text{and } \frac{b_{\prime}^2}{a_{\prime}^2}=\frac{1-m}{1-n}, \text{ (b.)}.$$

Let $c_{\prime}$ be the eccentricity of the hyperbolic base, and c that of the elliptic base, then

$$n^2(c_{\prime}^2-1)=i^2(1-c^2).$$

Comparing (a.) with (b.), $\sqrt{n}\frac{a_{\prime}}{a}=\sqrt{l}\frac{b_{\prime}}{b}=1+\frac{2m(1-n)}{(n-m)}$.

This equation gives at once the ratio between the axes of the hyperbolic and elliptic cylinders.

When the paraboloid becomes a plane, or when its parameter is infinite, $m=0$, S becomes an arc of a plane ellipse, $k\int\frac{dv}{\cos^3 v}$ is changed into a rectilinear asymptot, and the expression in (286.) is now transformed into $k\int\frac{dv}{\cos^3 v}-\Upsilon=S$; or the difference between the infinite branch of an hyperbola and its asymptot may be represented by an arc of a plane ellipse.

L. *On the rectification of the logarithmic hyperbola when the conjugate parameters are equal*, or $m=n$.

We have shown in XLII. that when $m=n$, the arc of the logarithmic ellipse is equivalent to an arc of a plane ellipse; so when $m=n$, the arc of a logarithmic hyperbola may be represented by an arc of a parabola, and an arc of a plane hyperbola.

In (262.), if we make $m=n$, or $l=1+j$, $n=1-j$, we shall have, writing N for M,

$$2j\int\frac{d\varphi}{L\sqrt{I}}-2j\int\frac{d\varphi}{N\sqrt{I}}=-\frac{2\sin\varphi\cos\varphi\sqrt{I}}{LN}-2\int\frac{d\varphi}{\sqrt{I}}+4\int\frac{dv}{\cos^3 v}; \quad \quad (a.)$$

and in (170.), if we make $m=n$, and $M=N$,

$$2(1-n)\int\frac{d\varphi}{N\sqrt{I}}=(2-n)\int\frac{d\varphi}{\sqrt{I}}-2\int\frac{d\tau}{\cos^3\tau}+\frac{n\sin\varphi\cos\varphi\sqrt{I}}{N^2}. \quad \quad (b.)$$

Adding these equations together, as $1-n=j$, we get

$$2j\int\frac{d\phi}{L\sqrt{I}}=-(1-j)\int\frac{d\phi}{\sqrt{I}}+4\int\frac{dv}{\cos^3 v}-2\int\frac{d\tau}{\cos^3\tau}+\frac{\sin\phi\cos\phi\sqrt{I}}{N}\left[\frac{n}{N}-\frac{2}{L}\right]. \quad \text{(c.)}$$

Now the arc of the logarithmic hyperbola, as in (233.), is

$$\frac{T}{k}=\frac{b^2}{\sqrt{B(A+C)}}\int\frac{\cos^2\phi\, d\phi}{L^2\sqrt{I}}. \quad \text{(d.)}$$

In this case, the coefficient $\frac{b^2}{\sqrt{B(A+C)}}=\frac{l}{2}$, as may be shown by putting in the general value for this expression, given in (249.), $m=n$; hence

$$\frac{2T}{k}=l\int\frac{\cos^2\phi\, d\phi}{L^2\sqrt{I}}. \quad \text{(e.)}$$

Now (257.) gives $\quad 2(l-i^2)\int\frac{\cos^2\phi\, d\phi}{L^2\sqrt{I}}=l\Phi_i-\int d\phi\sqrt{I}+\left(\frac{l-i^2}{l}\right)\int\frac{d\phi}{\sqrt{I}}+\frac{\delta}{l}\int\frac{d\phi}{L\sqrt{I}}; \quad \text{(f.)}$

and the general value of δ being $l^2+i^2-2li^2$, as in (256.), $\delta=2l(1-n)^2$, $l=2-n$, and $l-i^2=l(1-n)$, since $ln=i^2$.

The last equation may now be written, combining (e.) with it,

$$\frac{4T}{k}=\frac{l}{1-n}\Phi_i-\frac{1}{1-n}\int d\phi\sqrt{I}+\int\frac{d\phi}{\sqrt{I}}+2j\int\frac{d\phi}{L\sqrt{I}}. \quad \text{(287.)}$$

Adding this equation to (c.),

$$\frac{4T}{k}=4\int\frac{dv}{\cos^3 v}-2\int\frac{d\tau}{\cos^3\tau}+j\int\frac{d\phi}{\sqrt{I}}-\frac{1}{j}\int d\phi\sqrt{I}+\frac{l}{j}\Phi_i+\frac{\sin\phi\cos\phi}{N}\left[\frac{n}{N}-\frac{2}{L}\right]. \quad \text{(288.)}$$

Now $\qquad \frac{l\Phi_i}{j}=\frac{(1+j)\sin\phi\cos\phi\sqrt{I}}{jL}=\frac{\tan\phi\sqrt{I}}{j}+\frac{\tan\phi\sqrt{I}}{L}.$

Combining this value of Φ_i with the preceding equation, we get

$$\frac{4T}{k}=4\int\frac{dv}{\cos^3 v}-2\int\frac{d\tau}{\cos^3\tau}+\frac{1}{j}\left[\tan\phi\sqrt{I}-\int d\phi\sqrt{I}+j^2\int\frac{d\phi}{\sqrt{I}}\right]+\tan\phi\sqrt{I}\left[\frac{n\cos^2\phi}{N^2}-\frac{2\cos^2\phi}{LN}+\frac{1}{L}\right]; \quad \text{(289.)}$$

and this latter term, in this case, may be reduced to $-\frac{j\tan\phi\sqrt{I}}{N^2}$.

But, a and b being the semiaxes of the hyperbolic cylinder, (248.) gives $\frac{ab}{k^2}=\frac{mnij}{(m+n-2mn)^2}$, or in this case, as $m=n$, $\quad \frac{2\sqrt{ab}}{\sqrt{ij}}=\frac{k}{j}$.

Now $\sqrt{\frac{ab}{ij}}$ is the distance from the centre to the focus of an hyperbola, the squares of whose semiaxes are $\frac{i}{j}ab$ and $\frac{j}{i}ab$, hence

$$\frac{k}{2j}\left[\tan\phi\sqrt{I}-\int d\phi\sqrt{I}+j^2\int\frac{d\phi}{\sqrt{I}}\right]$$

represents an arc of an hyperbola the squares of whose semiaxes are $\frac{i}{j}ab$ and $\frac{j}{i}ab$.

Introduce this value of $\frac{k}{j}$, and divide by 2,

$$2\Upsilon=2k\int\frac{dv}{\cos^3 v}-k\int\frac{d\tau}{\cos^3\tau}+\sqrt{\frac{ab}{ij}}\left[\tan\varphi\sqrt{\mathrm{I}}-\int d\varphi\sqrt{\mathrm{I}}+j^2\int\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]-\frac{kj\tan\varphi\sqrt{\mathrm{I}}}{2\mathrm{N}^2}. \quad (290.)$$

Now when this equation is integrated between the limits $\varphi=0$, and $\varphi=\sin^{-1}\sqrt{\frac{1}{l}}$, or, taking the corresponding values, between $\tau=0$, and $\tau=\sin^{-1}\left(\frac{1-j}{1+j}\right)$, or between $v=0$, and $v=\frac{\pi}{2}$, Υ is infinite, and the arc of the asymptotic parabola $k\int\frac{dv}{\cos^3 v}$ is also infinite, but twice the difference Δ between those infinite quantities is finite. Let $\sin^2\varphi_,=\frac{1}{l}$, $\sin\tau_,=\frac{1-j}{1+j}$, then

$$\Delta=\frac{k(1+j)^2}{8j}+k\int_0^{\tau_,}\frac{d\tau}{\cos^3\tau}-\sqrt{\frac{ab}{ij}}\left[1-\int_0^{\varphi_,}d\varphi\sqrt{\mathrm{I}}+j^2\int_0^{\varphi_,}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]. \quad (291.)$$

Hence the difference between the two infinite arcs of the equilateral logarithmic hyperbola, and the corresponding infinite arcs of the asymptotic parabola, is equal to a right line + an arc of a plane parabola — an arc of a plane hyperbola.

LI. On the logarithmic hyperbola, when $l=\infty$. Case XII., p. 316.

Resume (233.), or $\frac{\Upsilon}{k}=\frac{b^2}{\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}}\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}$.

Now as $ln=i^2$, and as i is finite, while $l=\infty$, $n=0$.

The equation of condition $m+n-mn=i^2$, gives therefore $m=i^2$. Equations (248.) and (249.) give $a=0$, $b=k$.

And as $\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}=\frac{\mathrm{B}\sqrt{n}}{\sqrt{m}}$, we get $\frac{b^2}{\sqrt{\mathrm{B}(\mathrm{A}+\mathrm{C})}}=\frac{k^2\sqrt{m}\sqrt{l}}{k^2\sqrt{n}\sqrt{l}}=\sqrt{l}$, since $m=i^2=nl$;

hence $$\frac{\Upsilon}{k}=\sqrt{l}\int\frac{\cos^2\varphi d\varphi}{[1-l\sin^2\varphi]^2\sqrt{1-i^2\sin^2\varphi}}. \quad (a.)$$

Let $l\sin^2\varphi=\sin^2\psi$, therefore $\sqrt{l}\cos\varphi d\varphi=\cos\psi d\psi$, $[1-l\sin^2\varphi]^2=\cos^4\psi$,

$$\sqrt{1-i^2\sin^2\varphi}=\sqrt{1-\frac{i^2}{l}\sin^2\psi}=\sqrt{1-n\sin^2\psi},\text{ and }\cos\varphi=\sqrt{1-\frac{\sin^2\psi}{l}}.$$

Making these substitutions in the preceding equation, we get

$$\frac{\Upsilon}{k}=\frac{\sqrt{l}}{\sqrt{l}}\int\frac{d\psi}{\cos^3\psi}\frac{\sqrt{1-\frac{1}{l}\sin^2\psi}}{\sqrt{1-n\sin^2\psi}}.\quad\text{When } l=\infty,\ \frac{1}{l}=0,\ n=0;\ \text{ hence } \Upsilon=k\int\frac{d\psi}{\cos^3\psi}, \quad (292.)$$

or the logarithmic hyperbola in this case becomes a common parabola.

As $a=0$, $b=k$, the hyperbolic cylinder becomes a vertical plane, at right angles to the transverse axis.

Hence, comparing this result with (XIX.), we find that when the parameters are either $+\infty$ or $-\infty$, the corresponding hyperconic section is a plane principal section of the generating surface, *i.e.* either a circle or a parabola.

LII. By giving a double rectification of the common hyperbola, we shall the more readily discover the striking analogy which exists between this curve and the logarithmic hyperbola.

Let U be an arc of a common hyperbola, whose equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Assume $$x^2=\frac{a^4\cos^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda},\quad y^2=\frac{b^4\sin^2\lambda}{a^2\cos^2\lambda-b^2\sin^2\lambda}. \qquad \text{(a.)}$$

Differentiating these expressions, and substituting, we get

$$\frac{dU}{d\lambda}=\frac{b^2}{a\left[1-\frac{a^2+b^2}{a^2}\sin^2\lambda\right]^{\frac{3}{2}}}.\quad \text{Assume } \sin^2\varphi=\frac{a^2+b^2}{a^2}\sin^2\lambda, \text{ and let } i^2=\frac{a^2}{a^2+b^2}. \qquad \text{(b.)}$$

Finding from this equation the value of $\frac{d\lambda}{d\varphi}$, as $\frac{dU}{d\varphi}=\frac{dU}{d\lambda}\cdot\frac{d\lambda}{d\varphi}$, we shall finally obtain,

since $\frac{b^2}{\sqrt{a^2+b^2}}=\frac{a(1-i^2)}{i}$, $$\frac{U}{a}=\frac{(1-i^2)}{i}\int\frac{d\varphi}{[1-\sin^2\varphi]\sqrt{1-i^2\sin\varphi^2}}. \qquad \text{(c.)}$$

(31.) gives $-U=\int p\,d\lambda+\int\frac{d^2p}{d\lambda^2}d\lambda$, or $U=-\int p\,d\lambda-\frac{dp}{d\lambda}$. (d.)

Now as $p^2=a^2\cos^2\lambda-b^2\sin^2\lambda$, $\frac{dp}{d\lambda}=-\frac{(a^2+b^2)\sin\lambda\cos\lambda}{(a^2\cos^2\lambda-b^2\sin^2\lambda)^{\frac{1}{2}}}$, and as $\sin^2\varphi=\frac{a^2}{a^2+b^2}\sin^2\lambda$, (e.)

$$\frac{d\varphi}{d\lambda}=\frac{\sqrt{a^2+b^2}\sqrt{1-i^2\sin^2\varphi}}{a\cos\varphi}; \text{ (f.)} \qquad \text{hence } \frac{dp}{d\lambda}=-\sqrt{a^2+b^2}\tan\varphi\sqrt{1-i^2\sin^2\varphi}; \qquad \text{(g.)}$$

and as $p=a\cos\varphi$, $$p\,d\lambda=\frac{a^2\cos^2\varphi\,d\varphi}{\sqrt{(a^2+b^2)}\sqrt{1-i^2\sin^2\varphi}}=\frac{a}{i}\,\frac{\{1+i^2-i^2\sin^2\varphi-1\}}{\sqrt{1-i^2\sin\varphi}}; \qquad \text{(h.)}$$

whence, finally, $$\frac{i}{a}U=\tan\varphi\sqrt{I}-\int d\varphi\sqrt{I}+(1-i^2)\int\frac{d\varphi}{\sqrt{I}}. \qquad \text{(k.)}$$

This is the expression for an arc of an hyperbola referred to in (XLIX.).

The integral $\int\frac{d\varphi}{[1-i^2\sin^2\varphi]^{\frac{3}{2}}}=\frac{1}{(1-i^2)}\int d\varphi\sqrt{I}-\frac{i^2}{1-i^2}\,\frac{\sin\varphi\cos\varphi}{\sqrt{I}}$.

See HYMER's Integral Calculus, p. 195. Adding this integral to (k),

$$\frac{i}{a}U+(1-i^2)\int\frac{d\varphi}{I\sqrt{I}}=(1-i^2)\int\frac{d\varphi}{\sqrt{I}}+\tan\varphi\sqrt{I}-\frac{i^2\sin\varphi\cos\varphi}{\sqrt{I}}; \qquad \text{(m.)}$$

but $$\tan\varphi\sqrt{I}-\frac{i^2\sin\varphi\cos\varphi}{\sqrt{I}}=\frac{(1-i^2)\tan\varphi}{\sqrt{I}}.$$

Hence dividing by $(1-i^2)$, $$\frac{iU}{a(1-i^2)}+\int\frac{d\varphi}{I\sqrt{I}}=\frac{\tan\varphi}{\sqrt{I}}+\int\frac{d\varphi}{\sqrt{I}}; \qquad \text{(n.)}$$

but (c.) gives $$\frac{iU}{a(1-i^2)}=\int\frac{d\varphi}{[1-\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Eliminating U from these equations, we obtain

$$\int\frac{d\varphi}{[1-\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}+\int\frac{d\varphi}{[1-i^2\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}=\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}+\frac{\tan\varphi}{\sqrt{1-i^2\sin^2\varphi}}. \qquad \text{(293.)}$$

See HYMER's Integral Calculus, p. 245. The parameters are reciprocal in this equation, being 1 and i^2.

Now this is the extreme case of the formula for the comparison of elliptic integrals of the third order and logarithmic form. We perceive that this formula results from the comparison of two expressions for the same arc of a common hyperbola. We may also see that it is the limiting case of the general formula for the comparison of elliptic integrals of the third order having reciprocal parameters; a formula which in like manner has been deduced from the comparison of two expressions for the same arc of the logarithmic hyperbola. It is also evident that $j^2 \frac{\tan\phi}{\sqrt{I}}$ being the difference between $\tan\phi\sqrt{I}$ and $\frac{i^2 \sin\phi \cos\phi}{\sqrt{I}}$, it is the difference between tangents, one drawn to the hyperbola, the other to the plane ellipse, for $\tan\phi\sqrt{I}$ denotes the portion of a tangent to a hyperbola between the point of contact and the perpendicular on it from the centre; and $\frac{i^2 \cos\phi \sin\phi}{\sqrt{I}}$ denotes a similar quantity in an ellipse; this difference is precisely analogous to the expression which occurs in (284.) $\int\frac{dv}{\cos^3 v}-\int\frac{d\tau}{\cos^3\tau}$, which denotes the difference between two parabolic arcs, one drawn a tangent to the logarithmic hyperbola, the other a tangent to the logarithmic ellipse.

SECTION VII.—*On the Values of complete Elliptic Integrals of the third order.*

LIII. We have hitherto investigated the properties and lengths of elliptic curves, on the supposition that the generating surface, whether sphere or paraboloid, was invariable, and that the lengths of the curves were made up by the summation of the elements produced by the successive values given to the amplitude ϕ between certain limits, 0 and $\frac{\pi}{2}$, suppose, if the arcs are to be quadrants. Thus the length of the quadrant is obtained, by adding together the successive increments which result from the successive additions, indefinitely small, which are made to the amplitude. We may, however, proceed on another principle to effect the rectification of those curves. If, to fix our ideas, we want to determine the length of a quadrant of the spherical ellipse, we may imagine the vertical cylinder, which we shall suppose invariable, to be successively intersected by a series of all possible concentric spheres. Every quadrant will differ in length from the one immediately preceding it in the series, by an infinitesimal quantity; and if we take the least of these quadrants, and add to it the successive elements, by which one quadrant differs from the next immediately preceding, we shall thus obtain the length of a quadrant of the required spherical ellipse, equal to the least quadrant which can be described on the elliptic cylinder, plus the sum of all the elements between the least quadrant and the required one. Thus, for example, the least quadrant which can be drawn on an elliptic vertical cylinder, is its section by an horizontal plane, or a quadrant of the plane ellipse,

whose semiaxes are a and b. In this case the radius of the sphere is infinite. The least sphere is that whose radius is a, and which cuts the cylinder in its circular sections. Hence the greatest spherical elliptic quadrant is the quadrant of the circle whose radius is a. All the spherical elliptic quadrants will therefore be comprised between the quadrants of an ellipse, and of a circle whose radius is a. Any quadrant therefore of a given spherical ellipse is equal to a quadrant of a plane ellipse, plus a certain increment; or to a quadrant of a circle, minus a certain decrement. The same reasoning will hold as well when we take any other limits, besides 0 and $\frac{\pi}{2}$. These considerations naturally lead to the process of differentiation under the sign of integration, because we cannot express, under a finite known form, the arc of a spherical or logarithmic ellipse, and then differentiate the expression so found, with respect to a quantity which hitherto has been taken as a constant.

We may conceive the generation of successive curves of this kind to take place in another manner. Let us imagine an invariable sphere to be cut by a succession of concentric cylinders indefinitely near to each other, and generated after a given law. These cylinders will cut the sphere in a series of spherical ellipses, any one of which will differ from the one immediately preceding, by an indefinitely small quantity. If we sum all these indefinitely small quantities, or in other words, integrate the expression so found, we shall discover the finite difference between any two curves of the series separated by a finite interval. One of the limits being a known curve, the other may thus be determined.

To apply this reasoning.

In the following investigations we shall assume the generating sphere to be invariable, and the modulus i, with the amplitude φ to be constant. The intersecting cylinder we shall suppose to vary from curve to curve on the surface of the sphere. But i is constant, and $i^2=\frac{a^2-b^2}{a^2}$, see (27.). Now a and b being the semiaxes of the base of the cylinder, it follows that the bases of all the varying cylinders are concentric and similar ellipses. Hence in the elliptic integral of the third order, which represents the spherical ellipse, the parameter e^2 or m, and the criterion of sphericity $\sqrt{x}$ will vary.

In (17.) we found for a quadrant of a spherical hyperconic section, the expression

$$\sigma=\sqrt{x}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{[1-e^2\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Let k be the radius of the sphere.

Since $e^2=\frac{k^2i^2}{k^2-j^2a^2}$, e will vary, as being a function of a the transverse semiaxe of the variable cylinder. We have also

$$e^2x=(1-e^2)(e^2-i^2).$$

Hence $$\frac{dx}{de}=-2e\left(1-\frac{i^2}{e^4}\right); \quad (294.)$$

and if, as before, we write M for $1-m\sin^2\varphi$, or $1-e^2\sin^2\varphi$, we shall have

$$\sigma=\sqrt{x}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}.$$

Differentiating this expression on the hypothesis that i and φ are constant, while e is variable, we shall have

$$\frac{d\sigma}{de}=\frac{1}{2\sqrt{x}}\frac{dx}{de}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}+\frac{\sqrt{x}}{e}2\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M^2\sqrt{I}}-\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}\right]. \quad (295.)$$

Multiplying this equation by $\frac{\sqrt{x}}{e}$, and recollecting that $\frac{dx}{de}=-2e\left(1-\frac{i^2}{e^4}\right)$, we shall have

$$\frac{\sqrt{x}}{e}\frac{d\sigma}{de}=-\left(1-\frac{i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}+\frac{2x}{e^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M^2\sqrt{I}}-\frac{2x}{e^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}. \quad (296.)$$

But (see HYMER's Integral Calculus, p. 195)

$$\frac{2x}{e^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M^2\sqrt{I}}=\left[\frac{2}{e^2}(1+i^2)-1-\frac{3i^2}{e^4}\right]\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}-\left(\frac{e^2-i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}+\frac{1}{e^2}\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}. \quad (297.)$$

Introducing this value into the preceding equation, the coefficient of $\int_0^{\frac{\pi}{2}}\frac{d\varphi}{M\sqrt{I}}$ will vanish, and we shall have

$$\frac{\sqrt{x}}{e}\frac{d\sigma}{de}=-\left(\frac{e^2-i^2}{e^4}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}+\frac{1}{e^2}\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}. \quad (298.)$$

Dividing by $\frac{\sqrt{x}}{e}$, and integrating on the hypothesis that φ and i are constant,

$$\sigma=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{de}{e\sqrt{x}}-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\int\frac{de(e^2-i^2)}{e^3\sqrt{x}}+\text{constant};$$

or as $e\sqrt{x}=\sqrt{(1-e^2)(e^2-i^2)}$, we shall have

$$\sigma=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{de}{\sqrt{(1-e^2)(e^2-i^2)}}-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\int\frac{de}{e^2}\sqrt{\frac{e^2-i^2}{1-e^2}}+\text{constant}. \quad (299.)$$

We must recollect that the definite integrals within the brackets are functions, not of φ, but of i^2, 0, and $\frac{\pi}{2}$. They are therefore constants.

It is not a little remarkable that the coefficients of the *definite* elliptic integrals are themselves also elliptic integrals of the first and second orders. To show this, assume

$$e^2=\cos^2\theta+i^2\sin^2\theta. \quad (300.)$$

Therefore $1-e^2=j^2\sin^2\theta$, and $e^2-i^2=j^2\cos^2\theta$; we have also $ede=-j^2\sin\theta\cos\theta d\theta$.

Hence, if $1-j^2\sin^2\theta=J$, $$\int\frac{de}{\sqrt{(e^2-i^2)(1-e^2)}}=-\int\frac{d\theta}{\sqrt{1-j^2\sin^2\theta}}=-\int\frac{d\theta}{\sqrt{J}}, \quad (301.)$$

and $$\sqrt{x}=\frac{j^2\sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}. \quad (302.)$$

In the same manner we may show that

$$\int\sqrt{\frac{e^2-i^2}{1-e^2}}\,\frac{de}{e^2}=-\int\frac{d\theta}{\sqrt{1-j^2\sin^2\theta}}+i^2\int\frac{d\theta}{[1-j^2\sin^2\theta]^{\frac{3}{2}}}\,; \quad \ldots\ldots \quad (303.)$$

but

$$i^2\int\frac{d\theta}{[1-j^2\sin^2\theta]^{\frac{3}{2}}}=\int d\theta\sqrt{1-j^2\sin^2\theta}-j^2\frac{\sin\theta\cos\theta}{\sqrt{J}}. \quad \ldots\ldots \quad (304.)$$

Hence

$$\int\sqrt{\frac{e^2-i^2}{1-e^2}}\,\frac{de}{e^2}=\int d\theta\sqrt{J}-\int\frac{d\theta}{\sqrt{J}}-j^2\frac{\sin\theta\cos\theta}{\sqrt{J}}. \quad \ldots\ldots \quad (305.)$$

Substituting these values in (141.), we obtain

$$\sigma=\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\left[\int\frac{d\theta}{\sqrt{J}}-\int d\theta\sqrt{J}+j^2\frac{\sin\theta\cos\theta}{\sqrt{J}}\right]-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{d\theta}{\sqrt{J}}+\text{constant}. \quad (306.)$$

To determine this constant. We must not suppose $i=0$, in this case, as is generally done, to determine the constant. This would be to violate the supposition on which we have all along proceeded, namely, that the variable cylinders are all similar, and therefore that i must be constant. We must determine the constant from other considerations.

Since $e^2=\frac{i^2k^2}{k^2-j^2a^2}$, when $a=0$, $e^2=i^2$. But $e^2=\cos^2\theta+i^2\sin^2\theta$, therefore $\theta=\frac{\pi}{2}$. As a, the major semiaxe of the base of the cylinder, is supposed to vanish, the curve diminishes to a point, and therefore $\sigma=0$.

When $a=k$, $e^2=1$, and $\theta=0$. We have in this case $\sigma=\frac{\pi}{2}$; for the sections of a sphere by an elliptic cylinder, whose greater axis is equal to the diameter of the sphere, are two semicircles of a great circle of the sphere. Hence, when $\theta=0$, $\sigma=\frac{\pi}{2}$, $\sin\theta=0$, $\int d\theta\sqrt{J}=0$, $\int\frac{d\theta}{\sqrt{J}}=0$; therefore the constant is equal to σ, when $\theta=0$. But when $\theta=0$, $\sigma=\frac{\pi}{2}$, or the constant is equal to $\frac{\pi}{2}$.

The formula now becomes

$$\sigma=\frac{\pi}{2}-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{d\theta}{\sqrt{J}}+\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]-\left[\int\frac{d\theta}{\sqrt{J}}-\int d\theta\sqrt{J}+j^2\frac{\sin\theta\cos\theta}{\sqrt{J}}\right]. \quad \ldots \quad (307.)$$

When $\theta=\frac{\pi}{2}$, $e=i$, and $\sigma=0$, as the variable cylinder is in this case diminished to a right line; therefore the preceding formula will become, using the ordinary notation of elliptic integrals,

$$\frac{\pi}{2}=E_iF_j+E_jF_i-F_iF_j. \quad \ldots\ldots\ldots\ldots \quad (308.)$$

Hence we obtain the true geometrical meaning of this curious formula of verification discovered by LEGENDRE. In its general form (307.), it represents the *difference* between the quadrants of a great circle and of a spherical ellipse. When the spherical ellipse vanishes to a point, this expression must represent, as in (308.), the quadrant of a circle.

LIV. If we now apply the preceding investigations to the curve described on

the same sphere by the reciprocal cylinder, or by the cylinder which gives a function having a reciprocal parameter, we shall find

$$\sigma'=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\int\frac{de'}{\sqrt{(e'^2-i^2)(1-e'^2)}}-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\int\sqrt{\frac{e'^2-i^2}{1-e'^2}}\frac{de'}{e'^2}+\text{constant.} \quad (309.)$$

But by the conditions of the question, as

$$ee'=i,\ e'^2=\frac{i^2}{1-j^2\sin^2\theta},\qquad \int\frac{de'}{\sqrt{(e'^2-i^2)(1-e'^2)}}=\int\frac{d\theta}{\sqrt{1-j^2\sin^2\theta}}, \quad (310.)$$

and $$\int\frac{de'}{e'^2}\sqrt{\frac{e'^2-i^2}{1-e'^2}}=\int\frac{j^2\sin^2\theta d\theta}{\sqrt{1-j^2\sin^2\theta}}=\int\frac{d\theta}{\sqrt{1-j^2\sin^2\theta}}-\int d\theta\sqrt{1-j^2\sin^2\theta}.$$

Substituting these values of the integrals in (309.),

$$\sigma'=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{\mathrm{J}}}-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int\frac{d\theta}{\sqrt{\mathrm{J}}}-\int d\theta\sqrt{\mathrm{J}}\right]+\text{constant.} \quad (311.)$$

We shall now show that the constant $=0$.

When $\theta=0$, $e=1$, and therefore $e'=i$. Since $e'=i$, and σ is a quadrant of the vanishing spherical ellipse whose principal arcs, $\alpha=0$, $\beta=0$, we shall have $\sigma=0$. Hence also $\int d\theta\sqrt{\mathrm{J}}=0$, $\int\frac{d\theta}{\mathrm{J}}=0$; therefore the constant is 0. When $\theta=\frac{\pi}{2}$ $e'=1$, and (309.) becomes

$$\frac{\pi}{2}=\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\mathrm{J}}}\right]+\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{J}}\right]-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\left[\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\mathrm{J}}}\right], \quad (312.)$$

or, in the common notation, $\frac{\pi}{2}=\mathrm{E}_i\mathrm{F}_j+\mathrm{E}_j\mathrm{F}_i-\mathrm{F}_i\mathrm{F}_j$,

a formula already established in (308.).

If we add together (307.) and (312.), we shall have, since $\sqrt{\varkappa}=\frac{j^2\sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}$,

$$\sigma+\sigma'=\frac{\pi}{2}+\sqrt{\varkappa}\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]. \quad (313.)$$

Now $\sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}$, $\sigma'=\left(\frac{1-m_i}{m_i}\right)\sqrt{m_in_i}\int\frac{d\varphi}{[1-m_i\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}$, in which $mm'=i^2$.

Whence, as $\left(\frac{1-m}{m}\right)\sqrt{mn}=\left(\frac{1-m_i}{m_i}\right)\sqrt{m_in_i}=\sqrt{\varkappa}$, as we have shown in (113.),

$$\int_0^{\frac{\pi}{2}}\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}+\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\left[1-\frac{i^2}{m}\sin^2\varphi\right]\sqrt{1-i^2\sin\varphi}}=\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}+\frac{\pi}{2}. \quad (314.)$$

The reader will observe how very different are the geometrical origins of two algebraical formulæ apparently similar. In the logarithmic form of the elliptic integral, the formula for the comparison of elliptic integrals, with reciprocal parameters (one of which is greater, while the other is less than 1), resulted from putting in equation

two algebraical expressions for the *same* arc of the *one* logarithmic hyperbola. See Art. XLVIII. In the preceding case, that of the spherical ellipse, the analogous formula expresses the sum of the arcs of two inverse spherical ellipses, whose amplitudes are the same.

LV. We shall use the term *inverse spherical ellipses* to denote curves whose representative elliptic integrals have *reciprocal* parameters. The terms *reciprocal* and *supplemental* have long since been appropriated to curves otherwise related.

Let α and β, $\alpha_{\prime}$ and $\beta_{\prime}$ denote the principal semiarcs of two such curves. Since the modulus i is the same in both integrals, the orthogonal projections of these curves, on the base of the hemisphere, are similar ellipses. (15.) gives

$$e^2=i^2\sec^2\beta,\ e_{\prime}^2=i^2\sec^2\beta_{\prime}, \text{ and we assume } e^2e_{\prime}^2=i^2.$$

Hence
$$\sec\beta\sec\beta_{\prime}i=1. \qquad (315.)$$

Again, as $\tan^2\alpha(1-e^2)=\tan^2\beta=\sec^2\beta-1$, and $\tan^2\alpha_{\prime}(1-e_{\prime}^2)=\tan^2\beta_{\prime}=\sec^2\beta_{\prime}-1$; multiplying these expressions together, and introducing the relation in (315.),

$$\tan^2\alpha\tan^2\alpha_{\prime}i^2=\frac{i^2\sec^2\beta\sec^2\beta_{\prime}-i^2(\sec^2\beta+\sec^2\beta_{\prime})+i^2}{1+i^2-i^2(\sec^2\beta+\sec^2\beta_{\prime})}=1. \qquad (316.)$$

Hence the principal arcs of the inverse spherical ellipses are connected by the symmetrical relations

$$\tan\alpha\tan\alpha_{\prime}\,i=1, \quad \text{and } \sec\beta\sec\beta_{\prime}i=1. \qquad (317.)$$

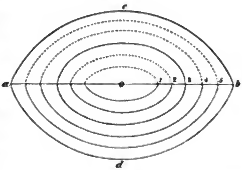
When the inverse curves coincide, $\alpha=\alpha_{\prime}$, $\beta=\beta_{\prime}$, and the last equations may be reduced to $\tan^2\alpha-\tan^2\beta=1$. Now we have shown in (59.) that when the principal arcs of a spherical hyperconic section are so related, the curve is the spherical parabola, or when the curve becomes its own inverse, it is the spherical parabola.

We have shown in (15.) that $i^2=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}=1-\frac{\sin^2\beta}{\sin^2\alpha}$, but (3.) gives $\cos\eta=\frac{\sin\beta}{\sin\alpha}$, 2η being the angle between the cyclic arcs of the spherical ellipse. Hence $i=\sin\eta$, but i is constant. Therefore all inverse spherical ellipses have the same cyclic arcs.

That portion of the surface of a sphere which lies between the cyclic circles may be called the *cyclic area.*

The spherical parabola divides the cyclic area into two regions. In the one, between the pole and the spherical parabola, lie all the inverse curves, whose parameters range from i^2 to i. In the other, between the spherical parabola and the cyclic circles, lie all the conjugate inverse curves, whose parameters range from i to 1.

Fig. 17.



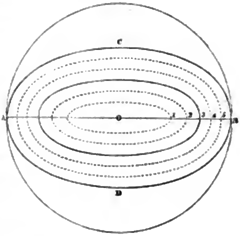
Let acb, adb be the cyclic circles, the intersection of the sphere by an elliptic cylinder, whose transverse axe is equal to the diameter of the sphere, and whose minor axe is $2j$. Let a series of concyclic spherical ellipses be described within this cyclic area, whose semitransverse arcs are 01, 02, 04, 05, and let 03 be the spherical parabola of the series. For every curve, 01, or 02, *within* the spherical parabola, there may be found another *without* it,

05, or 04, such that their principal arcs are connected by the equations

$$\tan\alpha \tan\alpha_i i = 1, \quad \sec\beta \sec\beta_i i = 1.$$

The algebraic expressions for the arcs of these curves, having the same amplitude, give elliptic integrals with *reciprocal* parameters.

Fig. 18.

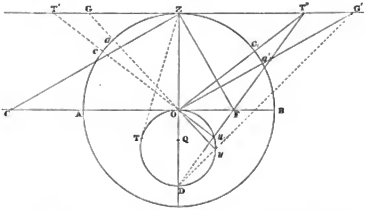


The concyclic spherical ellipses will be orthogonally projected on the base of the hemisphere into as many concentric and similar plane ellipses, whose semiaxes are 01, 02, 04, 05. The cyclic area will be projected into the plane ellipse, and the spherical parabola into the area of the plane ellipse, whose transverse semiaxe is $\frac{k}{\sqrt{1+i}}$. Let E be the area of the plane ellipse, the projection of the cyclic area, and Π the area of the plane ellipse, the projection of the spherical parabola. Then $E=\pi j$, and $\Pi = \frac{\pi j}{1+i}$, whence $\frac{E-\Pi}{\Pi} = i$, or the ellipse, the projection of the spherical parabola, divides the area of the ellipse, the projection of the cyclic area, into two portions, such that the outer is to the inner as $i:1$.

The reader must have observed the importance of this curve, the spherical parabola, in the discussion of the geometrical theory of elliptic integrals.

We may determine the principal arcs of two inverse spherical ellipses by a simple geometrical construction. Let AZB be a vertical section of the hemisphere, on which

Fig. 19.



the curves are to be described. Let F be the focus of the elliptic base of the maximum cylinder, whose principal transverse axis is accordingly equal to the diameter of the sphere. Join OZ, FZ, and draw ZC at right angles to ZF, meeting the line AO in C. Produce ZO until OD=AC, and on OD as diameter describe a circle. We are required, given one principal arc Za, to determine the corresponding principal arc Za' of the inverse hyperconic. Draw the tangent ZG. Through a draw

the line GOu. Through D draw the line $DuG_{\prime}$. Join $OG_{\prime}$, it will cut the sphere in $a_{\prime}$, the vertex of the principal arc $Za_{\prime}$. Let $OZ=k$, then $ZG=k\tan\alpha$, and as CZF is a right-angled triangle, $CO=ZD=\frac{k^2}{\sqrt{k^2-B^2}}=\frac{k}{i}$, k and B being the semiaxes of the maximum cylinder. As all the bases of the cylinders are similar, $\frac{k^2-B^2}{k^2}=\frac{a^2-b^2}{a^2}=i^2$.

Now as ZOG and ZDG′ are similar triangles, $ZG : ZO :: ZD : ZG'$, or $k\tan\alpha : k :: \frac{k}{i} : ZG'$, or $ZG'=\frac{k}{i\tan\alpha}$. But $ZG'=k\tan\alpha_{\prime}$, hence $\tan\alpha\tan\alpha_{\prime} i=1$, or the arcs α and $\alpha_{\prime}$ are connected by the equation established in (317.).

When we require to know which of these successive curves on this sphere is the spherical parabola, the same construction will enable us to determine it. Draw ZT, a tangent to the circle on OD, take $ZT'=ZT''=ZT$, and join T′ and T″ with O cutting the sphere in c and c'. $Zc=Zc'$ is the principal semitransverse arc of the spherical parabola, for $\overline{ZT}_{\prime}^2=k^2\tan^2\alpha=OZ.ZD=\frac{k^2}{i}$, or $\tan^2\alpha=\frac{1}{i}$. As $ZT' > ZO$, $cZc' > \frac{\pi}{2}$, or the principal arc of a spherical parabola is always greater than a right angle. Since in the spherical parabola $\gamma+2\iota=\frac{\pi}{2}$, the angle $COT'=2\iota$, or COT′ is equal to the distance between the foci of the curve.

LVI. If we revert to the general formula (307.) and take $\breve{\sigma}$ as the quadrant of a spherical parabola, the integrations with respect to θ must take place between $\theta=0$, and $\theta_{\prime}=\tan^{-1}\left(\frac{1}{\sqrt{i}}\right)$, for $e^2=i$, in (300.) gives $\tan\theta=\frac{1}{\sqrt{i}}$. Hence

$$\breve{\sigma}=\frac{\pi}{2}+\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]\left[\int_0^{\theta_{\prime}}\frac{d\theta}{\sqrt{J}}\right]-\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]\left[\int_0^{\theta_{\prime}}d\theta\sqrt{J}\right]-\left[\int_0^{\frac{\pi}{2}}d\phi\sqrt{I}\right]\left[\int_0^{\theta_{\prime}}d\theta\sqrt{J}\right]+(1-i)\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}\right]. \quad (318.)$$

Since $$\frac{j^2\sin\theta\cos\theta}{\sqrt{1-j^2\sin^2\theta}}=(1-i), \text{ when } \tan\theta=\left(\frac{1}{i}\right)^{\frac{1}{2}}.$$

Putting the sum of these integrals $=\Delta$, we shall have $\breve{\sigma}=\frac{\pi}{2}-\Delta$.

But (68.) gives for the quadrant of the spherical parabola

$$\breve{\sigma}=\frac{j^2}{(1+i)^2}\int_0^{\mu_{\prime}}\frac{d\mu}{\sqrt{1-\frac{4i}{(1+i)^2}\sin^2\mu}}+\frac{\pi}{4}.$$

Comparing these expressions for the same arc $\breve{\sigma}$,

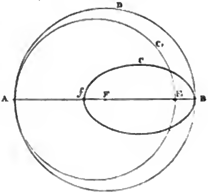
$$\frac{\pi}{4}=\frac{j^2}{(1+i)^2}\int_0^{\mu_{\prime}}\frac{d\mu}{\sqrt{1-\frac{4i}{(1+i)^2}\sin^2\mu}}+\Delta, \quad \ldots\ldots\ldots \quad (319.)$$

μ being taken between the limits $\mu=0$, and $\mu_{\prime}=\tan^{-1}\left(\frac{1}{\sqrt{j}}\right)$.

It is easy to show that the integrals of the first order in Art. LIII. may be represented

by two confocal spherical parabolas, having one common focus, and the nearer vertex of the one curve on the focus of the other. Thus let F be the pole of the hemisphere ABD. Let BCf and $AC_{,}F_{,}$ denote two spherical parabolas having one common focus at F; $F_{,}$ and f being the other foci. Let $Ff=\gamma$, and therefore $FF_{,}=\frac{\pi}{2}-\gamma$. Hence the modular angles of the two curves are γ, and $\frac{\pi}{2}-\gamma$, and if we make $\cos\gamma=i$, $\cos\left(\frac{\pi}{2}-\gamma\right)=j$.

Fig. 20.



Thus while the arc of the one is given by the integral $j\int\frac{d\phi}{\sqrt{1-i^2\sin^2\phi}}$ the arc of the other depends on the integral $i\int\frac{d\phi}{\sqrt{1-j^2\sin\phi^2}}$.

LVII. *On the value of the complete elliptic integral of the third order and logarithmic form.*

Let
$$\int_0^{\frac{\pi}{2}}\frac{d\phi}{[1-n\sin^2\phi]\sqrt{1-i^2\sin^2\phi}}=\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}. \quad \text{(320.)}$$

Assume κ the criterion of sphericity $=(1-n)\left(\frac{i^2}{n}-1\right)$,

then
$$\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}\right]=\frac{1}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N^2\sqrt{I}}-\frac{1}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}. \quad \text{(321.)}$$

Multiply by 2κ, then
$$2\kappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}\right]=\frac{2\kappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N^2\sqrt{I}}-\frac{2\kappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}} \quad \text{(322.)}$$

But (see HYMER's Integral Calculus, p. 195)

$$\frac{2\kappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N^2\sqrt{I}}=\left[1-\frac{2}{n}(1+i^2)+\frac{3i^2}{n^2}\right]\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}-\int_0^{\frac{\pi}{2}}d\phi\sqrt{I}, \quad \text{(323.)}$$

and
$$\frac{2\kappa}{n}\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}=\left[\frac{2i^2}{n^2}-\frac{2}{n}-\frac{2i^2}{n}+2\right]\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}.$$

Introducing the substitutions suggested by the two latter equations into (322.),

$$2\kappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}\right]=\left(\frac{i^2}{n^2}-1\right)\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}-\frac{1}{n}\int_0^{\frac{\pi}{2}}d\phi\sqrt{I}. \quad \text{(324.)}$$

Now $\frac{d\kappa}{dn}=-\left(\frac{i^2}{n^2}-1\right)$, whence

$$2\kappa\frac{d}{dn}\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}\right]+\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{N\sqrt{I}}\right]\frac{d\kappa}{dn}=-\left(\frac{i^2-n}{n^2}\right)\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{I}}-\frac{1}{n}\int_0^{\frac{\pi}{2}}d\phi\sqrt{I}. \quad \text{(325.)}$$

If we divide this equation by $2\sqrt{x}$, the first member will be the differential of $\sqrt{x}\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}\right]$. Integrating this equation,

$$2\sqrt{x}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}=-\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\int\frac{(i^2-n)dn}{n^2\sqrt{x}}-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{dn}{n\sqrt{x}}. \quad \ldots \quad (326.)$$

Assume $n=i^2\sin^2\theta$, then $x=\frac{1-i^2\sin^2\theta}{\tan^2\theta}$, $dn=2i^2\sin\theta\cos\theta d\theta$. . . . . . . (327.)

Hence
$$\int\frac{(i^2-n)dn}{n^2\sqrt{x}}=2\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}. \quad \ldots \quad (328.)$$

We must now integrate this expression,

$$\left.\begin{aligned}
\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}&=\int\frac{d\theta}{\sin^2\theta\sqrt{1-i^2\sin^2\theta}}-\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}},\\
\int\frac{d\theta}{\sin^2\theta\sqrt{1-i^2\sin^2\theta}}&=-\frac{\cot\theta}{\sqrt{1-i^2\sin^2\theta}}+\int\frac{i^2\cos^2\theta d\theta}{\sqrt{(1-i^2\sin^2\theta)^3}}\\
\int\frac{i^2\cos^2\theta d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}&=\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}-(1-i^2)\int\frac{d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}\\
-(1-i^2)\int\frac{d\theta}{(1-i^2\sin^2\theta)^{\frac{3}{2}}}&=\frac{i^2\sin\theta\cos\theta}{\sqrt{1-i^2\sin^2\theta}}-\int d\theta\sqrt{1-i^2\sin^2\theta};
\end{aligned}\right\} \quad \ldots \quad (329.)$$

adding these equations,

$$\left.\begin{aligned}
\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}&=\frac{i^2\sin\theta\cos\theta}{\sqrt{1-i^2\sin^2\theta}}-\frac{\cot\theta}{\sqrt{1-i^2\sin^2\theta}}-\int d\theta\sqrt{1-i^2\sin^2\theta};\\
\therefore -\int\frac{d\theta}{\tan^2\theta\sqrt{1-i^2\sin^2\theta}}&=\cot\theta\sqrt{1-i^2\sin^2\theta}+\int d\theta\sqrt{1-i^2\sin^2\theta}\ (299.).
\end{aligned}\right\} \quad \ldots \quad (330.)$$

We have next to compute the value of the integral $\int\frac{dn}{n\sqrt{x}}$.

Now
$$\int\frac{dn}{n\sqrt{x}}=\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}=\int\frac{d\theta}{\sqrt{_{\theta}I}}.$$

Substituting these values of the integrals in (326.),

$$\sqrt{x}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}=\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}\right]\left[\cot\theta\sqrt{_{\theta}I}+\int d\theta\sqrt{_{\theta}I}\right]-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I}\right]\int\frac{d\theta}{\sqrt{_{\theta}I}}. \quad \ldots \quad (331.)$$

If we now substitute this value of $\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}$ in the equation given in (175.) for a quadrant of the logarithmic ellipse, namely,

$$\frac{2\sqrt{1-i^2}}{\sqrt{1-n}}\frac{\Sigma}{a}=\frac{[2n-i^2-n^2]}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{N\sqrt{I}}+\frac{(i^2-n)}{n}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}+\int_0^{\frac{\pi}{2}}d\varphi\sqrt{I},$$

since $\frac{2n-i^2-n^2}{n}=(1-i^2\sin^2\theta)-\cot^2\theta$, we shall obtain the resulting equation,

$$\frac{2\sqrt{1-i^2}}{\sqrt{(\mathrm{I}_\theta)}}\frac{\Sigma}{a}=\left[\int_0^{\frac{\pi}{2}}d\phi\sqrt{\mathrm{I}}\right]+(\mathrm{I}_\theta)\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}\right]+\mathrm{H}\left[\frac{\sqrt{(\mathrm{I}_\theta)}}{\cot\theta}-\frac{\cot\theta}{\sqrt{(\mathrm{I}_\theta)}}\right]+\text{constant}, \quad . \quad (332.)$$

writing H for $\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}\right]\int d\theta\sqrt{(\mathrm{I}_\theta)}-\left[\int_0^{\frac{\pi}{2}}d\phi\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{(\mathrm{I}_\theta)}}$, ... (333.)

or in the ordinary notation,

$$\mathrm{H}=\mathrm{F}_i\mathrm{E}_i(\theta)-\mathrm{E}_i\mathrm{F}_i(\theta).$$

When we require to determine the constant, we must not suppose $\theta=0$, for this would render $n=0$, and so change the nature of the curve. Neither should we be justified in making $i=0$, (as some writers do), for this would be to violate the original supposition—and all the conclusions derived from it—namely, that i is constant, and less than 1. Moreover, since $m+n-mn=i^2=0$, on this hypothesis, $m+n=mn$; or m and n would each be greater than 1, which is inconsistent with the possible values of those quantities.

We have now to determine the value of the constant. In these investigations we have all along supposed $n>m$. The least value n can have is $n=m$. Were we to suppose n to be less than m, it would be nothing more than to write m for n, since m and n are connected by the equation $m+n-mn=i^2$. Hence if m is not equal to n, one of them must be the greater, and this one we agree to call n, writing m for the lesser. To determine the constant, let us assume $n=m$.

Now $n=i^2\sin^2\theta$, and n, when equal to m, is $=1-\sqrt{1-i^2}$, $(\mathrm{I}_\theta)=1-i^2\sin^2\theta=\sqrt{1-i^2}$, $\cot^2\theta=\sqrt{1-i^2}$, and $\tan\theta=\left(\frac{1}{j}\right)^{\frac{1}{2}}$. Hence the coefficient of H in the last equation, $\frac{\sqrt{(\mathrm{I}_\theta)}}{\cot\theta}-\frac{\cot\theta}{\sqrt{(\mathrm{I}_\theta)}}$, becomes 0, since in this case $\cot\theta=\sqrt[4]{1-i^2}$; and as $n=m$, the curve is the circular logarithmic ellipse. See Art. XLIII.

The last equation now becomes

$$2\sqrt[4]{1-i^2}\frac{\Sigma}{a}=\int_0^{\frac{\pi}{2}}d\phi\sqrt{\mathrm{I}}+\sqrt{1-i^2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}+\text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (334.)$$

Now if we turn to (176.), we shall find this, without the constant, to be the expression for the quadrant of a circular logarithmic ellipse, or the curve in which a circular cylinder, the radius of whose base is a, intersects at an infinite distance a paraboloid indefinitely attenuated. Hence the constant is 0.

To determine the value of the above integral, when $\theta=\frac{\pi}{2}$.

In this case, as $\mathrm{H}=\mathrm{F}_i\mathrm{E}_i-\mathrm{E}_i\mathrm{F}_i$, $\mathrm{H}=0$. And as $\cot\theta=0$, and $\sqrt{\mathrm{I}_\theta}=\sqrt{1-i^2}$, the equation (332.) will assume the form

$$2\frac{\Sigma}{a}=\left[\int_0^{\frac{\pi}{2}}d\phi\sqrt{\mathrm{I}}\right]+(1-i^2)\left[\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\mathrm{I}}}\right]+\frac{0}{0}\sqrt{1-i^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (335.)$$

How are we to interpret this expression?

To determine the value of the fraction $\frac{\mathrm{H}}{\cot\theta}$, which appears under the form of $\frac{0}{0}$ when $\theta=\frac{\pi}{2}$, we must take the first differentials of the numerator and denominator of this fraction. Now, as in (333.)

$$\mathrm{H}=\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right]\int d\theta\sqrt{1-i^2\sin^2\theta}-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]\int\frac{d\theta}{\sqrt{1-i^2\sin^2\theta}}. \quad . \quad . \quad . \quad . \quad \text{(a.)}$$

$$\text{Therefore } \frac{d\mathrm{H}}{d\theta}=\frac{\left[\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right](1-i^2\sin^2\theta)-\left[\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right]}{\sqrt{1-i^2\sin^2\theta}}, \text{ and } \frac{d\cot\theta}{d\theta}=-\frac{1}{\sin^2\theta}. \quad . \quad . \quad \text{(b.)}$$

$$\text{Hence, when } \theta=\frac{\pi}{2}, \quad \frac{\frac{d\mathrm{H}}{d\theta}}{\frac{d\cot\theta}{d\theta}}=\frac{\mathrm{H}}{\cot\theta}=\frac{\left[\left(\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right)(1-i^2)-\left(\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right)\right]}{-(\sqrt{1-i^2})}. \quad . \quad . \quad . \quad . \quad \text{(c.)}$$

Accordingly

$$\frac{\mathrm{H}}{\cot\theta}\sqrt{1-i^2\sin^2\theta}=\frac{0}{0}\sqrt{1-i^2}=-\left(\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{\mathrm{I}}}\right)(1-i^2)+\left(\int_0^{\frac{\pi}{2}}d\varphi\sqrt{\mathrm{I}}\right), \text{ when } \theta=\frac{\pi}{2}. \quad . \quad \text{(336.)}$$

Substituting this value in (332.), we get $\Sigma=a\int_0^{\frac{\pi}{2}}d\varphi\sqrt{1-i^2\sin^2\varphi}$, ... (337.)

the common expression for a quadrant of a plane ellipse, whose major axis is a, and eccentricity i. As it should be, for when $\theta=\frac{\pi}{2}$, or $n=i^2$, the section of the cylinder is a plane ellipse, as shown in Case VII. p. 316. In the spherical form, the limits of θ are 0 and $\frac{\pi}{2}$, while in the paraboloidal form, the limits of θ are $\tan^{-1}\left(\frac{1}{j}\right)^{\frac{1}{2}}$ and $\frac{\pi}{2}$.

SECTION VIII.—*On Conjugate Arcs of Hyperconic Sections.*

LVIII. Conjugate arcs of hyperconic sections may be defined, as arcs whose amplitudes φ, χ, ω are connected by the equation

$$\cos\omega=\cos\varphi\cos\chi-\sin\varphi\sin\chi\sqrt{1-i^2\sin^2\omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(338.)}$$

This is a fundamental theorem in the theory of elliptic integrals.

The angles φ, χ, ω may be called conjugate amplitudes.

When the hyperconic section is a circle, $i=0$, and $\cos\omega=\cos\varphi\cos\chi-\sin\varphi\sin\chi$, whence $\omega=\varphi+\chi$, or the conjugate amplitudes are $\varphi+\chi$, φ and χ. The development of this expression is the foundation of circular trigonometry.

On the Trigonometry of the Parabola.

When the hyperconic section is a parabola, $i=1$, and (338.) may be reduced to

$$\tan\omega=\tan\varphi\sec\chi+\tan\chi\sec\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(339.)}$$

If we make the imaginary transformations,

$\tan\omega=\sqrt{-1}\sin\omega'$, $\tan\varphi=\sqrt{-1}\sin\varphi'$, $\tan\chi=\sqrt{-1}\sin\chi'$, $\sec\varphi=\cos\varphi'$, $\sec\chi=\cos\chi'$. (340.)

The preceding formula will become, on substituting these values, and dividing by $\sqrt{-1}$,

$$\sin\omega' = \sin\varphi' \cos\chi' + \sin\chi' \cos\varphi',$$

the well-known trigonometrical expression for the sine of the sum of two circular arcs.

Hence, by the aid of imaginary transformations, we may interchangeably permute the formulæ of the trigonometry of the circle with those of the trigonometry of the parabola. In the trigonometry of the circle, $\omega=\varphi+\chi$, and in the trigonometry of the parabola ω is such a function of the angles φ and χ, as will render $\tan[(\varphi,\chi)]=\tan\varphi\sec\chi+\tan\chi\sec\varphi$. We must adopt some appropriate notation to represent this function. Let the function (φ,χ) be written $\varphi\perp\chi$, so that $\tan(\varphi\perp\chi)=\tan\varphi\sec\chi+\tan\chi\sec\varphi$. This must be taken as the *definition* of the function $\varphi\perp\chi$.

In like manner, we may represent by $\tan(\varphi\top\chi)$ the function $\tan\varphi\sec\chi-\tan\chi\sec\varphi$.

In applying the imaginary transformations, or while $\tan\varphi$ is changed into $\sqrt{-1}\sin\varphi$, $\sec\varphi$ into $\cos\varphi$, and $\cot\varphi$ into $-\sqrt{-1}\,\mathrm{cosec}\,\varphi$, $\perp$ must be changed into $+$ and $\top$ into $-$.

$\perp$ and $\top$ may be called logarithmic plus and minus. As examples of the analogy which exists between the trigonometry of the parabola and that of the circle, we give the following expressions in parallel columns; premising that the formulæ, marked by corresponding letters, may be derived singly, one from the other, by the help of the preceding imaginary transformations.

Trigonometry of the Parabola.		*Trigonometry of the Circle.*	(341.)
$\tan(\varphi\perp\chi)=\tan\varphi\sec\chi+\tan\chi\sec\varphi$.	$(\alpha.)$	$\sin(\varphi+\chi)=\sin\varphi\cos\chi+\sin\chi\cos\varphi$.	(a.)
$\tan(\varphi\top\chi)=\tan\varphi\sec\chi-\tan\chi\sec\varphi$.	$(\beta.)$	$\sin(\varphi-\chi)=\sin\varphi\cos\chi-\sin\chi\cos\varphi$.	(b.)
$\sec(\varphi\overset{\perp}{\top}\chi)=\sec\varphi\sec\chi\pm\tan\varphi\tan\chi$.	$(\gamma.)$	$\cos(\varphi\pm\chi)=\cos\varphi\cos\chi\mp\sin\varphi\sin\chi$.	(c.)
$\sin(\varphi\perp\chi)=\frac{\sin\varphi+\sin\chi}{1+\sin\varphi\sin\chi}$.	$(\delta.)$	$\tan(\varphi+\chi)=\frac{\tan\varphi+\tan\chi}{1-\tan\varphi\tan\chi}$.	(d.)
$\sin(\varphi\top\chi)=\frac{\sin\varphi-\sin\chi}{1-\sin\varphi\sin\chi}$.	$(\epsilon.)$	$\tan(\varphi-\chi)=\frac{\tan\varphi-\tan\chi}{1+\tan\varphi\tan\chi}$.	(e.)
Let $\varphi=\chi$.		Let $\varphi=\chi$.	
$\tan(\varphi\perp\varphi)=2\tan\varphi\sec\varphi$.	$(\eta.)$	$\sin2\varphi=2\sin\varphi\cos\varphi$.	(ē.)
$\sec(\varphi\perp\varphi)=\sec^2\varphi+\tan^2\varphi$.	$(\theta.)$	$\cos2\varphi=\cos^2\varphi-\sin^2\varphi$.	(th.)
$\sin(\varphi\perp\varphi)=\frac{2\sin\varphi}{1+\sin^2\varphi}$.	$(\iota.)$	$\tan2\varphi=\frac{2\tan\varphi}{1-\tan^2\varphi}$.	(*i*.)
$\sec\varphi=\frac{e^{\int\frac{d\varphi}{\cos\varphi}}+e^{-\int\frac{d\varphi}{\cos\varphi}}}{2}$, $\tan\varphi=\frac{e^{\int\frac{d\varphi}{\cos\varphi}}-e^{-\int\frac{d\varphi}{\cos\varphi}}}{2}$.	$(\kappa.)$	$\cos\varphi=\frac{e^{\varphi\sqrt{-1}}+e^{-\varphi\sqrt{-1}}}{2}$, $\sin\varphi=\frac{e^{\varphi\sqrt{-1}}-e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}}$.	(k.)
$1+\sqrt{-1}\tan(\varphi\perp\varphi)=(\sec\varphi+\sqrt{-1}\tan\varphi)^2$.	$(\lambda.)$	$1+\sin2\varphi=(\cos\varphi+\sin\varphi)^2$.	(l.)
$\tan^2\varphi=\frac{\sec(\varphi\perp\varphi)-1}{2}$.	$(\mu.)$	$\sin^2\varphi=\frac{1-\cos2\varphi}{2}$.	(m.)
Let the amplitudes be $\varphi\perp\chi$ and $\varphi\top\chi$.		Let the amplitudes be $\varphi+\chi$ and $\varphi-\chi$.	
$\tan(\varphi\perp\chi)\tan(\varphi\top\chi)=\tan^2\varphi-\tan^2\chi$.	$(\nu.)$	$\sin(\varphi+\chi)\sin(\varphi-\chi)=\sin^2\varphi-\sin^2\chi$.	(n.)

Since $\sec(\phi\perp\phi)=\sec^2\phi+\tan^2\phi$, and $\tan(\phi\perp\phi)=2\tan\phi\sec\phi$,

$$\sec(\phi\perp\phi)+\tan(\phi\perp\phi)=(\sec\phi+\tan\phi)^2.$$

Again, as $\sec(\phi\perp\phi\perp\phi)=\sec(\phi\perp\phi)\sec\phi+\tan(\phi\perp\phi)\tan\phi$,

and $\tan(\phi\perp\phi\perp\phi)=\tan(\phi\perp\phi)\sec\phi+\sec(\phi\perp\phi)\tan\phi$,

it follows that $\sec(\phi\perp\phi\perp\phi)+\tan(\phi\perp\phi\perp\phi)=(\sec\phi+\tan\phi)^3$,

and so on to any number of angles. Hence

$$\sec(\phi\perp\phi\perp\phi\ldots \text{ to } n\phi)+\tan(\phi\perp\phi\perp\phi\ldots \text{ to } n\phi)=(\sec\phi+\tan\phi)^n. \quad (342.)$$

Introduce into the last expression the imaginary transformation, $\tan\phi=\sqrt{-1}\sin\phi$, and we get DEMOIVRE's imaginary theorem for the circle,

$$\cos n\phi+\sqrt{-1}\sin n\phi=(\cos\phi+\sqrt{-1}\sin\phi)^n.$$

Let $\bar{\omega}$ be conjugate to ψ and ω, while ω, as before, is conjugate to ϕ and χ. Then we shall have $\tan\bar{\omega}=\tan(\phi\perp\chi\perp\psi)$, or

$$\tan(\phi\perp\chi\perp\psi)=\tan\phi\sec\chi\sec\psi+\tan\chi\sec\psi\sec\phi+\tan\psi\sec\phi\sec\chi+\tan\phi\tan\chi\tan\psi, \quad (\varpi.)$$

$$\sec(\phi\perp\chi\perp\psi)=\sec\phi\sec\chi\sec\psi+\sec\phi\tan\chi\tan\psi+\sec\chi\tan\psi\tan\phi+\sec\psi\tan\phi\tan\chi, \quad (\rho.)$$

and $$\sin(\phi\perp\chi\perp\psi)=\frac{\sin\phi+\sin\chi+\sin\psi+\sin\phi\sin\chi\sin\psi}{1+\sin\chi\sin\psi+\sin\psi\sin\phi+\sin\phi\sin\chi}; \quad (\sigma.)$$

whence, in the trigonometry of the circle,

$$\sin(\phi+\chi+\psi)=\sin\phi\cos\chi\cos\psi+\sin\chi\cos\psi\cos\phi+\sin\psi\cos\phi\cos\chi-\sin\phi\sin\chi\sin\psi, \quad (p.)$$

$$\cos(\phi+\chi+\psi)=\cos\phi\cos\chi\cos\psi-\cos\phi\sin\chi\sin\psi-\cos\chi\sin\psi\sin\phi-\cos\psi\sin\phi\sin\chi, \quad (r.)$$

$$\tan(\phi+\chi+\psi)=\frac{\tan\phi+\tan\chi+\tan\psi-\tan\phi\tan\chi\tan\psi}{1-\tan\chi\tan\psi-\tan\psi\tan\phi-\tan\phi\tan\chi} \quad (s.)$$

LIX. Let $(k.\omega)$, $(k.\phi)$, $(k.\chi)$ denote three parabolic arcs measured from the vertex of the parabola whose parameter is k.

The normal angles of these arcs are ω, ϕ, and χ; ω, ϕ and χ being conjugate amplitudes. Then

$$2(k.\phi)=k\tan\phi\sec\phi+k\int\frac{d\phi}{\cos\phi},\ 2(k.\chi)=k\tan\chi\sec\chi+k\int\frac{d\chi}{\cos\chi},\ 2(k.\omega)=k\tan\omega\sec\omega+k\int\frac{d\omega}{\cos\omega};$$

whence, since $\int\frac{d\omega}{\cos\omega}-\int\frac{d\phi}{\cos\phi}-\int\frac{d\chi}{\cos\chi}=0$, because ω, ϕ, and χ are conjugate amplitudes,

$$(k.\omega)-(k.\phi)-(k.\chi)=k\tan\omega\tan\phi\tan\chi. \quad (343.)$$

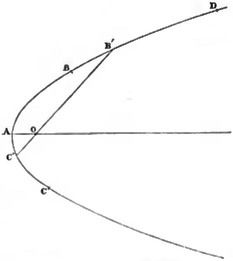
Let y, y', y'' be the ordinates of the arcs $(k.\phi)$, $(k.\chi)$, and $(k.\omega)$. Then $y=k\tan\phi$, $y'=k\tan\chi$, $y''=k\tan\omega$, and the last expression becomes

$$(k.\omega)-(k.\phi)-(k.\chi)=\frac{yy'y''}{k^2}. \quad (344.)$$

If we call an arc measured from the vertex of a parabola an *apsidal* arc, to distinguish it from an arc taken anywhere along the parabola, the preceding theorem

will enable us to express an arc of a parabola, taken anywhere along the curve, as the sum or difference of an apsidal arc and a right line.

Fig. 21.



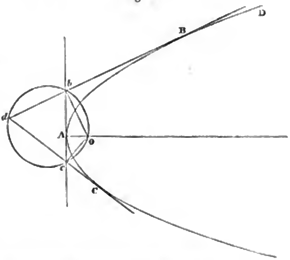
Thus let ACD be a parabola, O its focus and A its vertex. Let $AB=(k.\varphi)$, $AC=(k.\chi)$, $AD=(k.\omega)$ and $\frac{y_{\prime}y''}{k^2}=h$. Then (343.) shows that the parabolic arc $(AC+AB)$=apsidal arc $AD-h$; and the parabolic arc $(AD-AB)=BD$=apsidal arc $AC+h$. When the arcs AC′, AB′ together constitute a focal arc, or an arc whose cord passes through the focus, $\varphi+\chi=\frac{\pi}{2}$, and h is the ordinate of the conjugate arc AD. Hence we derive this theorem,

Any focal arc of a parabola is equal to the difference between the conjugate apsidal arc and its ordinate.

The relation between the amplitudes φ and ω in this case is $\sin 2\varphi=\frac{2\cos\omega}{1-\cos\omega}$. Thus when the focal cord makes an angle of 30° with the axis, we get $\cos\omega=\frac{1}{5}$, or $y=5k$. Here therefore the ordinate of the conjugate arc is five times the semiparameter.

LX. We may, in all cases, represent by a simple geometrical construction, the ordinates of the conjugate parabolic arcs, whose amplitudes are φ, χ and ω.

Fig. 22.



Let ABC be a parabola whose focus is O, and whose vertex is A. Let $AO=g=\frac{k}{2}$; moreover let AB be the arc whose amplitude is φ, and AC the arc whose amplitude is χ. At the points A, B, C draw tangents to the parabola, they will form a triangle circumscribing the parabola, whose sides represent the semi-ordinates of the conjugate arcs, AB, AC, AD.

We know that the circle, circumscribing this triangle, passes through the focus of the parabola.

Now $\quad Ab=g\tan\varphi,\; Ac=g\tan\chi,\; bd=g\tan\varphi\sec\chi,\; cd=g\tan\chi\sec\varphi;$

hence $\quad bd+cd=g(\tan\varphi\sec\chi+\tan\chi\sec\varphi)$, therefore $g\tan\omega=bd+cd$.

When AB, AC together constitute a focal arc, the angle bdc is a right angle.

The diameter of this circle is $g \sec\phi \sec\chi$.

The demonstration of these properties follows obviously from the figure.

LXI. In the trigonometry of the circle, we find the formula

$$\vartheta = \tan\vartheta - \frac{\tan^3\vartheta}{3} + \frac{\tan^5\vartheta}{5} - \frac{\tan^7\vartheta}{7} + \&c.; \quad \ldots \quad \text{(a.)}$$

and if we develop, by common division, the expression

$$\frac{1}{\cos\theta} = \frac{\cos\theta}{1-\sin^2\theta} = \cos\theta(1+\sin^2\theta+\sin^4\theta+\sin^6\theta+\ldots\&c.) \text{ and integrate,}$$

$$\int\frac{d\theta}{\cos\theta} = \sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \ldots\&c. \quad \ldots \quad \text{(b.)}$$

If we now inquire, what, in the circle, is the arc which differs from its protangent, by the distance between the vertex and its focus; or, as the protangent is 0 in the circle, and the focus is the centre; the question may be changed into what is the trigonometrical tangent of the arc of a circle equal to the radius. This question is answered by putting 1 for ϑ in (a.), and reverting the series

$$1 = \tan(1) - \frac{\tan^3(1)}{3} + \frac{\tan^5(1)}{5} - \frac{\tan^7(1)}{7} + \&c.;$$

we should get, in functions of the numbers of Bernouilli, the value of $\tan(1)$, as is shown in most treatises on trigonometry.

Let us now make a like inquiry in the case of the parabola, and ask what is the value of the amplitude which will give the difference, between the arc of the parabola and its protangent, equal to the distance between the focus and the vertex of the parabola. Now if θ be this angle, we must have $(k.\theta) - g\sec\theta\tan\theta = g$. But in general, $(k.\theta) - g\sec\theta\tan\theta = g\int\frac{d\theta}{\cos\theta}$. Hence we must have, in this case, $\int\frac{d\theta}{\cos\theta} = 1$. If we now revert the series (b.), putting 1 for $\int\frac{d\theta}{\cos\theta}$, we shall get from this particular value of the series,

$$1 = \sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \&c.,$$

an arithmetical value for $\sin\theta$. This will be found to be, $\sin\theta = \frac{e^1 - e^{-1}}{e^1 + e^{-1}}$, e being the base of the Napierian logarithms. Hence $\sec\theta + \tan\theta = e$, or if we write ϵ for this particular value of θ to distinguish it from every other, and call it the *angle of the base*,

$$\sec\epsilon + \tan\epsilon = e. \quad \ldots \quad (345.)$$

We are thus (for the first time it is believed) put in possession of the geometrical origin of that quantity, so familiarly known to mathematicians, the Napierian base. From the above equations we may derive

$$\sec\epsilon = \frac{e^1 + e^{-1}}{2}, \quad \tan\epsilon = \frac{e^1 - e^{-1}}{2}, \quad \ldots \quad (346.)$$

or $\tan\epsilon = 1{\cdot}175203015$, whence $\epsilon = {\cdot}8657606$, or $\epsilon = 49^\circ.\ 36'.\ 15''$.

The corresponding arc of the parabola will be $(k.\theta) = k\left[1 + \frac{2^1}{123} + \frac{2^3}{12345} + \&c.\right]$. (347.)

If we assume the theory of logarithms as known, we may at once arrive at this value,

for in general $$\int\frac{d\theta}{\cos\theta}=\log(\sec\theta+\tan\theta);$$

and as this is to be 1, we must have $\sec\theta+\tan\theta=e$, as before.

LXII. If we now extend this inquiry, and ask, what is the magnitude of the amplitude of the arc of the parabola which shall render the difference between the parabolic arc and its protangent equal to n times the distance between the focus and the vertex; we shall have, as before, by the terms of the question,

$$(k.\theta)-g\sec\theta\tan\theta=ng. \quad \ldots \quad (348.)$$

But in general $$(k.\theta)-g\sec\theta\tan\theta=g\int\frac{d\theta}{\cos\theta};$$

hence we must have $$n=\int\frac{d\theta}{\cos\theta}=\log(\sec\theta+\tan\theta),$$

or $$\sec\theta+\tan\theta=e^{n}. \quad \ldots \quad (349.)$$

Now we may solve this equation in two ways; either by making n a given number, and then determine the value of $\sec\theta+\tan\theta$, which may be called the *base*. Or we may assign an arbitrary value to $\sec\theta+\tan\theta$, and then derive the value of n. Taking the latter course, let, for example,

$$\sec\theta+\tan\theta=10. \quad \text{Then } n=\log 10,$$

or $\frac{1}{n}$ is the modulus of the second system of logarithms. Hence, if we assume any number of systems of logarithms on the same parabola, and take their bases

$$g(\sec\theta+\tan\theta), \quad g(\sec\theta'+\tan\theta'), \quad g(\sec\theta''+\tan\theta''), \ldots \&c.,$$

the moduli of these successive systems will be the ratios of half the semiparameter to the successive differences between the base parabolic arcs and their protangents.

In the Napierian system, g the distance from the focus to the vertex of the parabola, is taken as 1. The difference between the parabolic arc and its protangent, when equal to g, gives $g(\sec\theta+\tan\theta)=eg$. In the decimal system $g(\sec\theta_{,}+\tan\theta_{,})=10g$, and the difference between the corresponding parabolic arc and its protangent being ng, if we make this difference ng equal to the *arithmetical unit*, we shall have $ng=1$, or $g=\frac{1}{n}$=modulus of the system. Hence in every system of logarithms whatever, g the distance between the focus and the vertex of the parabola, is the modulus of the system. Every system of logarithms may be derived from the same parabola, but the Napierian system, in which the focal distance of the vertex is itself taken as the unit, may justly be taken as the *natural* system. In the same way we may consider that to be the *natural* system of circular trigonometry, in which the radius is taken as the unit. The modulus, in the trigonometry of the parabola, corresponds with the radius in the trigonometry of the circle. But while in the trigonometry of the parabola the base is real, in the circle it is imaginary. In the parabola, the angle of the base is given by the equation $\sec\theta+\tan\theta=e$. In the circle $\cos\vartheta+\sqrt{-1}\sin\vartheta=e^{\vartheta\sqrt{-1}}$, and

making $\vartheta=1$, we get $\cos(1)+\sqrt{-1}\sin(1)=e^{\sqrt{-1}}$. Hence while e^1 is the *parabolic* base, $e^{\sqrt{-1}}$ is the *circular* base. Or as $[\sec\epsilon+\tan\epsilon]$ is the Napierian base, $[\cos(1)+\sqrt{-1}\sin(1)]$ is the *circular* or imaginary base. Thus

$$[\cos(1)+\sqrt{-1}\sin(1)]^{\vartheta}=\cos\vartheta+\sqrt{-1}\sin\vartheta.$$

Hence, speaking more precisely, imaginary numbers have real logarithms, but an imaginary base. We may always pass from the real logarithms of the parabola, to the imaginary logarithms of the circle, by changing $\tan\theta$ into $\sqrt{-1}\sin\vartheta$, $\sec\theta$ into $\cos\vartheta$, and e^1 into $e^{\sqrt{-1}}$.

As in the parabola the angle θ is non-periodic, its limit being $\pm\frac{\pi}{2}$, while in the circle ϑ has no limit, it follows that while a number can have only one real or *parabolic* logarithm, it may have innumerable imaginary or *circular* logarithms.

In the parabola we thus can show the geometrical origin of the magnitudes known as the base and the modulus. We might too form systems of circular trigonometry analogous to different systems of logarithms. We might refer the arc of a circle not to the radius, but to some other arbitrary fixed line, the diameter or any other suppose. Let the circumference be referred to the diameter, then π will signify a whole circumference instead of a semicircle, and $\frac{\pi}{4}$ will represent a right angle. Having on this system, or any similar one, found the lengths of the arcs which correspond to certain functions, such as given sines or tangents, we should multiply the results by some fixed number, which we might call a modulus (2 in this example), to reduce them to the standard system; but such systems would obviously be useless.

If ϵ be the angle which gives the difference between the parabolic arc and its protangent equal to $g=\frac{k}{2}$; $(\epsilon\perp\epsilon)$ is the angle which will give this difference equal to $2g$, $(\epsilon\perp\epsilon\perp\epsilon)$ is the angle which will give this difference equal to $3g$, and so on to any number of angles. Hence, in the circle, if ϑ be the angle which gives the circular arc equal to the radius, 2ϑ is the angle which will give an arc equal to twice the radius, and so on for any number of angles. This is of course self-evident in the case of the circle, but it is instructive to point out the complete analogy which holds in the trigonometries of the circle and of the parabola.

LXIII. The geometrical origin of the exponential theorem may thus be shown.

Assume two known logarithmic bases $(\sec\alpha+\tan\alpha)$, and $(\sec\beta+\tan\beta)$, and let us investigate the ratio of the differences of the corresponding parabolic arcs and their protangents.

Let $\sec\epsilon+\tan\epsilon$ be the Napierian base, and let one difference be xg and the other yg. The ratio of these differences is therefore $\frac{y}{x}=z$, if we make $y=xz$. Hence

$$\sec\alpha+\tan\alpha=(\sec\epsilon+\tan\epsilon)^x=e^x, \text{ and } (\sec\beta+\tan\beta)=e^y. \text{ Therefore}$$

$$(\sec\alpha+\tan\alpha)^y=e^{x+y}=(\sec\beta+\tan\beta)^x.$$

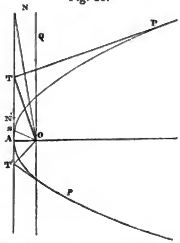
Or, as $y=xz$, $$(\sec\alpha+\tan\alpha)^z=\sec\beta+\tan\beta.$$

Let A be the first base, and B the second. Then $B=A^x$. This is the exponential theorem.

Let A be the Napierian base, then $x=1$, and $A=e$. Hence $B=e^x$.

LXIV. Given the number to find its logarithm, may be exhibited by the following geometrical construction.

Fig. 23.



Let OAP be a parabola. Through the focus O draw the perpendicular OQ to the axis AO. Through A let a tangent of indefinite length be drawn. On this tangent take the line AN to represent the given number. Join NO, and make the angle NOT always equal to the angle NOQ. Draw TP at right angles to TO. This line will touch the parabola in the point P, and the arc of the parabola AP—PT will be the logarithm of AN.

When $AN'=AO=$ the unit g, the angle N'OQ is equal to half a right angle. Hence the point T in this case will coincide with A. The parabolic arc therefore vanishes, or the logarithm of 1 is 0. When $\sec\theta+\tan\theta=1$, $\theta=0$.

When the number is less than 1, the point N will fall below N' in the position n. Hence nOQ is greater than half a right angle. Therefore T will fall *below* the axis in the point T'; and if we draw through T' a tangent T'p, it will give the *negative* arc of the parabola T'p, corresponding to the number An. Fractional numbers, or numbers between $+1$ and 0, must therefore be represented by the expression $g(\sec\theta-\tan\theta)$, since $\tan\theta$ changes its sign.

When the number is 0, n coincides with A, and the angle NOQ in this case is a right angle. Therefore the point T' will be the intersection of AT' and OQ. Hence T' is at an infinite distance below the axis, and therefore the logarithm of $+0$ is $-\infty$.

Hence negative numbers have no logarithms, at least no real ones; and imaginary ones can only be educed by the transformation so often referred to, and this leads us to seek them among the properties of the circle. For as θ always lies between 0 and a right angle, or between 0 and the half of $\pm\pi$, $\sec\theta\pm\tan\theta$ is *always* positive; hence *negative* numbers can have no real or *parabolic* logarithms, but they may have imaginary or *circular* logarithms; for in the expression $\log(\cos\vartheta+\sqrt{-1}\sin\vartheta)=\vartheta\sqrt{-1}$, we may make $\vartheta=(2n+1)\pi$, and we shall get $\log(-1)=(2n+1)\pi\sqrt{-1}$.

Hence also, as the length of the parabolic arc TP, without reference to the sign, depends solely on the amplitude θ, it follows that the logarithm of $\sec\theta-\tan\theta$ is equal to the logarithm of $\sec\theta+\tan\theta$. As $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1$, we may hence infer, that the logarithm of any number is equal to the logarithm of its reciprocal, with the sign changed.

When θ is very large, $\sec\theta+\tan\theta=2\tan\theta$, nearly. Hence if we represent a large number by an ordinate of a parabola whose focal distance to the vertex is 1, the difference between the corresponding arc and its protangent will represent its logarithm.

Along the tangent to the vertex of the parabola, as in the preceding figure, draw,

measured from the vertex, a series of lines in geometrical progression,

$$g\,(\sec\theta+\tan\theta),\quad g\,(\sec\theta+\tan\theta)^2,\quad g\,(\sec\theta+\tan\theta)^3\ \ldots\ldots\ g\,(\sec\theta+\tan\theta)^n.$$

Join N, the general representative of the extremities of these right lines, with the focus O. Erect the perpendicular OQ, and make the angle NOT *always* equal to the angle NOQ. The line OT will be $=g\sec\theta$, the line $OT_{,}=g\sec(\theta\perp\theta)$, the line $OT_{,,}=g\sec(\theta\perp\theta\perp\theta)$, &c., and we shall likewise have

$$AT=g\tan\theta,\quad AT_{,}=g\tan(\theta\perp\theta),\quad AT_{,,}=g\tan(\theta\perp\theta\perp\theta),\ \&c.$$

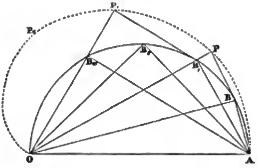
This follows immediately from (342.); for any integral power of $(\sec\theta+\tan\theta)$ may be exhibited as a linear function of $\sec\Theta+\tan\Theta$, if $\Theta=\theta\perp\theta\perp\theta\ldots$&c.,

since $\quad\sec(\theta\perp\theta\perp\theta\perp\theta$ &c. to $n\theta)+\tan(\theta\perp\theta\perp\theta\perp\theta$ &c. to $n\theta)=(\sec\theta+\tan\theta)^n$.

Hence the parabola enables us to give a graphical construction for the angle $(\theta\perp\theta\perp$&c.) as the circle does for the angle $n\theta$.

The analogous theorem in the circle may be developed as follows:—In the circle OBA, (fig. 24) take the arcs $AB=BB_{,}=B_{,}B_{,,}=B_{,,}B_{,,,}\ldots$&c.$=2\vartheta$. Let the diameter be G. Then $OB=G\cos\vartheta$, $OB_{,}=G\cos2\vartheta$, $OB_{,,}=G\cos3\vartheta\ldots$ &c. and $AB=G\sin\vartheta$, $AB_{,}=G\sin2\vartheta$, $AB_{,,}=G\sin3\vartheta\ldots$&c.

Fig. 24.



Now as the lines in the second group are always at right angles to those in the first, and as such a change is denoted by the symbol $\sqrt{-1}$, we get

$$OB+BA=G\,(\cos\vartheta+\sqrt{-1}\,\sin\vartheta),$$
$$OB_{,}+B_{,}A=G\,(\cos2\vartheta+\sqrt{-1}\,\sin2\vartheta)=G\,(\cos\vartheta+\sqrt{-1}\,\sin\vartheta)^2;$$
$$OB_{,,}+B_{,,}A=G\,(\cos3\vartheta+\sqrt{-1}\,\sin3\vartheta)=G\,(\cos\vartheta+\sqrt{-1}\,\sin\vartheta)^3\text{\&c.}$$

LXV. The known theorem, that a parabola is the reciprocal polar of a circle, whose circumference passes through the focus, suggests a transformation, which will exhibit a much closer analogy between the formulæ for the rectification of the parabola and the circle, than when the centre of the latter curve is taken as the origin.

Let OBA be a semicircle, let the origin be placed at O, let the angle $AOB=\vartheta$, and let G, as before, be the diameter of the circle. Through B draw the tangent BP; let fall on this tangent the perpendicular $OP=p$, and let BP the protangent be equal to t.

Now as $p=G\cos^2\vartheta$, and $t=G\sin\vartheta\cos\vartheta$, as also the angle $AOP=2\vartheta$, if we apply to the circle the formula for rectification in (33.), we shall have the arc

$$AB=s=2G\int\cos^2\vartheta d\vartheta-G\sin\vartheta\cos\vartheta.$$

Make the imaginary transformations $\cos\vartheta=\sec\theta$, and $\sin\vartheta=\sqrt{-1}\tan\theta$, and we shall have

$$\frac{s}{G\sqrt{-1}}=2\int\frac{d\theta}{\cos^3\theta}-\sec\theta\tan\theta.$$

The expression for an arc of a parabola, diminished by its protangent.

The protangent to the circle, which is exhibited in this formula, disappears in the actual process of integration; while in the parabola, the protangent which is involved in the differential, is evolved by the process of integration.

As in the parabola, the perpendicular, from the focus on the tangent, bisects the angle between the radius vector and the axis of the curve; so in the circle, the radius vector OB drawn from the extremity of the diameter, bisects the angle between the perpendicular OP and the diameter OA.

There are some curious analogies between the parabola and the circle, considered under this point of view.

In the parabola, the points T, $T_{,}$, $T_{,,}$, which divide the lines

$$g\,(\sec\theta+\tan\theta),\ g\,[\sec(\theta\perp\theta)+\tan(\theta\perp\theta)],\ \&c.$$

into their component parts, are upon tangents to the parabola. The corresponding points B, $B_{,}$, $B_{,,}$ in the circle, are on the circumference of the circle.

In the parabola the extremities of the lines $g\,(\sec\theta+\tan\theta)$ are on a right line AN; in the circle, the extremities of the bent lines G $(\cos\vartheta+\sqrt{-1}\,\sin\vartheta)$ are all in the point A.

The locus of the point T, the intersections of the tangents to the parabola with the perpendiculars from the focus, is a right line; or in other words, while one end of a protangent rests on the parabola, the other end rests on a right line. So in the circle, while one end of the protangent rests on the circle, the other end rests on a *cardioide*, whose diameter is equal to that of the circle, and whose cusp is at O. $OPP_{,}A$ is the cardioide.

The length of the tangent AT to any point T is $g\tan\theta$. The length of the cardioide is $2G\sin\vartheta$.

It is singular that the imaginary formulæ in trigonometry have long been discovered, while the corresponding real expressions have escaped notice. Indeed, it was long ago observed by LAMBERT, and by other geometers—the remark has been repeated in almost every treatise on the subject since—that the ordinates of an equilateral hyperbola might be expressed by real exponentials, whose exponents are sectors of the hyperbola; but the analogy, being illusory, never led to any useful results. And the analogy was illusory from this, that it so *happens* the length and area of a circle are expressed by the *same* function, while the area of an equilateral hyperbola is a function of an arc of a parabola. The true analogue of the circle is the parabola.

LXVI. Let $\bar{\varpi}$ be the conjugate amplitude of ϖ and ψ, while ϖ is the conjugate amplitude, as before, of ϕ and χ.

Then as $\int\frac{d\bar{\varpi}}{\cos\bar{\varpi}}=\int\frac{d\varpi}{\cos\varpi}+\int\frac{d\psi}{\cos\psi}$, and $\int\frac{d\varpi}{\cos\varpi}=\int\frac{d\phi}{\cos\phi}+\int\frac{d\chi}{\cos\chi}$,
we shall have

$$\int\frac{d\bar{\varpi}}{\cos\bar{\varpi}}=\int\frac{d\phi}{\cos\phi}+\int\frac{d\chi}{\cos\chi}+\int\frac{d\psi}{\cos\psi};$$

and if $(k.\bar{\omega})$, $(k.\phi)$, $(k.\chi)$ and $(k.\psi)$ are four corresponding parabolic arcs,

$$(k.\bar{\omega})-(k.\phi)-(k.\chi)-(k.\psi)=k\tan(\phi\perp\chi)\tan(\phi\perp\psi)\tan(\chi\perp\psi). \quad . \quad . \quad (350.)$$

which gives a simple relation between four conjugate parabolic arcs.

Let, in the preceding formula, $\phi=\chi=\psi$, and we shall have

$$(k.\bar{\omega})-3(k.\phi)=k\tan^3(\phi\perp\phi)=8k\tan^3\phi\sec^3\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (351.)$$

We are thus enabled to assign the difference between an arc of a parabola and three times another arc, $\bar{\omega}=(\phi\perp\phi\perp\phi)$.

If in (ϖ) (341.) we make $\phi=\chi=\psi$, $\tan\bar{\omega}=4\tan^3\phi+\tan\phi$.

Introduce into this expression, the imaginary transformation $\tan\phi=\sqrt{-1}\sin\theta$, and we shall get $\sin 3\theta=-4\sin^3\theta+\sin\theta$, which is the known formula for the trisection of a circular arc. (351.) may therefore be taken as the formula which gives the trisection of an arc of a parabola.

When there are five parabolic arcs, whose normal angles ϕ, χ, ψ, υ, Ω are related as above, namely,

$$\omega=\phi\perp\chi,\quad \bar{\omega}=\omega\perp\psi=\phi\perp\chi\perp\psi,\quad \Omega=\phi\perp\chi\perp\psi\perp\upsilon,$$

we get the following relation,

$$(k.\Omega)-(k.\phi)-(k.\chi)-(k.\psi)-(k.\upsilon)=k\tan(\phi\perp\chi\perp\upsilon)\tan(\chi\perp\psi\perp\upsilon)\tan(\psi\perp\phi\perp\upsilon), (352.)$$

a formula which connects five parabolic arcs, whose amplitudes are derived by the given law.

We might pursue this subject very much further; but enough has been done to show the analogy which exists between the trigonometry of the circle and that of the parabola. As the calculus of angular magnitude has always been referred to the circle as its type, so the calculus of logarithms may, in precisely the same way, be referred to the parabola as its type.

The obscurities, which hitherto have hung over the geometrical theory of logarithms, have it is hoped been now removed. It is possible to represent logarithms, as elliptic integrals usually have been represented, by curves devised to exhibit some special property only; and accordingly, such curves, while they place before us the properties they have been constructed to represent, fail generally to carry us any further. The close analogies which connect the theory of logarithms with the properties of the circle will no longer appear inexplicable*.

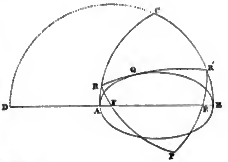
* The views above developed, on the trigonometry of the parabola, throw much light on a controversy long carried on between Leibnitz and J. Bernoulli on the subject of the logarithms of negative numbers. Leibnitz insisted they were imaginary, while Bernoulli argued they were real, and the same as the logarithms of equal positive numbers. Euler espoused the side of the former, while D'Alembert coincided with the views of Bernoulli. Indeed, if we derive the theory of logarithms from the properties of the hyperbola (as geometers always have done), it will not be easy satisfactorily to answer the argument of Bernoulli—that as an hyperbolic area represents the logarithm of a positive number, denoted by the positive abscissa $+x$, so a negative number, according to conventional usage, being represented by the negative abscissa $-x$, the corresponding hyperbolic area should denote its logarithm also. All this obscurity is cleared up by the theory developed in the text, which completely establishes the correctness of the views of Leibnitz and Euler.

On Conjugate Arcs of a Spherical Parabola.

LXVII. The well-known relations between elliptic integrals of the first order, whose amplitudes are conjugate, develope some very elegant geometrical theorems.

Thus in fig. (25.), since the arc $AQ=j\int\frac{d\varphi}{\sqrt{I_\varphi}}+QR$, and the arc $BQ=j\int\frac{d\chi}{\sqrt{I_\chi}}+QR'$, the arcs $AQ+BQ=j\left[\int\frac{d\varphi}{\sqrt{I_\varphi}}+\int\frac{d\chi}{\sqrt{I_\chi}}\right]+QR+QR'$. . . . . (a.)

Fig. 25.



Now $AQ+BQ=$ two quadrants of the spherical parabola, and $QR+QR'=\frac{\pi}{2}$, whence half the circumference, or $AQB=j\left[\int\frac{d\varphi}{\sqrt{I_\varphi}}+\int\frac{d\chi}{\sqrt{I_\chi}}\right]+\frac{\pi}{2}$.

In XXII. it has been shown that the complete integral represents the semicircumference, whence

$$AQB=j\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{I_\omega}}+\frac{\pi}{2}. \quad . \quad . \quad . \quad . \quad . \quad (b.)$$

Comparing these equations (a.) and (b.) together, we get

$$\int_0^{\frac{\pi}{2}}\frac{d\omega}{\sqrt{I_\omega}}=\int\frac{d\varphi}{\sqrt{I_\varphi}}+\int\frac{d\chi}{\sqrt{I_\chi}}.$$

Now as the triangle RR'P is a quadrantal right-angled triangle, the relation between the angles AFR, BER', or φ and χ, is easily discovered. Since FPE is a spherical triangle right-angled at P, and $FE=2\varepsilon=\frac{\pi}{2}-\gamma$, we get $j\tan\varphi\tan\chi=1$. When $AQ=BQ$, $\varphi=\chi$, and $\tan\varphi=\frac{1}{\sqrt{j}}$.

The locus of the point P is a spherical ellipse, supplemental to the former, having the extremities of its principal minor arc, in the foci F, E of the former.

LXVIII. Let $\sigma, \sigma_{,}, \sigma_{,,}$ be three arcs of a spherical parabola, corresponding to the conjugate amplitudes φ, χ, ω. Then successively substituting these amplitudes in (58.), the resulting equation becomes

$$\sigma+\sigma_{,}-\sigma_{,,}=j\left[\int\frac{d\varphi}{\sqrt{I_\varphi}}+\int\frac{d\chi}{\sqrt{I_\chi}}-\int\frac{d\omega}{\sqrt{I_\omega}}\right]+\tau+\tau_{,}-\tau_{,,}.$$

But as the amplitudes φ, χ, ω are conjugate, the sum of these integrals of the first order is 0, whence

$$\sigma+\sigma_{,}-\sigma_{,,}=\tau+\tau_{,}-\tau_{,,}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (353.)$$

Or, when the amplitudes of three arcs in the spherical parabola are conjugate amplitudes, the sum of the arcs is equal to the sum of the protangents. We use the word sum in its algebraic sense.

On Conjugate Arcs of a Spherical Ellipse.

LXIX. If, in (42.), we substitute successively ϕ, χ, ω, and add the resulting equations, we shall have

$$\sigma+\sigma_{,}-\sigma_{,,}=\left(\frac{1+n}{n}\right)\sqrt{mn}\left[\int\frac{d\phi}{N_{\phi}\sqrt{I_{\phi}}}+\int\frac{d\chi}{N_{\chi}\sqrt{I_{\chi}}}-\int\frac{d\omega}{N_{\omega}\sqrt{I_{\omega}}}\right]$$

$$-\frac{i^2}{\sqrt{mn}}\left[\int\frac{d\phi}{\sqrt{I_{\phi}}}+\int\frac{d\chi}{\sqrt{I_{\chi}}}-\int\frac{d\omega}{\sqrt{I_{\omega}}}\right]-\tau-\tau_{,}+\tau_{,,}. \quad (354.)$$

Now the conjugate relation between ϕ, χ and ω renders the sum of the integrals of the first order$=0$, and the sum of the integrals of the third order equal to a circular arc Θ, which is given by the equation

$$\tan\Theta=\frac{\sqrt{mn}\sin\phi\sin\chi\sin\omega}{1-\frac{n}{1+n}\cos\phi\cos\chi\cos\omega}. \quad (355.)$$

Hence $$\sigma+\sigma_{,}-\sigma_{,,}=\Theta-\tau-\tau_{,}+\tau_{,,}. \quad (356.)$$

Or, *when the amplitudes are conjugate, the sum of three arcs of a spherical ellipse may be expressed as the sum of four circular arcs.*

When one of the amplitudes ω is a right angle, $\sigma_{,,}$ becomes a quadrant of the spherical ellipse$=\breve{\sigma}$. $\tau_{,,}=0$, and $\Theta=\tau=\tau_{,}$, as we shall show presently, whence

$$(\sigma-\sigma_{,})-\sigma=\tau, \text{ which agrees with (52.).}$$

Or the difference between two arcs of a spherical ellipse, measured from the vertices of the curve, may be expressed by a circular arc. In (45.) we found

$$\tan\tau=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}},\quad \tan\tau_{,}=\frac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}}.$$

Now when $\omega=\frac{\pi}{2}$, (338.) gives $\sin\chi=\frac{\cos\phi}{\sqrt{1-i^2\sin^2\phi}}$, $\sin\phi=\frac{\cos\chi}{\sqrt{1-i^2\sin^2\chi}}$,

whence $$\sqrt{mn}\sin\phi\sin\chi=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}=\frac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}},$$

or $\Theta=\tau=\tau_{,}$, when $\tau_{,,}=0$, or $\omega=\frac{\pi}{2}$.

LXX. When we take the negative parameter m instead of the positive n, (17.) gives

$$\sigma+\sigma_{,}-\sigma_{,,}=\left(\frac{1-m}{m}\right)\sqrt{mn}\left[\int\frac{d\phi}{M_{\phi}\sqrt{I_{\phi}}}+\int\frac{d\chi}{M_{\chi}\sqrt{I_{\chi}}}-\int\frac{d\omega}{M_{\omega}\sqrt{I_{\omega}}}\right]. \quad (357.)$$

Now the sum of these arcs is equal to a circular arc$-\Theta_{,}$, which may be determined by the expression

$$\tan\Theta_{,}=\frac{\sqrt{mn}\sin\phi\sin\chi\sin\omega}{1+\frac{m}{1-m}\cos\phi\cos\chi\cos\omega}, \quad (358.)$$

whence $$\sigma+\sigma_{,}-\sigma_{,,}=-\Theta_{,}. \quad (359.)$$

If we compare together (356.) and (359.), we shall have the following simple rela-

tion between the five circular arcs Θ, $\Theta_{,}$, τ, $\tau_{,}$, $\tau_{,,}$,

$$\Theta+\Theta_{,}=\tau+\tau_{,}-\tau_{,,} \quad \ldots \ldots \ldots \quad (360.)$$

We may give an independent proof of this remarkable theorem.

The primary theorem (338.) $\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi\sqrt{I_\omega}$

gives
$$\frac{\sin\omega\cos\omega}{\sqrt{I_\omega}}=\frac{\sin\phi\sin\chi\sin\omega\cos\omega}{\cos\phi\cos\chi-\cos\omega},$$

and $\cos^2\phi+\cos^2\chi+\cos^2\omega=1+2\cos\phi\cos\chi\cos\omega-i^2\sin^2\phi\sin^2\chi\sin^2\omega$.

Let
$$\sin\phi\sin\chi\sin\omega=U, \quad \cos\phi\cos\chi\cos\omega=V. \quad \ldots \ldots \quad (361.)$$

Now
$$\tan\tau_{,,}=\frac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}=-\frac{\sqrt{mn}\,U\cos^2\omega}{\cos^2\omega-V},$$

whence
$$\tan\tau=\frac{\sqrt{mn}\,U\cos^2\phi}{\cos^2\phi-V}, \quad \tan\tau_{,}=\frac{\sqrt{mn}\,U\cos^2\chi}{\cos^2\chi-V},$$

and
$$\tan(\tau+\tau_{,}-\tau_{,,})=\frac{\tan\tau+\tan\tau_{,}-\tan\tau_{,,}+\tan\tau\tan\tau_{,}\tan\tau_{,,}}{1+\tan\tau_{,,}\tan\tau_{,}+\tan\tau\tan_{,,}-\tan\tau_{,}\tan\tau},$$

whence

$$\tan(\tau+\tau_{,}-\tau_{,,})=\frac{\sqrt{mn}\,U\left[\frac{\cos^2\phi}{\cos^2\phi-V}+\frac{\cos^2\chi}{\cos^2\chi-V}+\frac{\cos^2\omega}{\cos^2\omega-V}-\frac{mnU^2\cos^2\phi\cos^2\chi\cos^2\omega}{(\cos^2\phi-V)(\cos^2\chi-V)(\cos^2\omega-V)}\right]}{1-mnU^2\left[\frac{\cos^2\chi\cos^2\omega}{(\cos^2\chi-V)(\cos^2\omega-V)}+\frac{\cos^2\omega\cos^2\phi}{(\cos^2\omega-V)(\cos^2\phi-V)}+\frac{\cos^2\phi\cos^2\chi}{(\cos^2\phi-V)(\cos^2\chi-V)}\right]}.$$

If we reduce this expression, we shall have, on introducing the relations

$$\left.\begin{aligned}&\cos^2\phi+\cos^2\chi+\cos^2\omega=1+2V-i^2U^2,\\ \text{and}\quad &\cos^2\omega\cos^2\chi+\cos^2\phi\cos^2\omega+\cos^2\chi\cos^2\phi=V^2+2V+j^2U^2,\end{aligned}\right\} \quad \ldots \ldots \quad (362.)$$

$$\tan(\tau+\tau_{,}-\tau_{,,})=\frac{[2j^2+(i^2+mn)V]\sqrt{mn}\,U}{j^2+(i^2+mn)V-mn(V^2+j^2U^2)}. \quad \ldots \ldots \quad (363.)$$

If we now combine the values of $\tan\Theta$ and $\tan\Theta_{,}$, given in (355.) and (358.), we shall have

$$\tan(\Theta+\Theta_{,})=\frac{[2j^2+(i^2+mn)V]\sqrt{mn}\,U}{j^2+(i^2+mn)V-mn(V^2+j^2U^2)}, \quad \ldots \ldots \quad (364.)$$

whence
$$\Theta+\Theta_{,}=\tau+\tau_{,}-\tau_{,,},$$

as is evident from an inspection of the preceding formulæ.

On Conjugate Arcs of a Logarithmic Ellipse.

LXXI. In (162.) substitute χ and ω successively for ϕ. Let

$$\sqrt{z}=\left(\frac{1-n}{n}\right)\sqrt{mn}, \quad \Phi=\frac{\sin\phi\cos\phi\sqrt{I_\phi}}{1-n\sin^2\phi}, \quad X=\frac{\sin\chi\cos\chi\sqrt{I_\chi}}{1-n\sin^2\chi}, \quad \Omega=\frac{\sin\omega\cos\omega\sqrt{I_\omega}}{1-n\sin^2\omega}, \quad (365.)$$

we shall have, adding the three resulting equations together, and dividing by $\frac{n-m}{\sqrt{mn}}$,

$$\frac{2}{k}[\Sigma_\omega-\Sigma_\chi-\Sigma_\phi]=\frac{\sqrt{mn}}{n-m}\left[n\Phi+nX-n\Omega-\left(\int d\phi\sqrt{I_\phi}+\int d\chi\sqrt{I_\chi}-\int d\omega\sqrt{I_\omega}\right)\right]$$

$$-\frac{m}{n}\frac{(1-n)}{(n-m)}\sqrt{mn}\left[\int\frac{d\phi}{\sqrt{I_\phi}}+\int\frac{d\chi}{\sqrt{I_\chi}}-\int\frac{d\omega}{\sqrt{I_\omega}}\right]-\sqrt{z}\left[\int\frac{d\phi}{N_\phi\sqrt{I_\phi}}+\int\frac{d\chi}{N_\chi\sqrt{I_\chi}}-\int\frac{d\omega}{N_\omega\sqrt{I_\omega}}\right]. \quad (366.)$$

Now as φ, χ, and ω are conjugate amplitudes,

$$\int\frac{d\varphi}{\sqrt{I}}+\int\frac{d\chi}{\sqrt{I}}-\int\frac{d\omega}{\sqrt{I}}=0, \text{ and } \int d\varphi\sqrt{I}+\int d\chi\sqrt{I}-\int d\omega\sqrt{I}=i^2\sin\varphi\sin\chi\sin\omega.$$

See HYMER's Integral Calculus, p. 206.

Whence $\frac{2}{k}[\Sigma_\omega-\Sigma_\chi-\Sigma_\varphi]=\frac{\sqrt{mn}}{n-m}[n\Phi+nX-n\Omega-i^2\sin\varphi\sin\chi\sin\omega]$

$$-\sqrt{\varkappa}\Big[\int\frac{d\varphi}{N_\varphi\sqrt{I}}+\int\frac{d\chi}{N_\chi\sqrt{I}}-\int\frac{d\omega}{N_\omega\sqrt{I}}\Big]. \quad (367.)$$

We have now to compute the sum of $\Phi+X-\Omega$.

Since $\sqrt{I_\omega}=\frac{\cos\varphi\cos\chi-\cos\omega}{\sin\varphi\sin\chi}$, $\frac{\sin\omega\cos\omega\sqrt{I_\omega}}{1-n\sin^2\omega}=\Omega=-\frac{\sin^2\omega(\cos^2\omega-V)}{N_\omega U}$, if we make, as before, $\cos\varphi\cos\chi\cos\omega=V$, and $\sin\varphi\sin\chi\sin\omega=U$. Finding similar expressions for Φ and X, we shall have

$$n\Phi+nX-n\Omega=\frac{n}{U}\Big[\frac{\sin^2\varphi\cos^2\varphi}{N_\varphi}+\frac{\sin^2\chi\cos^2\chi}{N_\chi}-\frac{\sin^2\omega\cos^2\omega}{N_\omega}\Big]-\frac{V}{U}\Big[\frac{n\sin^2\omega}{N_\omega}+\frac{n\sin^2\chi}{N_\chi}+\frac{n\sin^2\varphi}{N_\varphi}\Big]. \quad (368.)$$

Now $$\frac{n\sin^2\varphi\cos^2\varphi}{UN}=\frac{\cos^2\varphi(1+n\sin^2\varphi-1)}{NU}=\frac{\cos^2\varphi}{NU}-\frac{\cos^2\varphi}{U},$$

and $$\frac{\cos^2\varphi}{NU}=\frac{1+n-n\sin^2\varphi-1}{nNU}=\frac{1}{nU}-\frac{(1-n)}{nNU},$$

whence $$\frac{n\sin^2\varphi\cos^2\varphi}{NU}=\frac{1}{nU}-\frac{\cos^2\varphi}{U}-\frac{(1-n)}{nNU}, \quad \text{and} -\frac{Vn\sin^2\varphi}{NU}=\frac{V}{U}-\frac{V}{NU}.$$

Finding similar expressions for the functions of ω and χ, and recollecting that, as in (362.), $\cos^2\varphi+\cos^2\chi+\cos^2\omega=1+2V-i^2U^2$, we shall have, making $W=1-n+nV$,

$$nU(n\Phi+nX-n\Omega)=3-n+nV+ni^2U^2-W\Big[\frac{1}{N_\varphi}+\frac{1}{N_\chi}+\frac{1}{N_\omega}\Big],$$

Now $$\int d\varphi\sqrt{I_\varphi}+\int d\chi\sqrt{I_\chi}-\int d\omega\sqrt{I_\omega}=i^2U, \text{ whence}$$

$$nU\Big[n\Phi+nX-n\Omega-\Big(\int d\varphi\sqrt{I}+\int d\chi\sqrt{I}-\int d\omega\sqrt{I}\Big)\Big]=2-W\Big[\frac{1}{N_\varphi}+\frac{1}{N_\chi}+\frac{1}{N_\omega}-1\Big]. \quad (369.)$$

We shall find, after some complicated calculations, $N_\varphi N_\chi N_\omega=W^2-n^2\varkappa U^2$, . (370.)

and $N_\chi N_\omega+N_\omega N_\varphi+N_\varphi N_\chi=W^2+2W-n(1-n)(i^2+m)U^2$. . . . . (371.)

Substituting the values hence derived, the whole expression becomes divisible by nU^2, and we shall obtain, finally, the following expression,

$$\frac{\sqrt{mn}}{n-m}[n\Phi+nX-n\Omega-i^2U]=\frac{n\sqrt{\varkappa}WU}{W^2-n^2\varkappa U^2}+\frac{2mn^2\sqrt{\varkappa}UV}{(n-m)(W^2-n^2\varkappa U^2)}. \quad (372.)$$

It may easily be shown, that

$$-\sqrt{\varkappa}\Big[\int\frac{d\varphi}{N_\varphi\sqrt{I}}+\int\frac{d\chi}{N_\chi\sqrt{I}}-\int\frac{d\omega}{N_\omega\sqrt{I}}\Big]=\frac{1}{2}\log\Big[\frac{1-n+nV+n\sqrt{\varkappa}U}{1-n+nV-n\sqrt{\varkappa}U}\Big]. \quad (373.)$$

or writing, as before, W for $1-n+n$V, and multiplying numerator and denominator by the numerator,

$$-\sqrt{\varkappa}\left[\int\frac{d\varphi}{N_\varphi\sqrt{I}}+\int\frac{d\chi}{N_\chi\sqrt{I}}-\int\frac{d\omega}{N_\omega\sqrt{I}}\right]=\log\left[\frac{W+n\sqrt{\varkappa}U}{\sqrt{W^2-n^2\varkappa U^2}}\right]. \quad . \quad . \quad . \quad . \quad (374.)$$

Now let
$$\frac{n\sqrt{\varkappa}U}{W}=\sin\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (375.)$$

and the preceding logarithm becomes $\log(\sec\xi+\tan\xi)$, which is, we know, the integral of $\int\frac{d\xi}{\cos\xi}$.

Now
$$\frac{n\sqrt{\varkappa}WU}{W^2-n^2\varkappa U^2}=\sec\xi\tan\xi;\ \text{and as}\ 2\int\frac{d\xi}{\cos^3\xi}=\sec\xi\tan\xi+\int\frac{d\xi}{\cos\xi},$$

we shall have, dividing by 2,

$$\Sigma_\omega-\Sigma_\varphi-\Sigma_\chi=k\int\frac{d\xi}{\cos^3\xi}+\frac{kmn^2\sqrt{\varkappa}UV}{(n-m)(W^2-n^2\varkappa U^2)}. \quad . \quad . \quad . \quad . \quad . \quad (376.)$$

Hence the sum of three arcs of a logarithmic ellipse may be expressed by an arc of a parabola and a right line.

When one of the arcs Σ_ω is a quadrant, $V=0$, and the equation becomes

$$\left[\Sigma_{\frac{\pi}{2}}-\Sigma_\chi\right]-\Sigma_\varphi=k\int\frac{d\xi}{\cos^3\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (377.)$$

which coincides with (160.).

If we apply to (163.) the same process, step by step, and make $\sin\zeta=\frac{m\sqrt{\varkappa_{,}}U}{W_{,}}$, in which $W_{,}=1-m+mV$, we shall have

$$\Sigma_\omega-\Sigma_\chi-\Sigma^\varphi=-k\int\frac{d\zeta}{\cos^3\zeta}+\frac{km^2n\sqrt{\varkappa_{,}}UV}{(n-m)(W_{,}^2-m^2\varkappa_{,}U^2)}+k\int\frac{d\tau}{\cos^3\tau}+k\int\frac{d\tau_{,}}{\cos^3\tau_{,}}-k\int\frac{d\tau_{,,}}{\cos^3\tau_{,,}}. \quad . \quad (378.)$$

If we subtract this equation from (376.), we shall have

$$\int\frac{d\xi}{\cos^3\xi}+\int\frac{d\zeta}{\cos^3\zeta}=\int\frac{d\tau}{\cos^3\tau}+\int\frac{d\tau_{,}}{\cos^3\tau_{,}}-\int\frac{d\tau_{,,}}{\cos^3\tau_{,,}}+\frac{mn}{n-m}UV\left[\frac{m\sqrt{\varkappa_{,}}}{W_{,}^2-m\varkappa_{,}U^2}-\frac{n\sqrt{\varkappa}}{W^2-n^2\varkappa U^2}\right]. \quad . \quad (379.)$$

Now this last term is divisible by $(n-m)$, and may be reduced to the expression

$$\frac{mn\sqrt{mn}UV\left[V^2+j^2mnU^2-j^2(1-V)^2\right]}{\left[W^2-n^2\varkappa U^2\right]\left[W_{,}^2-m^2\varkappa_{,}U^2\right]}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (380.)$$

If in (170.), which gives the relation between conjugate elliptic integrals of the third order, we substitute successively φ, χ and ω, and add the equations thence resulting, we shall have

$$\int\frac{d\xi}{\cos\xi}+\int\frac{d\zeta}{\cos\zeta}=\int\frac{d\tau}{\cos\tau}+\int\frac{d\tau_{,}}{\cos\tau_{,}}-\int\frac{d\tau_{,,}}{\cos\tau_{,,}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (381.)$$

in which

$$\left.\begin{array}{l}\sin\xi=\dfrac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\frac{n}{1-n}\cos\varphi\cos\chi\cos\omega},\quad \sin\zeta=\dfrac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\frac{m}{1-m}\cos\varphi\cos\chi\cos\omega}\\[2ex] \sin\tau=\dfrac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}},\quad \sin\tau_{,}=\dfrac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}},\quad \sin\tau_{,,}=\dfrac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}.\end{array}\right\} \quad . \quad . \quad . \quad (382.)$$

If, in these equations, we change n into $-n$, and therefore $\sin\xi$ into $\sqrt{-1}\tan\Theta$, $\sin\zeta$ into $\sqrt{-1}\tan\Theta'$,

$$\sin\tau \text{ into } \sqrt{-1}\tan\tau,\ \sin\tau_{,} \text{ into } \sqrt{-1}\tan\tau_{,},\ \text{and } \sin\tau_{,,} \text{ into } \sqrt{-1}\tan\tau_{,,},$$

the preceding equations will become

$$\left.\begin{aligned}\tan\Theta=\frac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1-\frac{n}{1+n}\cos\varphi\cos\chi\cos\omega},\quad \tan\Theta'=\frac{\sqrt{mn}\sin\varphi\sin\chi\sin\omega}{1+\frac{m}{1-m}\cos\varphi\cos\chi\cos\omega}\\ \tan\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}},\quad \tan\tau_{,}=\frac{\sqrt{mn}\sin\chi\cos\chi}{\sqrt{1-i^2\sin^2\chi}},\quad \tan\tau_{,,}=\frac{\sqrt{mn}\sin\omega\cos\omega}{\sqrt{1-i^2\sin^2\omega}}.\end{aligned}\right\}\quad . \quad . \quad (383.)$$

and $\Theta+\Theta_{,}=\tau+\tau_{,}-\tau_{,,}$, as in (360.), values which coincide with those found in LXIX. for the circular form. Or we may pass from the logarithmic to the circular form, or from the paraboloid to the sphere, or inversely, by the imaginary transformations above referred to.

We shall find on trial, that the angles υ, υ' and τ in (279.) fulfil the condition of conjugate amplitudes.

Section IX.—*On the Maximum Protangent Arcs of Hyperconic Sections.*

LXXII. Since the protangents vanish at the summits of these curves, there must be some intermediate position at which they attain their maximum. When the curve has but one summit, as is the case in the parabola, the hyperbola, the logarithmic parabola, and the logarithmic hyperbola, there evidently can be no maximum*.

In the plane ellipse, the protangent $t=\frac{ai^2\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}$. If we differentiate this expression with respect to φ, and make the differential coefficient $\frac{dt}{d\varphi}=0$, we shall get

$$\tan\varphi=\frac{1}{\sqrt{j}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (384.)$$

Substituting this value of $\tan\varphi$ in the preceding expression,

$$t=a-b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (385.)$$

In this case, the arcs drawn from the vertices of the curve, and which are compared together, have a common extremity, or they together constitute the quadrant.

The coordinates x, y of the arc measured from the vertex of the minor axis are $x=a\sin\vartheta$, $y=b\cos\vartheta$, therefore $\frac{y}{x}=\frac{b}{a}\cot\vartheta=j\cot\vartheta$, since $ja=b$. If we now make $\cot\vartheta=\sqrt{j}$, $\frac{y}{x}=j^{\frac{3}{2}}$. Again, as $\tan\lambda=\frac{a^2}{b^2}\frac{y'}{x'}$, $\frac{y'}{x'}=j^2\tan\lambda$; or making $\lambda=\vartheta$, or $\tan\lambda=\frac{1}{\sqrt{j}}$,

* The investigation of these particular values of those portions of the tangent arcs to the curves, which lie between the points of contact and the perpendicular arcs from the origin upon them—or as they have been termed in this paper, protangent arcs—is of importance; because, as we shall show in the next section, in the different series of derived hyperconic sections, the maximum protangent arc of any curve in the series, becomes a parameter in the integral of the curve immediately succeeding.

$\frac{y'}{x'}=j^2$, or $\frac{y'}{x'}=\frac{y}{x}$. Therefore the arcs have a common extremity. We have also $\tan^2\lambda=\frac{a}{b}$. This property of the plane ellipse, called FAGNANO's theorem, may be found in any elementary treatise on elliptic functions. See HYMER's Integral Calculus, p. 209.

On the Maximum Protangent Arc in the Spherical Hyperconic Section.

LXXIII. If we assume the expression found for this arc τ in (45.),

$$\tan\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}, \quad \ldots \ldots \ldots \quad (386.)$$

and differentiate it, as in the last article, and make $\frac{d\tau}{d\varphi}=0$, we shall find, as before,

$$\tan\varphi=\frac{1}{\sqrt{j}}=\sqrt{\frac{\sin\alpha}{\sin\beta}}. \quad \ldots \ldots \ldots \quad (387.)$$

If we substitute this value of $\tan\varphi$ in the preceding expression, we shall obtain

$$\tan\bar{\tau}=\tan\alpha\sec\beta-\tan\beta\sec\alpha, \quad \ldots \ldots \ldots \quad (388.)$$

writing $\bar{\tau}$ to denote the maximum protangent.

Now if we turn to Art. LVIII., we shall there find that this equation connects the amplitudes of three conjugate arcs of a plane parabola. Or if $\bar{\tau}$, β, and α are made the three normal angles of a plane parabola, and $(k.\bar{\tau})$, $(k.\beta)$, $(k.\alpha)$ the three corresponding arcs of the parabola, we shall have

$$(k.\alpha)-(k.\beta)-(k.\tau)=k\tan\alpha\tan\beta\tan\tau.$$

If in (386.) we substitute for $\sin\varphi$ and $\cos\varphi$ their values $\frac{1}{\sqrt{1+j}}$ and $\frac{\sqrt{j}}{\sqrt{1+j}}$, the expression will become

$$\tan\bar{\tau}=\frac{\sqrt{mn}}{(1+j)}. \quad \ldots \ldots \ldots \quad (389.)$$

We shall see the importance of this value of $\bar{\tau}$ in the next section.

In the spherical parabola, as $m=n=i$, $\tan^2\bar{\tau}=\frac{1-j}{1+j}=i_{,}$.

Precisely in the same manner as in the plane ellipse, we may show that when $\tan\tau$ has the preceding value, the arcs drawn from the vertices of the curve have a common extremity. This will be shown by proving that the vector arcs, drawn from the centre of the curve to the extremities of the compared arcs, have the same inclination to the principal arc 2α. Now ψ and ψ' being these inclinations, as in XIV., we find

$$\tan^2\lambda=\frac{\tan^4\alpha}{\tan^4\beta}\tan^2\psi,$$

and (39.) shows that $\tan\varphi=\cos\epsilon\tan\lambda$. Hence reducing,

$$\tan^2\psi=\frac{\tan^2\beta}{\tan^2\alpha}\frac{\sin^2\beta}{\sin^2\alpha}\tan^2\varphi. \quad \ldots \ldots \ldots \quad (a.)$$

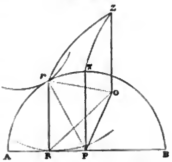
Again, (49.) shows, when we measure the arc from the minor principal arc, that $\cot\theta=\frac{a}{b}\frac{y}{x}$; or $\cot\theta=\frac{\sin\alpha}{\sin\beta}\tan\psi'$. Now in order that we may compare these arcs together, we must have $\theta=\lambda$. Hence

$$\tan^2\psi'=\frac{\tan^2\beta}{\tan^2\alpha}\cdot\frac{1}{\tan^2\varphi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b.)$$

When we substitute for φ any particular value, (a.) and (b.) will give the corresponding values of $\tan\psi$ and $\tan\psi'$; but when we make $\tan^2\varphi=\frac{\sin\alpha}{\sin\beta}=\frac{1}{j}$, the values of ψ and ψ' become equal, or the compared arcs together constitute the quadrant.

LXXIV. To determine the inclination, to the horizontal plane, of the tangent drawn to any point of the spherical ellipse. The spherical ellipse being taken as the curve of intersection of a cylinder by a sphere as in (X.), through a side Rr of the cylinder let a plane be drawn, it will cut the sphere in a *small* circle, which will touch the spherical ellipse in the point r, and will cut the base of the hemisphere in the right line RP, which touches the base of the cylinder at the point R. Let O be the centre of the sphere and Z the centre of the spherical hyperconic. Through the line OZ let a plane be drawn at right angles to the plane of the small circle R$r\varpi$P, it will cut the sphere in the arc of a great circle Zϖ at right angles to the arc $r\varpi$; and as the three planes, namely, the horizontal plane, the plane of the small circle, and the plane of the great circle ZOPϖ, are mutually at right angles, the right lines in which they intersect PR, Pϖ, PO are mutually at right angles, therefore P is the foot of the perpendicular drawn from the centre O of the base of the cylinder, to the tangent RP which touches the curve. P is also the centre of the small circle A$r\varpi$, since AB is a cord of the sphere. Hence Aϖ is a quadrant, and therefore, $r\varpi$ or ν is the inclination of the element of the spherical ellipse at r to the base of the hemisphere. Now ZO is the radius of the sphere, and Pr that of the small circle. RPO is a right angle, and therefore $\overline{OR}^2=\overline{OP}^2+\overline{PR}^2$. Hence $\overline{Rr}^2=\overline{Or}^2-\overline{OR}^2$. Now for the moment putting A and B for the semiaxes of the base of the cylinder, $\overline{OP}^2=A^2\cos\lambda+B^2\sin\lambda$, and

Fig. 26.



$$\overline{RP}^2=\frac{(A^2-B^2)^2\sin^2\lambda_{\prime}\cos^2\lambda_{\prime}}{A^2\cos^2\lambda_{\prime}+B^2\sin^2\lambda_{\prime}}. \quad \text{Whence } \overline{OR}^2=\frac{A^4\cos^2\lambda_{\prime}+B^4\sin^2\lambda_{\prime}}{A^2\cos^2\lambda_{\prime}+B^2\sin^2\lambda_{\prime}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a.)$$

and therefore $\overline{Rr}^2=\overline{Or}^2-\frac{A^4\cos^2\lambda_{\prime}+B^4\sin^2\lambda_{\prime}}{A^2\cos^2\lambda_{\prime}+B^2\sin^2\lambda_{\prime}}$. Let $Or=1$, $A=\sin\alpha$, $B=\sin\beta$, . . (b.)

and as $\tan^2\nu=\frac{\mathrm{RP}^2}{\overline{Rr}^2}$, $\tan^2\nu=\frac{(\sin^2\alpha-\sin^2\beta)^2\sin^2\lambda_{\prime}\cos^2\lambda_{\prime}}{\sin^2\alpha\cos^2\alpha\cos^2\lambda_{\prime}+\sin^2\beta\cos^2\beta\sin^2\lambda_{\prime}}$.

Let, as in (25.), $\tan\lambda_{,}=\cos\iota\tan\varphi=\frac{\cos\alpha}{\cos\beta}\tan\varphi$. Substituting, we get the expression

$$\tan\nu=\frac{\sin\iota\sin\eta\sin\varphi\cos\varphi}{\sqrt{(1-\sin^2\iota\sin^2\varphi)(1-\sin^2\eta\sin^2\varphi)}}. \quad \ldots \ldots \ldots \quad (390.)$$

In supplemental spherical ellipses, since $\sin\eta$ and $\sin\iota$* are respectively equal to $\sin\iota'$ and $\sin\eta'$, we infer, therefore, that in supplemental spherical ellipses the inclinations to the plane of xy of the tangents to the curves are the same, when the amplitudes φ are the same.

If we now differentiate this expression, and make $\frac{d\nu}{d\varphi}=0$, we shall find that $\tan^2\varphi=\frac{\tan\alpha}{\tan\beta}$. If we substitute this value of $\tan\varphi$ in (390.), we shall get

$$\tan\nu=\tan(\alpha-\beta), \text{ or } \nu=\alpha-\beta. \quad \ldots \ldots \ldots \quad (391.)$$

Hence the maximum inclination to the plane of xy of the tangent to the spherical ellipse is equal to the difference between the principal semiarcs. It is remarkable that the point of the curve which gives the maximum difference between the arcs, which together constitute the quadrant of the spherical ellipse, is not the point of greatest inclination. For this point is found by making $\tan^2\varphi=\frac{\tan\alpha}{\tan\beta}$; while the point of maximum difference is obtained by putting $\tan^2\varphi=\frac{\sin\alpha}{\sin\beta}$. This is the more worthy of notice, as we shall find the two points—the point of maximum division, and the point of greatest inclination—to coincide in the logarithmic ellipse.

If we take the two plane ellipses which are the projections of the spherical ellipse, one being the perspective, and the other the orthogonal projection, and seek on these plane ellipses their points of maximum division, we shall find that the angles, which the perpendiculars on the tangents, through these points of maximum division of those plane curves, make with the principal arc, are the values which must be assigned to the amplitude φ, to determine the point where the tangent to the curve has the greatest inclination to the plane of xy, and the point which divides the quadrant into two parts, such that their difference shall be a maximum. This is plain; for the semiaxes of one ellipse are $k\tan\alpha$, $k\tan\beta$; while the semiaxes of the other are $k\sin\alpha$ and $k\sin\beta$. And these angles are given by the equations

$$\tan^2\lambda=\frac{\tan\alpha}{\tan\beta}; \text{ and } \tan^2\lambda_{,}=\frac{\sin\alpha}{\sin\beta}.$$

On the Maximum Protangent in the Logarithmic Ellipse.

LXXV. If we follow the steps previously indicated, and differentiate the expression found in (165),

$$\sin\tau=\frac{\sqrt{mn}\sin\varphi\cos\varphi}{\sqrt{1-i^2\sin^2\varphi}}, \quad \ldots \ldots \ldots \quad (a.)$$

* Theory of Elliptic Integrals, p. 19.

$\bar{\tau}$ being the normal angle of the tangent parabolic arc to the logarithmic ellipse, this, evidently, will be a maximum when the parabolic arc is a maximum. Put the differential coefficient $\frac{d\tau}{d\varphi}=0$. This gives, as before, $\tan\varphi=\frac{1}{\sqrt{j}}$. Substituting this expression in (a.), we get

$$\sin\bar{\tau}=\frac{\sqrt{mn}}{(1+j)}. \qquad (392.)$$

We shall find the importance of this expression in the next section.

From (392.) we derive $\tan^2\bar{\tau}=\frac{mn}{(1+j)^2-mn}$.

Now $(1+j)^2=2+2j-i^2=2+2j-m-n+mn$. Hence as

$$j=\sqrt{(1-m)(1-n)},\ (1+j)^2-mn=[\sqrt{1-m}+\sqrt{1-n}]^2.$$

Whence we get $\tan\bar{\tau}=\frac{\sqrt{mn}}{\sqrt{1-m}+\sqrt{1-n}}$. Multiply this equation, numerator and denominator by $\sqrt{1-m}-\sqrt{1-n}$, and the last expression will become

$$\tan\bar{\tau}=\frac{\sqrt{mn}\sqrt{1-m}}{n-m}-\frac{\sqrt{mn}\sqrt{1-n}}{n-m}.$$

In (171.) we found for the semiaxes of the cylinder, whose intersection with the paraboloid is the logarithmic ellipse, $\frac{a}{k}=\frac{\sqrt{mn}\sqrt{1-m}}{n-m}$, $\frac{b}{k}=\frac{\sqrt{mn}\sqrt{1-n}}{n-m}$.

Hence $$\tan\bar{\tau}=\left(\frac{a-b}{k}\right). \qquad (393.)$$

This gives a simple expression for the tangent of the maximum parabolic arc, analogous to (385.) and (391.). We have only to take in the parabola, whose semiparameter is k, an arc whose ordinate is $a-b$, to determine the maximum protangent parabolic arc.

The value $\tan\varphi=\frac{1}{\sqrt{j}}$, which fixes the position and magnitude of the maximum protangent arc to the logarithmic ellipse, renders $\tan^2\lambda=\frac{a}{b}$. For (150.) gives $\tan^2\varphi=\frac{\alpha+\beta}{\alpha}\tan^2\lambda$. But (152.) gives $\frac{\alpha+\beta}{\alpha}=\frac{C}{C-B}$, and $\frac{C}{C-B}=\frac{1}{1-m}$,

hence $\tan^2\varphi=\frac{\tan^2\lambda}{1-m}$. If we now make

$$\tan^2\varphi=\frac{1}{j}=\frac{1}{\sqrt{(1-n)(1-m)}} \quad \tan^2\lambda=\sqrt{\frac{1-m}{1-n}}=\frac{a}{b},$$

as we may infer from (171.). Now substituting this value of $\tan^2\lambda$ in (155.), we shall get

$$\tan\tau=\frac{a-b}{k}.$$

Again, if we differentiate the values of x, y, z given in (158.), the coordinates of the extremity of the arc measured from the minor axis, and substitute them in the general

expression for the tangent of the inclination of any curve to the plane of xy, namely, $\frac{dz}{\sqrt{dx^2+dy^2}}$, and make $\vartheta=\lambda$, we shall get, as before, putting for $\tan^2\lambda=\tan^2\vartheta$, the value $\frac{a}{b}$, $\frac{dz}{\sqrt{dx^2+dy^2}}=\frac{a-b}{k}$. Hence the arcs have a common extremity, since they have the same inclination to the plane of xy. As $\frac{a}{b}=\tan^2\lambda$ is the value of $\tan^2\lambda$, which gives the maximum protangent $=a-b$ in the plane ellipse, the base of the cylinder; it follows that the point of maximum division on the logarithmic ellipse is orthogonally projected into the point of maximum division on the plane ellipse; and the corresponding protangent in the latter $a-b$ is the ordinate of the parabolic arc, which expresses the difference between the corresponding arcs of the former. Thus, while the arcs which together constitute the quadrant on the plane ellipse, differ by the difference of the semiaxes $a-b$, the corresponding arcs of the logarithmic ellipse will differ by an arc of a parabola whose ordinate is $a-b$.

LXXVI. When the amplitude φ is given by the equation $\tan\varphi=\frac{1}{\sqrt{j}}$, or when the protangent is a maximum, the corresponding arc of the spherical ellipse, or of the logarithmic ellipse, may be expressed by functions of the first and second orders only. This may be shown as follows. When $\tan\varphi=\frac{1}{\sqrt{j}}$ the arcs σ and $\sigma_{\prime}$ of the spherical ellipse, or the arcs Σ and S of the logarithmic ellipse, together make up the quadrant C. Hence $\sigma+\sigma_{\prime}=C$, or $\Sigma+S=C$. But we have also $\sigma_{\prime}-\sigma=\tau$, as in (52.), and $S-\Sigma=\tau$, as in (160.). Therefore

$$\sigma=\frac{C-\tau}{2}\quad \sigma_{\prime}=\frac{C+\tau}{2},\quad S=\frac{C+\tau}{2},\quad \Sigma=\frac{C-\tau}{2}.$$

Or σ and $\sigma_{\prime}$, or Σ and S may be expressed as simple functions of C and τ. Now C, the quadrant, as we have shown in the last section, may be expressed by functions of the first and second orders only, while τ is an arc either of a circle or of a parabola. Hence an elliptic integral of the third order, whose amplitude $\varphi=\tan^{-1}\left(\frac{1}{\sqrt{j}}\right)$, may be expressed by functions of the first and second orders only.

Section X.—*On Derivative Hyperconic Sections.*

LXXVII. We shall now proceed to show that, when a hyperconic section is given, whether it be spherical or paraboloidal, we may from it derive a series of curves, whose moduli and parameters shall decrease or increase according to a certain law; so that ultimately the rectification of these curves may be reduced to the calculation of circular or parabolic arcs, or in other words, to circular functions or logarithms. We shall also show that all these derived curves, together with the original curve, may be traced on the *same* generating surface, *i. e.* on the same sphere or paraboloid.

In (186.) we have shown that the rectification of a plane ellipse whose semiaxes are a and b, may be reduced to the rectification of another plane ellipse whose semiaxes $a_{\prime}$, $b_{\prime}$ are given by the equations $a_{\prime}=a+b$, $b_{\prime}=2\sqrt{ab}$, of which the eccentricity is less than that of the former. $a+b$ is that portion of the tangent, drawn through the point of maximum division, which lies between the axes; and $\sqrt{ab}$ is the perpendicular from the centre on it.

We have shown in (63.) and (74.), that if φ and ψ are connected by the equation

$\tan(\psi-\varphi)=j\tan\varphi$; while i and $i_{\prime}$ are so related, that $i_{\prime}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}=\frac{1-j}{1+j}$,

we shall have

$$\int\frac{d\varphi}{\sqrt{1-i^2\sin^2\varphi}}=\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{1-i_{\prime}^2\sin^2\psi}}=\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}}.$$

Let us now introduce this suggested transformation into the elliptic integral of the third order, *circular* form and *negative* parameter. In (191.) we found

$$2\sin^2\varphi=1+i_{\prime}\sin^2\psi-\cos\psi\sqrt{\mathrm{I}_{\prime}}.$$

Now
$$\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=\int\frac{d\varphi}{[1-m\sin^2\varphi]\sqrt{1-i^2\sin^2\varphi}}.$$

Or replacing φ by its equivalent functions in ψ, and recollecting that $m-n+mn=i^2$, since m and n are conjugate parameters, we shall find

$$\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=(1+i_{\prime})\int\frac{d\psi}{[2-m-mi_{\prime}\sin^2\psi+m\cos\psi\sqrt{\mathrm{I}_{\prime}}]\sqrt{\mathrm{I}_{\prime}}}. \quad . \quad . \quad . \quad . \quad (394.)$$

We may eliminate the radical $m\cos\psi\sqrt{\mathrm{I}_{\prime}}$ from the denominator of this expression, by treating it as the sum of two terms.

Multiplying and dividing the function by their difference, since $1+i_{\prime}=\frac{2}{1+j}$,

$$4(1-m)\int\frac{d\varphi}{\mathrm{M}\sqrt{\mathrm{I}}}=(1+i_{\prime})\int\frac{d\psi[2-m-mi_{\prime}\sin^2\psi-m\cos\psi\sqrt{\mathrm{I}_{\prime}}]}{\left[1+\frac{mn}{(1+j)^2}\sin^2\psi\right]\sqrt{\mathrm{I}_{\prime}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (395.)$$

Now it is truly remarkable that whether the parameter of the original function we start from be positive or negative, the parameter of the first derived integral will always be positive. Indeed it is necessary that this should be the case, because the parameters of the derived functions, increasing or diminishing as they do, must at length pass from between the limits 1 and i^2. Should they do so, the integral would be no longer of the circular form, but of the logarithmic. Now we cannot pass from one of these forms to the other by any but an imaginary transformation. This objection does not hold when the parameter is positive, because the limits of the positive parameter are 0 and ∞. It is, too, worthy of remark, that the first derived parameter is always the same, whether we transform from positive or negative parameters. Write

$$n_{\prime}=\frac{mn}{(1+j)^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (396.)$$

$n_{\prime}$ is the first derived parameter.

We may transform (395.) into

$$4(1-m)\int\frac{d\phi}{M\sqrt{I}}=(1+i_{\prime})\int\frac{d\psi\left[2-m-\frac{mi_{\prime}}{n_{\prime}}(1+n_{\prime}\sin^2\psi-1)-m\cos\psi\sqrt{I}\right]}{[1+n_{\prime}\sin^2\psi]\sqrt{I_{\prime}}}.$$

Now $\qquad \frac{mi_{\prime}}{n_{\prime}}=\frac{i^2}{n}$, and $2-m+\frac{mi_{\prime}}{n_{\prime}}=\frac{m+n}{n}$. . . . . (397.)

Hence

$$2\frac{(1-m)}{m}\int\frac{d\phi}{M\sqrt{I}}=\frac{(m+n)\sqrt{n_{\prime}}}{mn.\sqrt{mn}}\int\frac{d\psi}{[1+n_{\prime}\sin^2\psi]\sqrt{I_{\prime}}}-\frac{(1+i_{\prime})}{2}\frac{i^2}{mn}\int\frac{d\psi}{\sqrt{I}}-\frac{(1+i_{\prime})}{2\sqrt{n_{\prime}}}\tan^{-1}(\sqrt{n_{\prime}}\sin\psi). \quad (398.)$$

We shall now show that $\qquad \frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}=\sqrt{n_{\prime}}\sin\psi$. . . . . (399.)

If we revert to (189.) and (193.), we there find

$$2\sin\phi\cos\phi=\sin\psi[\sqrt{I_{\prime}}+i_{\prime}\cos\psi], \text{ and } 2\sqrt{I}=(1+j)[\sqrt{I_{\prime}}+i_{\prime}\cos\psi].$$

(396.) gives $\sqrt{mn}=\sqrt{n_{\prime}}(1+j)$; therefore $\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}=\sqrt{n_{\prime}}\sin\psi$.

If we replace $\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{I_{\prime}}}$ in the preceding equation by its value $\int\frac{d\phi}{\sqrt{I}}$, and put $N_{\prime}$ for $1+n_{\prime}\sin^2\psi$,

$$2\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}=\frac{(m+n)}{mn}\sqrt{\frac{n_{\prime}}{mn}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}-\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}-\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]. \quad (400.)$$

Now the common formula for comparing circular integrals with conjugate parameters is, we know, see (47.),

$$\left(\frac{1+n}{n}\right)\int\frac{d\phi}{N\sqrt{I}}-\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}=\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{I}}+\frac{1}{\sqrt{mn}}\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right].$$

Adding these equations we obtain this new formula

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{N\sqrt{I}}+\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{M\sqrt{I}}=\left(\frac{m+n}{mn}\right)\sqrt{n_{\prime}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}. \quad (401.)$$

By the help of this important formula we may establish a simple relation between the sum of the original conjugate functions of the third order, and the first derived function of this order.

LXXVIII. If σ be the arc of a spherical ellipse, it is shown in (46.) that

$$\sigma=\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{N\sqrt{I}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right],$$

and in (17.) that $\qquad \sigma=\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{M\sqrt{I}}.$

Adding these equations together, and introducing the relation just now established,

$$2\sigma=\frac{(m+n)}{mn}\sqrt{n_{\prime}}\int\frac{d\psi}{N_{\prime}\sqrt{I_{\prime}}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}\right]. \quad (402.)$$

Now as $m-n=i^2-mn$, $\quad (m+n)^2=i^4-2i^2mn+m^2n^2+4mn$.

We have also $mn=n_{,}(1+j)^2$, $i_{,}=\frac{1-j}{1+j}$, and $i^2=(1-j)(1+j)$, hence

$$m+n=(1+j)^2(1+n_{,})\sqrt{m_{,}}, \quad \ldots \quad (403.)$$

and therefore
$$\left(\frac{m+n}{mn}\right)\sqrt{n_{,}}=\left(\frac{1+n_{,}}{n_{,}}\right)\sqrt{m_{,}n_{,}}. \quad \ldots \quad (404.)$$

It is worthy of especial remark that this coefficient of $\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}$ is precisely the same in form as the coefficient of $\int\frac{d\phi}{N\sqrt{I}}$.

The preceding equation (402.) may now be written,

$$2\sigma=\left(\frac{1+n_{,}}{n_{,}}\right)\sqrt{m_{,}n_{,}}\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}-\frac{i^2}{\sqrt{mn}}\int\frac{d\phi}{\sqrt{I}}-\tan^{-1}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]. \quad \ldots \quad (405.)$$

Let $\sigma_{,}$, $n_{,}$, $i_{,}$, ψ be analogous quantities for the derived spherical ellipse $\sigma_{,}$,

$$\sigma_{,}=\left(\frac{1+n_{,}}{n_{,}}\right)\sqrt{m_{,}n_{,}}\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}-\frac{i^2_{,}}{\sqrt{m_{,}n_{,}}}\int\frac{d\psi}{\sqrt{I_{,}}}-\tan^{-1}\left[\frac{\sqrt{m_{,}n_{,}}\sin\psi\cos\psi}{\sqrt{I_{,}}}\right]. \quad \ldots \quad (406.)$$

Let q, $q_{,}$, $q_{,,}$, $q_{,,,}$, &c. denote $\frac{i^2}{\sqrt{mn}}$, $\frac{i^2_{,}}{\sqrt{m_{,}n_{,}}}$, $\frac{i^2_{,,}}{\sqrt{m_{,,}n_{,,}}}$, &c., and put r, $r_{,}$, $r_{,,}$, $r_{,,,}$, &c. for $(1+j)$, $(1+j)(1+j_{,})$, $(1+j)(1+j_{,})(1+j_{,,})$, $(1+j)(1+j_{,})(1+j_{,,})(1+j_{,,,})$, &c. Let also Ω, Ψ, $\Psi_{,}$, $\Psi_{,,}$, &c. denote the arcs, whose tangents are

$$\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}, \quad \frac{\sqrt{m_{,}n_{,}}\sin\psi\cos\psi}{\sqrt{1-i^2_{,}\sin^2\psi}}, \quad \frac{\sqrt{m_{,,}n_{,,}}\sin\psi_{,}\cos\psi_{,}}{\sqrt{1-i^2_{,,}\sin^2\psi_{,}}}, \text{ \&c.}$$

Making these substitutions, and writing Q, $Q_{,}$, $Q_{,,}$, &c. for the coefficients of

$\int\frac{d\phi}{N\sqrt{I}}$, $\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}$, $\int\frac{d\psi_{,}}{N_{,,}\sqrt{I_{,,}}}$, (405.) and (406.) become

$$2\sigma=Q_{,}\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}-q\int\frac{d\phi}{\sqrt{I}}-\Omega \quad \ldots \quad (a.) \qquad \sigma_{,}=Q_{,}\int\frac{d\psi}{N_{,}\sqrt{I_{,}}}-q_{,}r\int\frac{d\phi}{\sqrt{I}}-\Psi. \quad \ldots \quad (a_{,}.)$$

Taking the derivatives of these expressions, we may write

$$2\sigma_{,}=Q_{,,}\int\frac{d\psi_{,}}{N_{,,}\sqrt{I_{,,}}}-q_{,}r\int\frac{d\phi}{\sqrt{I}}-\Psi. \quad \ldots \quad (b.) \qquad \sigma_{,,}=Q_{,,}\int\frac{d\psi_{,}}{N_{,,}\sqrt{I_{,,}}}-q_{,,}r_{,}\int\frac{d\phi}{\sqrt{I}}-\Psi_{,}. \quad \ldots \quad (b_{,}.)$$

$$2\sigma_{,,}=Q_{,,,}\int\frac{d\psi_{,,}}{N_{,,,}\sqrt{I_{,,,}}}-q_{,,}r_{,}\int\frac{d\phi}{\sqrt{I}}-\Psi_{,}. \quad (c.) \qquad \sigma_{,,,}=Q_{,,,}\int\frac{d\psi_{,,}}{N_{,,,}\sqrt{I_{,,,}}}-q_{,,,}r_{,,}\int\frac{d\phi}{\sqrt{I}}-\Psi_{,,}. \quad \ldots \quad (c_{,}.)$$

Subtract $(a_{,}.)$ from $(a.)$, $(b_{,}.)$ from $(b.)$, and $(c_{,}.)$ from $(c.)$, the integrals of the third order disappear, and we shall have

$$\left.\begin{aligned}
2\sigma-\sigma_{,}&=(q_{,}r-q)\int\frac{d\phi}{\sqrt{I}}+\Psi-\Omega\\
2\sigma_{,}-\sigma_{,,}&=(q_{,,}r_{,}-q_{,}r)\int\frac{d\phi}{\sqrt{I}}+\Psi_{,}-\Psi\\
2\sigma_{,,}-\sigma_{,,,}&=(q_{,,,}r_{,,}-q_{,,}r_{,})\int\frac{d\phi}{\sqrt{I}}+\Psi_{,,}-\Psi_{,}\\
2\sigma_{,,,}-\sigma_{,,,,}&=(q_{,,,,}r_{,,,}-q_{,,,}r_{,,})\int\frac{d\phi}{\sqrt{I}}+\Psi_{,,,}-\Psi_{,,}
\end{aligned}\right\} \quad \ldots \quad (407.)$$

If we add these equations together,

$$\sigma+\sigma_{,}+\sigma_{,,}+\sigma_{,,,}+(\sigma-\sigma_{,,,,})=(q_{,,,}r_{,,,}-q)\int\frac{d\varphi}{\sqrt{I}}+\Psi_{,,,}-\Omega. \quad . \quad . \quad . \quad (408.)$$

If we multiply the first of (407.) by 2^3, the second by 2^2, the third by 2, and the fourth by 2^0, and add the results,

$$2^4\sigma-\sigma_{,,,,}=(q_{,,,}r_{,,,}+q_{,,}r_{,,}+2q_{,}r_{,}+4qr-8q)\int\frac{d\varphi}{\sqrt{I}}+(\Psi_{,,,}+\Psi_{,,}+2\Psi_{,}+4\Psi-8\Omega), \quad . \quad (409.)$$

an integral which enables us to approximate with ease to the value of the integral of the third order and circular form, in terms of an integral of the first order.

We have shown in XXVIII. how the integral of the first order may be reduced.

The above expressions may be reduced to simpler forms, when the functions are complete. In this case $\Omega=0$, $\Psi=0$, $\Psi_{,}=0$, $\Psi_{,,}=0$, &c.; and when σ is a quadrant, $\sigma_{,}$ will be two quadrants, $\sigma_{,,}$ will be four quadrants, $\sigma_{,,,}$ will be eight quadrants, and so on; the preceding expression may now be written, denoting a quadrant by the symbol $\breve{\sigma}$,

$$16(\breve{\sigma}-\breve{\sigma}_{,,,,})=(q_{,,,}r_{,,,}+q_{,,}r_{,,}+2q_{,}r_{,}+4qr-8q)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{I}}. \quad . \quad . \quad . \quad (410.)$$

In (396.) we found for the parameter of the derived integral of the third order, the expression $n_{,}=\frac{mn}{(1+j)^2}$. Or, referring to the geometrical representatives of these expressions, we found for the focal distance $\epsilon_{,}$ of this derived curve, the expression $n_{,}=\tan^2\epsilon_{,}=\frac{mn}{(1+j)^2}$; but if we turn to (389.) we shall see that this is the expression for the maximum protangent to the original spherical ellipse, which is given by the equation $\tan^2\tau=\frac{mn}{(1+j)^2}$. We thus arrive at this curious relation between the curves successively derived, that the maximum protangent of any one of the spherical ellipses becomes the focal distance of the one immediately succeeding in the series.

LXXIX. Given m, n and i, we may determine $m_{,}$, $n_{,}$ and $i_{,}$, for $i_{,}=\frac{1-j}{1+j}$, $n_{,}=\frac{mn}{(1+j)^2}$. Substituting these values of $i_{,}$ and $n_{,}$ in the equation which connects the parameters, $m_{,}-n_{,}+m_{,}n_{,}=i_{,}^2$,

$$m_{,}=\left[\frac{\sqrt{1+n}-\sqrt{1-m}}{\sqrt{1+n}+\sqrt{1-m}}\right]^2. \quad . \quad . \quad . \quad . \quad (411.)$$

Hence given m, n and i, we can easily compute the values of $m_{,}$, $n_{,}$ and $i_{,}$, and then of $m_{,,}$, $n_{,,}$ and $i_{,,}$; and so on as far as we please.

Given the semiaxes a and b of the elliptic cylinder, whose intersection with the sphere is the original spherical ellipse, to determine the semiaxes $a_{,}$ and $b_{,}$ of the cylinder, whose intersection with the sphere shall be the first derived spherical ellipse.

We may derive from (53.) and (54.) the values of a and b in terms of m, n and i, or eliminating i, in terms of m and n only. Now

$$\frac{a^2}{k^2}=\frac{n}{m(1+n)},\ \frac{b^2}{k^2}=\frac{n(1-m)}{m}. \quad \text{Hence } \frac{a_{,}^2}{k^2}=\frac{n_{,}}{m_{,}(1+n_{,})},\ \frac{b_{,}^2}{k^2}=\frac{n_{,}(1-m_{,})}{m_{,}}.$$

Or substituting the values of $m_{\prime}$ and $n_{\prime}$ in terms of m and n, and therefore of a and b,

$$a_{\prime}=\frac{a+b}{1+\frac{ab}{k^2}},\ b_{\prime}=\frac{2\sqrt{ab}}{1+\frac{ab}{k^2}}. \quad (412.)$$

When the radius of the sphere is infinite, or the derived curve is a plane ellipse, $a_{\prime}=a+b$, $b_{\prime}=2\sqrt{ab}$, as in LXXVII.

When $m=n=i$; $m_{\prime}=n_{\prime}=i_{\prime}$, or when the given curve is a spherical parabola, the derived curve will also be a spherical parabola. Hence all the curves of the series will be spherical parabolas.

If we take the corresponding integral of the third order with a reciprocal parameter l, such that $lm=i^2$, and derive by the foregoing process the first derived function of the third order, we shall find the parameter $l_{\prime}$ of this function to be positive also, and reciprocal to $n_{\prime}$, so that $l_{\prime}\,n_{\prime}=i^2_{\prime}$.

Hence, if we deduce a series of derived functions from two primitive functions of the third order and circular form, having either *positive* or *negative* reciprocal parameters, the parameters of all the derived functions $l_{\prime}$, $l_{\prime\prime}$, $l_{\prime\prime\prime}$, $n_{\prime}$, $n_{\prime\prime}$, $n_{\prime\prime\prime}$ will be *positive*, and reciprocal in pairs, so that $l_{\prime}n_{\prime}=i^2_{\prime}$, $l_{\prime\prime}n_{\prime\prime}=i^2_{\prime\prime}$, $l_{\prime\prime\prime}n_{\prime\prime\prime}=i^2_{\prime\prime\prime}$, &c.

LXXX. We may apply the same method of proceeding to the logarithmic ellipse, or to the logarithmic integral of the third order,

$$\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}},\ \text{in which } i^2>m.$$

If on this function we perform the operations effected on the similar integral in (394.), we shall have, after like reductions,

$$\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{(1+i_{\prime})}{4(1-m)}\int\frac{d\psi[2-m-mi_{\prime}\sin^2\psi-m\cos\psi\sqrt{\mathrm{I}_{\prime}}]}{[1-m_{\prime}\sin^2\psi]\sqrt{\mathrm{I}_{\prime}}}. \quad (413.)$$

We must recollect that

$$\mathrm{M}=1-m\sin^2\phi,\ \mathrm{M}_{\prime}=1-m_{\prime}\sin^2\psi,\ \mathrm{I}=1-i^2\sin^2\phi,\ \mathrm{I}_{\prime}=1-i^2_{\prime}\sin^2\psi,\ \text{and } m_{\prime}=\frac{mn}{(1+j)^2}. \quad (414.)$$

We may reduce this expression.

The numerator may be put under the form

$$2-m+\frac{mi_{\prime}}{m_{\prime}}\{1-m_{\prime}\sin^2\psi-1\}-m\cos\psi\sqrt{\mathrm{I}_{\prime}}.$$

Now $\quad 2-m-\frac{mi_{\prime}}{m_{\prime}}=\frac{(n-m)}{n}$, and $\frac{mi_{\prime}}{m_{\prime}}=\frac{i^2}{n}$. We have also $\frac{\sqrt{i_{\prime}}}{i}=\frac{1}{1+j}$.

Hence, making the necessary transformations,

$$2\frac{(1-m)}{m}\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{(n-m)}{mn}\frac{\sqrt{i_{\prime}}}{i}\int\frac{d\psi}{\mathrm{M}_{\prime}\sqrt{\mathrm{I}_{\prime}}}+\frac{i\sqrt{i_{\prime}}}{mn}\int\frac{d\psi}{\sqrt{_{\prime}\mathrm{I}}}-\frac{\sqrt{i_{\prime}}}{i}\int\frac{\cos\psi d\psi}{\mathrm{M}_{\prime}}.$$

If into this expression we introduce the relation given in (74.), $\int\frac{d\phi}{\sqrt{\mathrm{I}}}=\frac{(1+i_{\prime})}{2}\int\frac{d\psi}{\sqrt{\mathrm{I}_{\prime}}}$,

we shall have $2\left(\frac{1-m}{m}\right)\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}=\frac{(n-m)}{mn}\frac{\sqrt{i_{\prime}}}{i}\int\frac{d\psi}{\mathrm{M}_{\prime}\sqrt{\mathrm{I}_{\prime}}}+\frac{i^2}{mn}\int\frac{d\phi}{\sqrt{\mathrm{I}}}-\frac{\sqrt{i_{\prime}}}{i}\int\frac{\cos\psi d\psi}{\mathrm{M}_{\prime}}. \quad (415.)$

Now in (399.) it has been shown that $\sqrt{m_{,}}\sin\psi=\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}$, and as $\sqrt{mn}=\sqrt{m_{,}}(1+j)$, the last term of the preceding equation may be written

$$\frac{1}{\sqrt{mn}}\int\frac{\frac{d}{d\phi}\left[\frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{I}}\right]d\phi}{1-\frac{mn\sin^2\phi\cos^2\phi}{I}}.$$

Substituting this value in the preceding equation and comparing it with (169.) or (170.), we shall find

$$\left(\frac{1-m}{m}\right)\int\frac{d\phi}{M\sqrt{I}}-\left(\frac{1-n}{n}\right)\int\frac{d\phi}{N\sqrt{I}}=\frac{(n-m)}{nm}\frac{\sqrt{i}}{i}\int\frac{d\psi}{M_{,}\sqrt{I_{,}}}. \quad (416.)$$

This equation is analogous to (401.). By the help of it and the last equation we can always express

$$\int\frac{d\phi}{M\sqrt{I}} \text{ or } \int\frac{d\phi}{N\sqrt{I}} \text{ in terms of } \int\frac{d\psi}{M_{,}\sqrt{I_{,}}}.$$

Since $m_{,}=\frac{mn}{(1+j)^2}$ is symmetrical with respect to n and m, we should have obtained the same value for the derived parameter had it been deduced from $\int\frac{d\phi}{N\sqrt{I}}$ instead of $\int\frac{d\phi}{M\sqrt{I}}$.

Since $$m_{,}=\frac{mn}{(1+j)^2}, \quad n_{,}=\frac{(1-j)^2-mn}{(1+j)^2-mn}, \quad \text{or } n_{,}=\left[\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}\right]^2. \quad (417.)$$

LXXXI. We may express $m_{,}$ and $n_{,}$ simply, in terms of a and b, the semiaxes of the base of the elliptic cylinder, whose curve of section with the paraboloid is the logarithmic ellipse.

In (171.) we have found the values of m and n in terms of a, b and k, namely,

$$\frac{a}{k}=\frac{\sqrt{mn(1-m)}}{n-m}, \quad \frac{b}{k}=\frac{\sqrt{mn(1-n)}}{n-m}. \quad (a.)$$

Hence $\frac{a-b}{a+b}=\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}$, or assuming the value of $n_{,}$ in (417.) $n_{,}=\left(\frac{a-b}{a+b}\right)^2$.

Now $$n-m=(1-m)-(1-n)=(\sqrt{1-m}+\sqrt{1-n})(\sqrt{1-m}-\sqrt{1-n}).$$

Or as $$m_{,}=\frac{mn}{(1+j)^2}, \quad 1-m_{,}=\frac{(1+j)^2-mn}{(1+j)^2}=\frac{(\sqrt{1-m}+\sqrt{1-n})^2}{(1+j)^2},$$

and (a.) gives $$\frac{a-b}{k}=\frac{\sqrt{mn}}{\sqrt{1-m}+\sqrt{1-n}},$$

therefore $\frac{1-m_{,}}{m_{,}}=\frac{(\sqrt{1-m}-\sqrt{1-n})^2}{mn}$. Hence reducing, $m_{,}=\frac{(a-b)^2}{k^2+(a-b)^2}$.

If we now compare together these expressions for $m_{,}$ and $n_{,}$, namely,

$$m_{,}=\frac{(a-b)^2}{k^2+(a-b)^2}, \quad n_{,}=\left(\frac{a-b}{a+b}\right)^2, \quad (418.)$$

we shall find that $n_{,}>m_{,}$, so long as $k>2\sqrt{ab}$; that when $k=2\sqrt{ab}$, $m_{,}=n_{,}$; and that when $k<2\sqrt{ab}$, $n_{,}<m_{,}$.

To determine the axes of the base of the cylinder, whose intersection with the paraboloid gives the derived logarithmic ellipse.

Since $\frac{a_{,}^2}{k^2}=\frac{m_{,}n_{,}(1-m_{,})}{(n_{,}-m_{,})^2}$, $\frac{b_{,}^2}{k^2}=\frac{m_{,}n_{,}(1-n_{,})}{(n_{,}-m_{,})^2}$, as we may infer from (171.),

we shall have, substituting the preceding values of $m_{,}$ and $n_{,}$,

$$\frac{a_{,}^2}{k^2}=\frac{(a+b)^2k^2}{[k^2-4ab]^2},\quad \frac{b_{,}^2}{k^2}=\frac{4ab[k^2+(a-b)^2]}{[k^2-4ab]^2},\quad \text{and } i_{,}^2=\left(\frac{a-b}{a+b}\right)^2\left[\frac{k^2+(a+b)^2}{k^2+(a-b)^2}\right]. \quad . \quad . \quad (419.)$$

When $k=\infty$, or when the paraboloid is a plane, $a_{,}=(a+b)$, $b_{,}=2\sqrt{ab}$, which are the values of the semiaxes of a plane ellipse, whose eccentricity is $\frac{a-b}{a+b}=\frac{1-\sqrt{1-i^2}}{1+\sqrt{1-i^2}}$, as we should have anticipated, for these are the values found in LXXVII. and LXXIX. for the axes of the derived plane ellipse.

When $\quad m=n=1-j,\quad m_{,}=\frac{mn}{(1+j)^2}=\left(\frac{1-j}{1+j}\right)^2=i_{,}^2,\quad$ and $n_{,}=0$.

Hence, when the original logarithmic ellipse is of the circular form, the first derived ellipse is a plane ellipse.

When $k^2=4ab$, (418.) shows that $m_{,}=n_{,}$, or $\frac{a_{,}}{k}=\frac{b_{,}}{k}=\infty$, as in XLIII.; but $m_{,}=n_{,}$ is equivalent to $n=m(\sqrt{1+j}+\sqrt{j})^2$.

Whenever therefore this relation exists between the parameters and modulus of the original integral, the first derived integral will represent the circular logarithmic ellipse, which may be integrated by functions of the first and second orders. Accordingly whenever the above relation exists between the parameters, the integral of the third order may be reduced to others of the first and second orders.

If in the second, third, or any other of the derived logarithmic ellipses, we can make the parameters equal, this derived ellipse will be of the circular form, and its rectification may be effected by integrals of the first and second orders only; accordingly the rectification of all the ellipses which precede it in the scale, may be effected by integrals of the first and second orders only.

We may repeat the remark made in LXXIX. The derived functions of two integrals of the logarithmic form with reciprocal parameters, have themselves reciprocal parameters.

LXXXII. If we now add together (162.) and (163.), we shall have

$$\left.\begin{aligned}\frac{4(n-m)}{\sqrt{mn}}\,\frac{\Sigma}{k}&=-\left[n\Phi_{n}+m\Phi_{m}\right]+\left[\frac{i^2}{m}+\frac{i^2}{n}-2\right]\int\frac{d\phi}{\sqrt{\mathrm{I}}}\\ &-(n-m)\left[\frac{(1-m)}{m}\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}-\left(\frac{1-n}{n}\right)\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]+2\int d\phi\sqrt{\mathrm{I}}-2\frac{(n-m)}{\sqrt{mn}}\int\frac{d\tau}{\cos^2\tau}\end{aligned}\right\}. \quad . \quad . \quad (420.)$$

We must now reduce this equation into functions of ψ instead of ϕ; ψ and ϕ being connected, as before, by the fundamental equation

$$\tan(\psi-\phi)=j\tan\phi.$$

The elements of these transformations are given at page 358, namely,

$$2\sin^2\phi=1+i_{,}\sin^2\psi-\cos\psi\sqrt{\mathrm{I}_{,}}, \text{ and } \frac{\sqrt{mn}\sin\phi\cos\phi}{\sqrt{1-i^2\sin^2\phi}}=\sqrt{n_{,}}\sin\psi.$$

From this last equation we derive $(1-n\sin^2\phi)(1-m\sin^2\phi)=\mathrm{I}(1-m_{,}\sin^2\psi)$.

Now as $\Phi_n=\frac{\sin\phi\cos\phi\sqrt{\mathrm{I}}}{1-n\sin^2\phi}$, we shall have $n\Phi_n=\frac{\sqrt{m_{,}}\sin\psi}{2\sqrt{mn}}\left[\frac{2n-2nm\sin^2\phi}{1-m_{,}\sin^2\psi}\right]$. . . . (421.)

Or putting for $\sin^2\phi$ its value, $n\Phi_n=\frac{\sqrt{m_{,}}\sin\psi}{2\sqrt{mn}}\frac{[2n-nm-nmi_{,}\sin^2\psi+mn\cos\psi\sqrt{\mathrm{I}_{,}}]}{[1-m_{,}\sin^2\psi]}$. (422.)

In the same manner, we may find

$$m\Phi_m=\frac{\sqrt{m_{,}}\sin\psi}{2\sqrt{mn}}\frac{[2m-mn-mni_{,}\sin^2\psi+mn\cos\psi\sqrt{\mathrm{I}_{,}}]}{[1-m_{,}\sin^2\psi]}. \quad . . . . . . (423.)$$

Adding those equations together, and recollecting that $m+n-mn=i^2$, we shall get

$$n\Phi_n+m\Phi_m=\frac{\sqrt{m_{,}}i^2\sin\psi}{\sqrt{mn}}+\frac{\sqrt{m_{,}}\sqrt{mn}\cos\psi\sin\psi\sqrt{\mathrm{I}_{,}}}{[1-m_{,}\sin^2\psi]}. \quad . . . . (424.)$$

Now as $\qquad i^2=(1+j)(1-j)$, and $\sqrt{mn}=\sqrt{m_{,}}(1+j)$

$$-\{n\Phi_n+m\Phi_m\}=-(1-j)\sin\psi-(1+j)\frac{m_{,}\sin\psi\cos\psi\sqrt{\mathrm{I}_{,}}}{(1-m_{,}\sin^2\psi)}. \quad . . . (425.)$$

In (186.) we found

$$2\int d\phi\sqrt{\mathrm{I}}=(1+j)\int d\psi\sqrt{\mathrm{I}_{,}}-\frac{2j}{1+j}\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}+(1-j)\sin\psi. \quad . . . . (426.)$$

Adding this expression to the preceding, the terms involving $\sin\psi$ will disappear.

We must now compute the sum of the coefficients of $\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}$.

Since $\int\frac{d\phi}{\sqrt{\mathrm{I}}}=\frac{(1+i_{,})}{2}\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}$, this coefficient becomes $\frac{(1+i)}{2}\left[\frac{i^2}{m}+\frac{i^2}{n}-2(1+j)\right]$.

Or as $m+n=i^2+mn$, this coefficient may be written $\left[\frac{i^4}{mn}+i^2-2(1+j)\right]\frac{(1+i_{,})}{2}$.

Or as $mn=m_{,}(1+j)^2$, it becomes finally, $\frac{2}{1+i_{,}}\left[\frac{i_{,}^2}{m_{,}}-1\right]$. . . . . (427.)

Hence $\left[\frac{i^2}{m}+\frac{i^2}{n}-2(1+j)\right]\left(\frac{1+i_{,}}{2}\right)\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}=\frac{2}{1+i_{,}}\left[\frac{i_{,}^2}{m_{,}}-1\right]\int\frac{d\psi}{\sqrt{\mathrm{I}_{,}}}$. . . . . . (428.)

And $(n-m)\left[\left(\frac{1-m}{m}\right)\int\frac{d\phi}{\mathrm{M}\sqrt{\mathrm{I}}}-\frac{(1-n)}{n}\int\frac{d\phi}{\mathrm{N}\sqrt{\mathrm{I}}}\right]=\frac{(n-m)}{\sqrt{mn}}\frac{(n-m)}{\sqrt{mn}}\frac{1}{(1+j)}\int\frac{d\psi}{[1-m_{,}\sin^2\psi]\sqrt{\mathrm{I}_{,}}}$. (429.)

Now as $\qquad n+m=i^2-mn, \quad (n+m)^2=i^4-2mni^2+m^2n^2.$

Hence $\qquad (n-m)^2=i^4+2mni^2+m^2n^2-4mn,$

and as $\qquad i^4=(1+j)^2(1-j)^2, \quad mn=m_{,}(1+j)^2$, substituting

$$(n-m)^2=(1+j)^2(1-j)^2+2m_{,}(1+j)^3(1-j)+m_{,}^2(1+j)^4-4m_{,}(1+j)^2,$$

therefore $$(n-m)^2=(1+j)^4\left[\left(\frac{1-j}{1+j}\right)^2+2m_i\left(\frac{1-j}{1+j}\right)+m_i^2-\frac{4m_i}{(1+j)^2}\right],$$

and as $\frac{4}{(1+j)^2}=(1+i_i)^2$, the expression will finally become

$$n-m=(1+j)^2(1-m_i)\sqrt{n_i}, \text{ hence } \frac{n-m}{\sqrt{mn}}\frac{i_i}{\sqrt{i}}=\left(\frac{1-m_i}{m_i}\right)\sqrt{m_i n_i}. \quad \ldots \quad (430.)$$

If now we add together (420.), (425.), (426.), (428.) and (429.), we shall have, dividing by $\frac{(n-m)}{\sqrt{mn}}$,

$$\left.\begin{aligned}\frac{4\Sigma}{k}=&-\frac{m_i\sqrt{m_i}}{(1-m_i)\sqrt{n_i}}\Phi_{m_i}+\frac{\sqrt{m_i}}{(1-m_i)\sqrt{n_i}}\int d\psi\sqrt{I_i}\\&-\left(\frac{1-m_i}{m_i}\right)\sqrt{m_i n_i}\int\frac{d\psi}{M_i\sqrt{I_i}}+\sqrt{\frac{n_i}{m_i}}\int\frac{d\psi}{\sqrt{I_i}}-2\int\frac{d\tau}{\cos^3\tau}\end{aligned}\right\} \quad \ldots \quad (431.)$$

Let us now take the logarithmic ellipse whose equation contains m_i, n_i, i_i, ψ instead of m, n, i and φ, we shall have from (163.),

$$\begin{aligned}\frac{2\Sigma'}{k_i}=&-\frac{m_i\sqrt{m_i n_i}}{n_i-m_i}\Phi_{m_i}-\left(\frac{1-m_i}{m_i}\right)\sqrt{m_i n_i}\int\frac{d\psi}{N_i\sqrt{I_i}}\\&+\frac{\sqrt{m_i n_i}}{n_i-m_i}\int d\psi\sqrt{I_i}+\frac{n_i}{m_i}\left(\frac{1-m_i}{n_i-m_i}\right)\sqrt{m_i n_i}\int\frac{d\psi}{\sqrt{I_i}}-2\int\frac{d\tau_i}{\cos^3\tau_i}. \quad \ldots \quad (432.)\end{aligned}$$

If we now subtract these equations one from the other, combining together like integrals, the integral of the third order will vanish and we shall have,

$$\frac{2\Sigma_i}{k}-\frac{4\Sigma}{k}=\frac{m_i(1-n_i)\sqrt{m_i n_i}}{n_i(n_i-m_i)(1-m_i)}\left[\int d\psi\sqrt{I_i}+\frac{n_i}{m_i}(1-m_i)\int\frac{d\psi}{\sqrt{I_i}}-m_i\Psi\right]+2\int\frac{d\tau}{\cos^3\tau}-2\int\frac{d\tau_i}{\cos^3\tau_i}. \quad . \quad (433.)$$

Hence, as we may express an arc of a plane ellipse by an arc of a derived ellipse, an integral of the first order, and a right line—a known theorem—so we may extend this analogy and express an arc of a logarithmic ellipse by an arc of a derived logarithmic ellipse, by functions of the first and second orders, by an arc of a parabola and by a right line. The relations between the moduli and amplitudes are the same in both cases,

$$i_i=\frac{1-j}{1+j}, \text{ and } \tan(\psi-\varphi)=j\tan\varphi.$$

Let m_{ii}, n_{ii}, i_{ii}, ψ_i be derived from m_i, n_i, i_i, ψ, by the same law as these latter are derived from m, n, i, φ, namely,

$$i_i=\frac{1-j}{1+j}, \quad \tan(\psi-\varphi)=j\tan\varphi, \quad m_i=\frac{mn}{(1+j)^2}, \quad n_i=\left[\frac{\sqrt{1-m}-\sqrt{1-n}}{\sqrt{1-m}+\sqrt{1-n}}\right]^2,$$

and derive an arc of a third logarithmic ellipse, we shall have, putting A, B, C, D for the coefficients of the integrals, and Π for the parabolic arc,

$$\frac{2\Sigma_i}{k}-\frac{4\Sigma}{k}=A\int d\psi\sqrt{I_i}+B\int\frac{d\psi}{\sqrt{I_i}}-C\Psi+D\Pi,$$

$$\frac{2\Sigma_{ii}}{k}-\frac{4\Sigma_i}{k}=A'\int d\psi_i\sqrt{I_{ii}}+B'\int\frac{d\psi_{ii}}{\sqrt{I_{ii}}}C-'\Psi_i+D'\Pi_i.$$

Multiply the first of these equations by 2 and add them, Σ will be eliminated. In this way we may successively eliminate $\Sigma_{,}$, $\Sigma_{,,}$, $\Sigma_{,,,}$ until ultimately we shall have

$$\frac{2\Sigma_{\nu}}{k}-2^{\nu+1}\frac{\Sigma}{k}={}_{\nu}\mathrm{E}+{}_{\nu}\mathrm{F}+{}_{\nu}\overline{\Psi}-{}_{\nu}\overline{\Pi},$$

ν being the number of operations, and denoting by F and E, the sum of the integrals of the first and second orders, by $\overline{\Psi}$ the sum of the right lines, and by $\overline{\Pi}$ the sum of the parabolic arcs.

If in (401.) and (416.) we substitute the coefficients of the derived integrals as transformed in (404.) and (430.), the relation between the original and the derived integrals of the third order will be,

$$\left(\frac{1+n}{n}\right)\sqrt{mn}\int\frac{d\phi}{(1+n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}+\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}=\left(\frac{1+n_{,}}{n_{,}}\right)\sqrt{m_{,}n_{,}}\int\frac{d\psi}{(1+n_{,}\sin^2\psi)\sqrt{1-i_{,}^2\sin^2\psi}}, \quad (434.)$$

for the circular form or the spherical ellipse, and

$$\left(\frac{1-m}{m}\right)\sqrt{mn}\int\frac{d\phi}{(1-m\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}-\left(\frac{1-n}{n}\right)\sqrt{mn}\int\frac{d\phi}{(1-n\sin^2\phi)\sqrt{1-i^2\sin^2\phi}}=\left(\frac{1-m_{,}}{m_{,}}\right)\sqrt{m_{,}n_{,}}\int\frac{d\psi}{(1-m_{,}\sin^2\psi)\sqrt{1-i_{,}^2\sin^2\psi}}, \quad (435.)$$

for the logarithmic form, or the logarithmic ellipse.

LXXXIII. There are several plane curves, whose lengths we may express by elliptic integrals of the third order. For example, the length of the elliptic lemniscate, or the locus of the intersections of central perpendiculars on tangents to an ellipse, is equal to that of a spherical ellipse, which is supplemental to itself, or the sum of whose principal arcs is equal to π. We cannot represent elliptic integrals of the third order generally, by the arcs of curves, whose equations in their simplest forms contain only two constants. Thus let a and b be the constants. We shall have two equations between the constants the parameter and the modulus of the function, $i=f(a, b)$, $n=f'(a, b)$. Assume a as invariable, and eliminate b, we shall have one resulting equation between i, n, and a, or $\mathrm{F}(a, i, n)=0$; or n depends on i.

When there are three independent constants, as in the preceding investigations, a, b, and k, we shall have $i=f(a, b, k)$, $n=f'(a, b, k)$. Eliminating successively b and k, we shall have two resulting equations, instead of one, $\mathrm{F}(a, k, i, n)=0$, and $\mathrm{F}'(a, b, i, n)=0$, or i and n depend on two equations, and may therefore be independent.

ERRATA.

Page 319, last line, *dele* n.
—— 320, line 5, *for* page 6 *read* page 316.
—— 328, line 12, *for* (47.) *read* (46.).
—— 329, line 15, *for* (32.) *read* (31.).
—— 331, line 7 from bottom, *for* $n=m=i$, *read* $n=m=\frac{1-j}{1+j}$.
—— 338, to the last line *add*, i being here the eccentricity of the base of the elliptic cylinder.
—— 372, line 11, *for* Case XII. *read* Case XIII.
—— 389, line 14, *for* Bernouilli *read* Bernoulli.

XIX. *On a New Series of Organic Bodies containing Metals.*

By Dr. E. Frankland, *F.C.S., Professor of Chemistry, Owen's College, Manchester. Communicated by* B. C. Brodie, *Esq., F.R.S.*

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UNDER the above title I described, more than three years ago, some preliminary experiments† which proved the existence of certain organic compounds highly analogous to cacodyl, and, like that body, consisting of a metal, or in some cases phosphorus, associated with the groups C_2H_3, C_4H_5, C_6H_7, &c., and possessing, in most instances, highly remarkable powers of combination. I fixed the composition and studied some of the reactions of two of these bodies, to which the names Zincmethyl (C_2H_3Zn) and Zincethyl (C_4H_5Zn) were provisionally assigned, besides giving methods for procuring similar compounds containing tin, arsenic and phosphorus, by acting upon the iodides of the alcohol radicals with these elements, and expressing a belief, founded upon the similarity of functions existing between hydrogen and the groups of the form $C_nH_{(n+1)}$, that most, if not the whole, of the compounds contained in the following series might be formed; those marked thus * being at that time already known.

Hydrogen series.	Methyl series.	Ethyl series.	Butyl series.	Valyl series.	Amyl series.	Phenyl series.
ZnH	$Zn\,C_2H_3$*	$Zn\,C_4H_5$*	$Zn\,C_6H_7$	$Zn\,C_8H_9$	$Zn\,C_{10}H_{11}$	$Zn\,C_{12}H_5$
AsH_3*	$As(C_2H_3)_2$*	$As(C_4H_5)_2$	$As(C_6H_7)_2$	$As(C_8H_9)_2$	$As(C_{10}H_{11})_2$	$As(C_{12}H_5)_2$
SbH_3*	$Sb(C_2H_3)_3$	$Sb(C_4H_5)_3$	$Sb(C_6H_7)_3$	$Sb(C_8H_9)_3$	$Sb(C_{10}H_{11})_3$	$Sb(C_{12}H_5)_3$
PH_3*	$P(C_2H_3)_3$*	$P(C_4H_5)_3$	$P(C_6H_7)_3$	$P(C_8H_9)_3$	$P(C_{10}H_{11})_3$	$P(C_{12}H_5)_3$

More recently Löwig and Schweitzer‡ have commenced labouring in the same field, and have filled up one of the gaps in the foregoing table by the formation of stibethyl ($Sb(C_4H_5)$), in acting upon iodide of ethyl with an alloy of antimony and potassium; the same chemists state also the probable formation of similar compounds containing methyl and amyl in place of ethyl, and bismuth and phosphorus instead of antimony.

I have continued my researches upon the organo-metallic bodies formed as above described, and having succeeded in increasing the list by the addition of several new members, I propose, in a series of papers, of which this is the first, to lay before the Royal Society the results of my experiments on the formation of bodies of this class.

The agents which I have hitherto employed in the formation of these organo-

† Annalen der Chemie und Pharmacie, Bd. LXXI. s. 213, and Journal of the Chemical Society, vol. ii. p. 297.

‡ Annalen der Chemie und Phamacie, Bd. LXXV. s. 315.

metallic compounds are two, viz. heat and light; in many cases either of these can be used, in others only one can be made to effect the desired combination, whilst more rarely the assistance of both appears to be essential. In those experiments in which heat was employed the materials were subjected to its action in sealed glass tubes, about 12 inches long, and varying in diameter from half an inch to 1 inch, the thickness of the glass being about one-eighth of an inch*. To preserve the gaseous products of the operation in a state of perfect purity for subsequent investigation, the tubes were well exhausted before being sealed; they were then immersed to about half their depth in an oil-bath, and heated to the required temperature. In cases where the influence of light was employed, the materials, confined in tubes of precisely similar dimensions, were exposed to the sun's rays, concentrated in most cases by an 18-inch parabolic reflector, near the focus of which the tubes were placed, either naked or surrounded by a solution of sulphate of copper to absorb the calorific rays. By this arrangement the light and heat could be increased, diminished or modified at pleasure, which was found very convenient in several of the operations.

Action of Tin upon Iodide of Ethyl.

When iodide of ethyl and metallic tin are exposed to the action of either heat or light, the tin gradually dissolves in the ethereal liquid, which finally solidifies to a mass of nearly colourless crystals. This reaction is effected most conveniently by the action of light, an excess of tinfoil, cut into narrow slips, being employed: the sealed tubes containing these ingredients should be placed near the focus of a large parabolic reflector, the temperature being prevented, if necessary, from rising too high by immersing them in water or in a solution of sulphate of copper. The unconcentrated rays of the sun, or even diffused daylight, are quite sufficient to determine the formation of the crystalline body; but an exposure of several weeks, or even months, would be necessary for the completion of the change, which is effected by the use of the reflector in a few days of bright sunshine. The liquid gradually assumes a straw-yellow colour, but its solidification is prevented as long as possible at the end of the operation, by allowing the temperature to rise 20° or 30° C. above that of the atmosphere; thus nearly the whole of the iodide of ethyl becomes united with tin. When heat instead of light is employed to effect the combination, the tubes should not be more than half an inch in diameter, and to avoid the risk of explosion, should not be more than one-fourth filled with the materials: the combination takes place at about 180° C. The agency of heat is therefore much less convenient than that of light in the production of this reaction, which is also never so complete as when the latter agent is employed; I have satisfied myself, however, that the results are the same in both cases.

* A minute account of the construction and use of these tubes is given in the Journal of the Chemical Society, vol. ii. p. 265.

Examination of Solid Products.

The capillary extremities of the tubes in which the iodide of ethyl had been exposed to the action of tin, were broken off under sulphuretted water and beneath a jar filled with the same liquid*; the gases evolved were preserved for eudiometrical investigation. The crystalline product of the reaction was then withdrawn from the tubes, and after being exposed to a gentle heat for a few minutes to expel the iodide of ethyl that had escaped combination, was treated with alcohol, in which the crystals readily dissolved, leaving only a small residue of a bright red colour, which proved to be protoiodide of tin. The filtered alcoholic solution was then placed over sulphuric acid *in vacuo*, where it soon deposited a large crop of long needle-like crystals, which, when freed from the mother-liquor, washed with a small quantity of dilute alcohol, dried between folds of bibulous paper, and finally over sulphuric acid *in vacuo*, yielded the following analytical results :—

I. 1·6806 grm., treated with aqueous solution of ammonia, was immediately decomposed, iodide of ammonium being formed, whilst the iodine in the original compound became, as I shall show below, replaced by oxygen; this oxide, which is almost absolutely insoluble in ammonia, collected on a filter and dried at 100° C., weighed ·7263 grm.; decomposed by boiling nitric acid it gave ·5811 grm. peroxide of tin. The ammoniacal solution, acidulated with nitric acid and precipitated by nitrate of silver, yielded 1·8418 grm. iodide of silver. After precipitation of the excess of nitrate of silver by hydrochloric acid, sulphuretted hydrogen was passed through the solution, and the slight precipitate formed was washed, dried, ignited and added to the above peroxide of tin, in the weight of which it is included.

II. 1·4254 grm., burnt with oxide of copper, 2 inches of metallic copper being placed in front of the combustion-tube, gave ·5858 grm. carbonic acid and ·2975 grm. water.

III. 1·2209 grm. gave 0·5008 grm. carbonic acid and ·2580 grm. water.

IV. 2·0980 grms., treated as described in No. I., yielded ·9218 grm. of the body produced by the action of solution of ammonia, which yielded ·7239 grm. peroxide of tin. The ammoniacal solution, precipitated by nitrate of silver as in No. I., produced 2·2883 grms. iodide of silver.

V. ·9113 grm. gave ·3735 grm. carbonic acid and ·1908 grm. water†.

These numbers show that the crystalline body is a compound of one atom of ethyl, one atom of tin, and one atom of iodine. The formula—

$$C_4 H_5 Sn I$$

* Journal of Chemical Society, vol. ii. p. 267.

† The substance used in Nos. IV. and V. was produced by the action of light, that used in the other analyses by the agency of heat.

requires the following values:—

	Calculated.		Found.				
			I.	II.	III.	IV.	V.
C_4	24	11·18	——	11·21	11·19	——	11·18
H_5	5	2·33	——	2·32	2·35	——	2·33
Sn	58·82	27·40	27·18	——	——	27·12	
I	126·84	59·09	59·21	——	——	58·76	
		100·00					

For reasons described below, I propose to call this compound *iodide of stanethylium*.

Iodide of stanethylium crystallizes in transparent, slightly straw-coloured needles, which are right rectangular prisms, frequently one-twelfth of an inch broad and 2 or 3 inches in length. They are very soluble in ether and in boiling alcohol; less so in cold alcohol and in water; the watery solution is decomposed on boiling, oxide of stanethylium being precipitated and hydriodic acid formed. Iodide of stanethylium fuses at 42° C., and boils at 240° C., undergoing at the same time partial decomposition: it possesses, at common temperatures, a peculiar pungent odour, somewhat resembling the volatile oil of mustard, and which irritates the eyes and lining membrane of the nose, causing a discharge which continues for several hours or even days, especially if the vapour from the heated iodide of stanethylium be inhaled; yet this compound can scarcely be said to be volatile at common temperatures, since a few grains may be exposed to the air for several weeks without any appreciable loss of weight.

Oxide of Stanethylium.

In contact with solutions of the alkalies, iodide of stanethylium is immediately decomposed, oxide of stanethylium and an alkaline iodide being formed; with solutions of potash and soda the oxide of stanethylium dissolves in an excess of the precipitant, but is reprecipitated, unchanged, by cautious neutralization of the alkaline solution; with solution of ammonia the precipitated oxide remains undissolved on the addition of an excess of the alkali; a quantity of the oxide of stanethylium, prepared in this latter manner, was heated for a few minutes with an excess of ammonia, thrown on a filter and washed with distilled water until all iodide of ammonium was removed. Submitted to analysis it yielded the following results:—

I. ·3497 grm., burnt with oxide of copper, gave ·3218 grm. carbonic acid and ·1630 grm. water.

II. ·7296 grm., decomposed by nitric acid and ignited, gave ·5778 grm. peroxide of tin.

III. ·9218 grm. gave, when similarly treated, ·7239 grm. peroxide of tin.

These numbers agree closely with the formula—

$$C_4 H_5 SnO,$$

as is seen by the following comparison :—

		Calculated.		Found.	
			I.	II.	III.
C_4	24	25·05	25·09	—	—
H_5	5	5·22	5·18	—	—
Sn	58·82	61·39	—	62·25	61·73
O	8	8·34	—	—	—
		100·00			

Analyses Nos. I. and IV. of iodide of stanethylium also clearly show the transformation of the iodide into the oxide of stanethylium by ammonia; in analysis No. I. 1·6806 grm. of iodide of stanethylium gave ·7263 grm. of oxide of stanethylium, and in analysis No. IV. 2·0980 grms. of the iodide yielded ·9218 grm. of the oxide of stanethylium. Hence

	Calculated.	Found.	
		I.	II.
100 parts of iodide of stanethylium yield of oxide of stanethylium	44·64	43·22	43·93

The numbers obtained by experiment correspond sufficiently well with the theoretical one, when it is considered that oxide of stanethylium is not absolutely insoluble in excess of ammonia.

Oxide of stanethylium presents the appearance of a somewhat cream-white amorphous powder, closely resembling peroxide of tin, but less heavy than that oxide; it has a peculiar though slight ethereal odour and a bitter taste; it is insoluble in water, alcohol and ether, but readily dissolves in solutions of acids and of the fixed alkalies; with acids it forms salts, which are, however, for the most part difficultly crystallizable; those with strong acids exhibit an acid reaction. The nitrate deflagrates when heated to about 120° C., and on the application of a higher heat becomes pure peroxide of tin. The salts of oxide of stanethylium behave with reagents so nearly like the salts of peroxide of tin, that the two are very difficult to distinguish from each other.

Sulphide of Stanethylium.

When sulphuretted hydrogen is passed through an acid solution of a salt of stanethylium, a cream-coloured precipitate falls, which is insoluble in dilute acids and ammonia, but soluble in concentrated hydrochloric acid, solutions of the fixed alkalies, and alkaline sulphides; from its solutions in the fixed alkalies and alkaline sulphides, it is reprecipitated, unchanged, on the addition of an acid. I have made no analyses of this body, but there is no doubt that its formula is—

$$C_4H_5SnS,$$

and that it is produced by the following reaction—

$$C_4H_5SnO + HS = C_4H_5SnS + HO.$$

Sulphide of stanethylium presents the appearance of an amorphous cream-coloured

powder, having a pungent and very nauseous smell, resembling decayed horse-radish: when heated it fuses, froths up and decomposes, emitting vapours of a most insupportable odour. Heated with nitric acid it is decomposed with the formation of peroxide of tin.

Chloride of Stanethylium.

$$C_4 H_5 Sn Cl.$$

This salt is best prepared by dissolving oxide of stanethylium in dilute hydrochloric acid: on evaporation at a gentle heat or over sulphuric acid *in vacuo*, the chloride crystallizes out in long colourless needles, isomorphous with the iodide, which salt it also closely resembles in all its properties; it is however more volatile, and therefore emits a more intensely pungent and irritating odour than the iodide.

Stanethylium.

When a strip of zinc is immersed in a solution of a salt of stanethylium (a solution of the chloride of stanethylium is the best for this purpose), it speedily becomes covered with dense oily drops of a yellow colour, which finally separate from the lower extremity of the zinc and accumulate at the bottom of the vessel; the formation of the oily liquid is much favoured by the application of a gentle heat. The yellow oil was separated from the supernatant liquid by means of a pipette, and well washed with successive large portions of cold water; being then dried over chloride of calcium and submitted to analysis, it yielded the following results:—

I. ·3150 grm., burnt with oxide of copper and oxygen gas, gave ·3498 grm. carbonic acid and ·1757 grm. water.

These numbers correspond sufficiently with the formula

$$C_4 H_5 Sn,$$

when it is considered that the stanethylium, as thus prepared, contains traces of undecomposed chloride of stanethylium, which I did not succeed in removing by the most protracted washing; and as stanethylium does not crystallize and cannot be distilled without decomposition, I could not avail myself of these means of purification. The above formula requires the following numbers:—

		Calculated.	Found.
C_4	24	27·32	26·95
H_5	5	5·69	5·51
Sn	58·82	66·99	——
		100·00	

The isolation of stanethylium from its chloride by zinc, is therefore expressed by the following simple equation:—

$$\left.\begin{matrix} C_4 H_5 Sn Cl \\ Zn \end{matrix}\right\} = \left\{\begin{matrix} Zn Cl \\ C_4 H_5 Sn. \end{matrix}\right.$$

Stanethylium exists at the ordinary atmospheric temperature, as a thick, heavy, oily liquid, of a yellow or brownish-yellow colour, and an exceedingly pungent odour, resembling that of its compounds, but much more powerful. It is insoluble in water,

but soluble in alcohol and ether. At about 150° C. it enters into ebullition, a quantity of metallic tin is deposited, and a colourless liquid distils over, having a peculiar odour, containing a considerable quantity of tin, and exhibiting no tendency to combine with iodine or bromine: the composition and properties of this liquid I have not further ascertained; it possibly consists of or contains *binethide of tin* ($Sn(C_4H_5)_2$). In contact with the air stanethylium rapidly attracts oxygen and is converted into a white powder, which has all the properties of oxide of stanethylium. Chloride, bromide and iodide of stanethylium are immediately formed by the action of chlorine, bromine and iodine, or their hydrogen acids respectively, upon stanethylium; the first and third are in every respect identical with the salts above described. I have analytically examined the bromide prepared by adding an alcoholic solution of bromine to an alcoholic solution of stanethylium until the colour of the bromine no longer disappears; by spontaneous evaporation the bromide of stanethylium is deposited in long white needles, which closely resemble, both in appearance and properties, the chloride of stanethylium. These crystals, pressed between folds of bibulous paper, and dried over sulphuric acid *in vacuo*, gave the following analytical results:—

·9730 grm., burnt with oxide of copper, yielded ·5108 grm. carbonic acid and ·2582 grm. water.

These numbers agree very closely with the formula—

$$C_4H_5SnBr,$$

as is seen from the following comparison:—

	Equivs.		Calculated.	Found.
Carbon	4	24	14·30	14·32
Hydrogen	5	5	2·98	2·95
Tin	1	58·82	35·05	——
Bromine	1	80·00	47·67	——
		167·82	100·00	

These results show that stanethylium perfectly resembles cacodyl in its reactions, combining directly with the electro-negative elements and regenerating the compounds from which it has been derived.

Examination of Gases.

The examination of the gases evolved on opening the tubes in which iodide of ethyl and tin had been submitted to the action of heat, and which were allowed to stand over sulphuretted water for twelve hours, yielded the following results. Specific gravity:—

Weight of flask filled with gas	35·4712 grms.
Temperature of room	20·8° C.
Height of barometer	761·2 mm.
Height of internal column of mercury	15·2 mm.
Temperature in balance case	22°·6 C.

Weight of flask filled with dry air . . .	35·4703 grms.
Temperature in balance case	22·8° C.
Capacity of flask	140·50 cubic centimetres.

From these data the specific gravity was calculated to be 1·0384.

The remainder of the gas was submitted to eudiometrical analysis: the following numbers were obtained:—

I. In Short Eudiometer.

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0° C. and 1 metre press.
Volume of gas used (dry)	143·4	21·0° C.	16·3	760·5	99·09
Volume after action of fuming SO_3 (dry) .	122·5	18·2	37·0	754·7	82·42

II. In Combustion Eudiometer.

Volume of gas used (moist)	110·3	18·1 C.	570·7	755·2	17·48
Volume after admission of O (moist)	383·6	18·3	274·7	755·7	167·31
Volume after explosion (moist)	326·9	18·4	332·4	755·8	124·86
Volume after absorption of CO_2 (dry) . . .	264·5	17·0	398·3	760·7	90·23
Volume after admission of H (dry)	592·5	17·0	77·9	761·6	381·33
Volume after explosion (moist)	303·5	17·7	355·9	762·7	111·64

As the gas, left unabsorbed by fuming sulphuric acid, was soluble in about its own volume of alcohol, with the exception of a very small per-centage due to the nitrogen introduced by diffusion through the sulphuretted water, it could not contain either hydrogen or hydride of methyl; and the result of the above combustion proves that it is hydride of ethyl, for I have shown that 1 vol. of hydride of ethyl consumes 3·5 vols. oxygen, and generates 2 vols. carbonic acid, numbers which almost exactly correspond with those obtained.

17·48 vols. of the gas, containing 17·15 vols. of combustible gas and ·33 vol. of nitrogen, consumed 59·93 vols. oxygen and generated 34·63 vols. carbonic acid; hence

Volume of combustible gas.		Oxygen consumed.		Carbonic acid generated.
17·15		59·93		34·63
1	:	3·49	:	2·01

Further, the gas agrees in all its chemical and physical properties, with the hydride of ethyl prepared by the action of zinc upon iodide of ethyl in presence of water.

The composition of the gas absorbed by fuming sulphuric acid was determined

by exploding a known volume of the original gas with excess of oxygen, and determining the quantities of oxygen consumed, and carbonic acid generated.

This determination gave the following numbers:—

III.

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0° C. and 1 metre press.
Volume of gas used (moist)	122·0	19·7° C.	563·0	764·2	20·95
Volume after admission of O (moist)	424·5	20·0	240·4	764·3	200·33
Volume after explosion (moist)	363·5	20·4	301·0	763·5	150·40
Volume after absorption of CO_2 (dry)	296·7	19·7	366·4	760·4	109·03
Volume after admission of H (dry)	683·1	21·7	4·0	760·8	478·89
Volume after explosion (moist)	368·6	21·8	293·3	761·2	153·09

Hence 20·95 vols. containing 20·62 vols. combustible gas, consumed 70·78 vols. oxygen, and generated 41·37 vols. carbonic acid: now as 20·62 vols. of this gas must contain, according to analyses Nos. I. and II., 17·10 vols. hydride of ethyl, which would consume 59·85 vols. oxygen and generate 34·20 vols. carbonic acid, it is evident that the volumes of oxygen consumed, and carbonic acid generated, by the gas absorbed by fuming sulphuric acid, must bear the following relation to each other:—

Volume of gas absorbable by fuming SO_3.		Oxygen consumed.		Carbonic acid generated.
3·52		10·93		7·17
1	:	3·10	:	2·03

The body removed by fuming sulphuric acid is therefore olefiant gas, 1 vol. of which consumes 3 vols. oxygen, and generates 2 vols. carbonic acid, numbers which correspond sufficiently with those obtained in the above determination.

The last analysis can also be employed to control Nos. I. and II.; for if we represent the volume of nitrogen, contained in the original gas, by x, that of hydride of ethyl by y, and that of olefiant gas by z; and further, the volume of mixed gases, oxygen consumed, and carbonic acid generated, respectively by A, B and C, we have the following equations:—

$$x+y+z=A$$

$$\tfrac{7}{2}y+3z=B$$

$$2y+2z=C,$$

from which the following values for x, y and z are derived:—

$$\begin{aligned} x &= {\cdot}27 \\ y &= 17{\cdot}06 \\ z &= 3{\cdot}62 \\ \hline & 20{\cdot}95 \end{aligned}$$

The per-centage composition of the gases evolved by the action of heat upon iodide of ethyl and tin is therefore the following:—

	I. and II.	III.
Hydride of ethyl	81·61	81·43
Olefiant gas	16·82	17·28
Nitrogen	1·57	1·29
	100·00	100·00

This result is also confirmed by the determination of the specific gravity of the gaseous mixture, as is seen from the following calculation:—

$$\begin{array}{lll} C_4H_5H & \ldots\ldots & 81{\cdot}61 \times 1{\cdot}03652 = 84{\cdot}590 \\ C_4H_4 & \ldots\ldots & 16{\cdot}82 \times {\cdot}96742 = 16{\cdot}272 \\ N & \ldots\ldots & 1{\cdot}57 \times {\cdot}96740 = 1{\cdot}519 \\ \hline & & 100{\cdot}00 \qquad \dfrac{102{\cdot}381}{100} = 1{\cdot}02381 \end{array}$$

Specific gravity found by experiment . . . . $=1{\cdot}0384$

The presence of hydride of ethyl and olefiant gas amongst the products of the action of heat upon iodide of ethyl and tin, shows that the combination of tin with iodide of ethyl is not the only reaction which takes place, but that a portion of the iodide of ethyl is also decomposed by the tin, with the production of iodide of tin and ethyl; the latter body being transformed at the moment of its liberation into hydride of ethyl and olefiant gas, a catalysis to which this radical is so prone,

$$\left.\begin{array}{r} 2C_4H_5I \\ 2Sn \end{array}\right\} = \left\{\begin{array}{l} C_4H_5, H \\ C_4H_4 \\ 2Sn\,I \end{array}\right.$$

It was ascertained that protoiodide of tin was present amongst the solid products of the reaction.

The large excess of hydride of ethyl exhibited in the above analysis, may have been caused, either by the greater solubility of olefiant gas in iodide of ethyl (a further and considerable amount of gas being expelled from the tube by the application of a gentle heat), or by the presence of moisture in the materials, which would give rise to the formation of oxyiodide of tin and hydride of ethyl,

$$\left.\begin{array}{r} C_4H_5I \\ HO \\ 2Sn \end{array}\right\} = \left\{\begin{array}{l} C_4H_5H \\ Sn\,O + Sn\,I. \end{array}\right.$$

Both these causes probably contribute to produce the excess of hydride of ethyl; but the very small amount of gaseous products, compared with the solid ones, convinced me that the production of the former is only an accidental circumstance, which, however it may be interpreted, does not at all affect the principal reaction, viz. the formation of iodide of stanethylium. The gases, evolved by the action of light upon iodide of ethyl and tin, are perfectly similar to those obtained by the action of heat.

Stanmethylium and stanamylium are formed when the iodides of methyl and amyl respectively are exposed to the action of light in contact with tin; their salts are isomorphous with those of stanethylium; but I have not yet completed the investigation of these bodies.

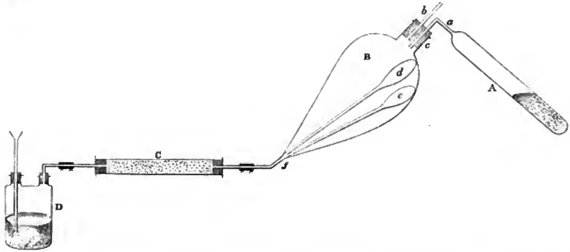
Action of Zinc upon Iodide of Methyl.

When iodide of methyl and zinc are exposed to a temperature of about 150° C. in a sealed tube, the zinc gradually dissolves with the evolution of gas, whilst a mass of white crystals and a colourless mobile liquid, refracting light strongly, occupy, after a few hours, the place of the original materials. The gas, evolved on breaking off the capillary extremity of the previously exhausted decomposition tube, was collected and preserved over sulphuretted water in the manner already described: I will refer to this gas again under the name of α. On cutting off the upper portion of the decomposition tube and pouring cold distilled water upon the mobile liquid and white mass of crystals just mentioned, a very violent action ensued, and a column of flame several feet high shot up momentarily from the mouth of the tube; but the action soon became more moderate, and a cork and gas-delivering tube being fitted into the decomposition tube, the gas, after all atmospheric air had been expelled, was collected and preserved in an apparatus similar to that used for the gas α. I will call this second gas β.

Zincmethylium.

From a preliminary experiment, it was ascertained that the gas evolved on opening the decomposition tube possessed, before contact with water, a most insupportable and very peculiar odour, and that, when ignited or brought in contact with pure oxygen gas, it burnt with a greenish blue flame, producing dense white fumes: when a porcelain plate was held in this flame, it immediately became coated with a jet black deposit, surrounded by a white ring; this black deposit dissolved in dilute hydrochloric acid with evolution of hydrogen gas, and the solution was found to contain chloride of zinc. Hence it was evident that a volatile or gaseous compound of zinc was present amongst the products of decomposition, and this was soon found to reside in the mobile liquid above mentioned; for on inverting the tube and allowing a few drops of the liquid to escape, it inflamed spontaneously the instant it came in contact with the air, and produced, by its combustion, large quantities of oxide of

zinc. In order to obtain this liquid in a state of purity, another tube was charged with iodide of methyl and excess of zinc, and subjected to a heat of 150° or 160° C. until every trace of iodide of methyl was decomposed. The drawn out extremity of the tube being broken off, the included gas was allowed to escape, and the liquid contents were then separated from the solid ones by distillation at a gentle heat, in an atmosphere of dry hydrogen. This was accomplished as shown in the following figure.



A is the decomposition tube bent at an obtuse angle at *a*, and connected with the receiver B by the doubly perforated cork *c*, which also contains the small tube *b*, open at both ends. The receiver B is drawn out at *f* until its internal diameter is diminished to about $\frac{1}{20}$th of an inch, and this drawn out extremity is connected, by means of a caoutchouc joint, with the chloride of calcium tube C, which at its opposite extremity is in connection with a hydrogen gas apparatus D. *d*, *e* are two small glass bulbs for preserving the condensed liquid. The apparatus being thus arranged, hydrogen is evolved in D, and becoming perfectly desiccated in passing through the chloride of calcium tube C, enters the receiver B at *f*, expelling the atmospheric air through the tube *b*. When the gas has thus streamed through the apparatus for at least a quarter of an hour, and every trace of air has been expelled from B and from the bulbs *d*, *e* by diffusion, the extremity of the tube *b* is hermetically sealed, at the same moment that the evolution of gas from D is interrupted. The drawn out extremity of the receiver B being then quickly sealed at *f*; B, *d* and *e* remain filled with pure dry hydrogen, and A with a mixture of gases free from oxygen, as any trace of this element, which might have penetrated there, would be instantaneously absorbed by its contents. B is then immersed to its neck in cold water, and a gentle heat cautiously applied to the whole length of A by means of a spirit lamp. The mobile fluid in A soon enters into ebullition, and distils over into the receiver B; as soon as the distillation is finished and A become cold, its capillary extremity is fused off at *a* by means of a blowpipe, *a* remaining hermetically sealed.

The receiver B is then removed from the water and dried; heat is applied to the side adjacent to the bulbs *d*, *e*, so as to expel a portion of the enclosed gas from their open ends at *f*; on subsequent cooling, a certain quantity of the liquid rises into these bulbs, which are alternately heated and cooled, until every trace of the liquid has not only entered them but passed entirely into their expanded portion, so as to leave the capillary limbs filled with hydrogen. It is of importance that the whole of the liquid should be forced to enter these bulbs, otherwise, on subsequently opening the mouth of the receiver, it inflames, causing the expulsion of the liquid from the bulbs, and thus rendering the experiment abortive. The cork *c* is then removed, and the bulbs *d*, *e* extracted as quickly as possible, the open capillary extremities being immediately sealed before the blowpipe. The bulbs, having been previously weighed, the increase denotes the weight of the included liquid. The residue in A was found scarcely to effervesce with water, and consisted of iodide of zinc mixed with the excess of metallic zinc employed.

I have fixed the composition of the liquid obtained as above described, and proved it to be a compound of one atom of zinc and one atom of methyl, by the following experiments:—

I. One of the bulbs above mentioned was opened beneath an inverted receiver filled with recently boiled distilled water; its contents were rapidly resolved into hydrated oxide of zinc and a permanent gas, which last was submitted to eudiometrical examination; the following results were obtained:—

The action of fuming sulphuric acid did not produce any diminution of volume.

The gas was nearly insoluble in absolute alcohol.

A eudiometrical combustion yielded the following results:—

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0° C. and 1 metre press.
Volume of gas used (moist)	122·5	18·7° C.	602·2	747·8	14·86
Volume after admission of O (moist)	287·9	18·7	418·4	747·6	84·39
Volume after explosion (moist)	232·1	18·6	479·7	747·4	54·71
Volume after absorption of CO_2 (dry)	188·2	18·6	519·4	747·5	40·19
Volume after admission of H (dry)	549·4	18·4	162·7	747·6	301·04
Volume after explosion (moist)	425·7	18·4	279·0	747·6	180·62

Volume of comb. gas.		Oxygen consumed.		CO_2 generated.
14·86		29·39		14·52
1	:	1·98	:	·98

In order to ascertain whether the gas was a single compound or a gaseous mix-

ture, and also to determine its specific gravity, it was submitted to diffusion in an apparatus which I have already described*: the following results were obtained:—

I. In Diffusion Eudiometer.

	Observed volume.	Temp.	Difference of mercury level.	Barom.	Corrected vol. at 0° C. and 1 metre press.
			mm	mm	
Volume of gas used (dry)	173·0	19·0° C.	20·1	741·1	116·61
Volume after diffusion (dry)	144·3	19·2	46·3	740·8	93·63

II. Estimation of Oxygen in residual gas.

	Observed volume.	Temp.	Difference of mercury level.	Barom.	Corrected vol. at 0° C. and 1 metre press.
Volume of gas used (moist)	117·5	19·3	53·1	740·6	75·44
Volume after absorption of O (dry)	101·2	17·0	69·3	742·8	64·16

III. Combustion of Gas remaining after absorption of Oxygen.

	Observed volume.	Temp.	Difference of mercury level.	Barom.	Corrected vol. at 0°C. and 1 metre press.
			mm	mm	
Volume of gas used (moist)	155·9	17·3° C.	558·7	743·3	24·91
Volume after admission of O (moist)	302·7	17·7	399·0	744·0	93·78
Volume after explosion (moist)	270·2	17·6	433·2	744·2	75·13
Volume after absorption of CO_2 (dry)	246·3	17·8	458·7	744·7	66·13
Volume after admission of H (dry)	522·3	17·9	182·4	745·1	275·80
Volume after explosion (moist)	349·5	18·0	350·5	745·0	124·30

The gas, remaining after diffusion and subsequent absorption of oxygen, therefore contained in 24·91 vols. 15·63 vols. nitrogen and 9·28 vols. combustible gas, which last was a single gas and not a mixture, since it consumed the same amount of oxygen and generated the same amount of carbonic acid after as before diffusion:—

Volume of combustible gas.		Oxygen consumed.		CO_2 generated.
9·28		18·37		9·00
1	:	1·98	:	·97

Experiments Nos. I., II. and III., taken together, enable us to ascertain the volume of the gas which escaped and that of the air which entered during the diffusion experiment; these volumes are as follow:—

Volume of gas escaped	86·95
Volume of air entered	63·97

* Quarterly Journal of Chemical Society, vol. ii. p. 283.

Hence, according to the well-known law of diffusion, the specific gravity of the gas must be ·5413.

The gas is therefore hydride of methyl (light carburetted hydrogen), 1 vol. of which consumes 2 vols. oxygen and generates 1 vol. carbonic acid, and the specific gravity of which is ·5528, numbers which correspond almost exactly with those obtained by experiment.

II. A glass jar, graduated in cubic centimetres, was filled with recently boiled distilled water, to which about twenty drops of sulphuric acid had been added, and inverted in a shallow glass dish containing the same liquid; the other bulb was then introduced into the inverted jar, and its capillary extremity broken off against the side of the vessel; the water now slowly gained access to the liquid in the bulb and steady decomposition ensued, the oxide of zinc dissolving as fast as formed, in the dilute sulphuric acid, and the hydride of methyl collecting in the inverted jar. When the decomposition was quite complete, the volume of gas was read off with the usual corrections for temperature and pressure, the graduated jar rinsed out and removed, and the solution of sulphate of zinc in the glass dish, after being evaporated to a smaller bulk, was treated with carbonate of potash and the zinc precipitated as basic carbonate, and weighed as oxide with the usual precautions: the following results were obtained:—

·3109 grm. gave ·2660 grm. oxide of zinc, and 138·15 cubic centimetres dry hydride of methyl, at 0° C. and 760 mm. pressure, equivalent to ·0930 grm.

These numbers agree sufficiently with those calculated from the formula

$$C_2H_3Zn,$$

when we consider that every trace of oxygen, which gained admission to the ethereal fluid before its decomposition, would diminish the volume of hydride of methyl, which would also be liable to further diminution, from the solvent action of the fluid over which it was determined:—

	Calculated.		Found.
1 equiv. of Methyl	15	31·56	29·91
1 equiv. of Zinc	32·52	68·44	68·67
	47·52	100·00	98·58

This compound, for which I propose the name Zincmethylium, possesses the following properties:—It is a colourless, transparent and very mobile liquid, refracting light strongly, and possessing a peculiar penetrating and insupportable odour; it is very volatile, but I have not yet been able to determine its boiling-point with accuracy.

Zincmethylium combines directly with oxygen, chlorine, iodine, &c., forming somewhat unstable compounds, a description of which I reserve for a future communication. Its affinity for oxygen is even more intense than that of potassium; in contact with atmospheric air it instantaneously ignites, burning with a beautiful greenish-blue flame, and forming white clouds of oxide of zinc; in contact with pure oxygen it

burns with explosion, and the presence of a small quantity of its vapour in combustible gases gives them the property of spontaneous inflammability in oxygen. Thrown into water, zincmethylium decomposes that liquid with explosive violence and with the evolution of heat and light; when this action is moderated, so as to prevent any great rise of temperature, the sole products of the decomposition are oxide of zinc and hydride of methyl,

$$\left.\begin{matrix} C_2\,H_3\,Zn \\ HO \end{matrix}\right\} = \left\{\begin{matrix} C_2\,H_3\,H \\ Zn\,O. \end{matrix}\right.$$

The extraordinary affinity of zincmethylium for oxygen, its peculiar composition, and the facility with which it can be procured, cannot fail to cause its employment for a great variety of transformations in organic compounds; by its agency there is every probability that we shall be able to replace oxygen, chlorine, &c., atom for atom, by methyl, and thus produce entirely new series of organic compounds, and obtain clearer views of the rational constitution of others. I intend to pursue this branch of the subject whilst studying the compounds of zincmethylium and the corresponding bodies containing ethyl and amyl.

Examination of the Gas α.

A quantity of this gas, after standing over sulphuretted water until all traces of iodide of methyl vapour had been absorbed, was transferred into a suitable flask for the determination of its specific gravity; the following numbers were obtained:—

Temperature of room	18°·6 C.
Height of barometer	754·2 mm.
Height of inner column of mercury . . .	15·2 mm.
Weight of flask and gas	35·4161 grms.
Temperature in balance case	19°·6 C.
Weight of flask and dry air	35·4500 grms.
Temperature in balance case	20°·2 C.
Capacity of flask	140·51 cubic centimetres.

From these data the specific gravity was calculated to be ·79598.

The eudiometrical analysis of the gas gave the following results:—

I. In Short Eudiometer.

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0°C. and 1 metre press.
Volume of gas used (dry)	194·7	18°·6 C.	2·1	754·2	137·06
Volume after action of fuming SO_3 (dry). . .	194·8	18·7	2·2	753·5	137·03
Volume after removal of specimen for combustion (dry)	153·8	19·0	22·0	741·0	103·38
Volume after action of alcohol	82·0	18·7	7·0	741·4	53·72

II. In Combustion Eudiometer.

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0°C. and 1 metre press.
Volume of gas used (moist)	101·5	18·9°C.	621·7	752·4	10·86
Volume after admission of O (moist)	388·1	18·6	311·7	752·0	154·20
Volume after explosion (moist)	354·6	18·5	345·8	752·2	129·71
Volume after absorption of CO_2 (dry)	321·1	18·3	376·9	752·0	112·88
Volume after admission of H (dry)	745·1	18·7	·8	751·1	523·19
Volume after explosion (moist)	428·3	18·7	274·1	750·5	184·54

III.

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0° C. and 1 metre press.
Volume of gas used (moist)	104·2	18·8°C.	619·2	750·5	11·23
Volume after admission of O (moist)	371·2	18·9	330·6	750·2	140·04
Volume after explosion (moist)	335·0	18·9	367·7	750·2	114·76
Volume after absorption CO_2 (dry)	299·0	18·2	401·3	750·3	97·83
Volume after admission of H (dry)	707·0	18·4	26·9	750·6	479·33
Volume after explosion (moist)	431·6	18·7	273·4	749·8	185·96

Analysis No. I. proves the absence of all the members of the olefiant gas family, and also that the mixture consists of—

Gas absorbable by alcohol	48·04
Gas unabsorbable by alcohol	51·96
	100·00

The behaviour of the iodides of ethyl and amyl in contact with zinc*, led me to expect that the gaseous products of the decomposition of iodide of methyl by the same metal would consist of methyl, hydride of methyl, and the first member of the olefiant gas series, methylene; but in addition to the proof of the absence of this latter body afforded by the absence of all absorption by fuming sulphuric acid, analyses Nos. II. and III. demonstrate the impossibility of methylene being a constituent of the gaseous mixture; for on constructing three equations in which the volumes of

* Journal of the Chemical Society, vol. ii. p. 265, and vol. iii. p. 30.

methyl, hydride of methyl, and methylene are expressed, the value obtained for the last gas is invariably a small negative quantity. The volumes of methyl and hydride of methyl are readily found by the two following equations, in which the volume of combustible gas is represented by A, the contraction produced by explosion with excess of oxygen by B, and the volumes of methyl and hydride of methyl respectively, by x and y, the contraction produced by the explosion of methyl with excess of oxygen being 2·5 times its own volume, and that produced by the explosion of hydride of methyl twice its own volume:—

$$x+y=A,$$
$$\frac{5}{2}x+2y=B.$$

The values of x and y may therefore be thus expressed:—

$$x=2B-4A,$$
$$y=5A-2B.$$

According to analysis No. II., 10·88 vols. of combustible gas produced a contraction, on explosion with oxygen, equal to 24·49 vols.; and in analysis No. III., 11·23 vols. of combustible gas produced a contraction, on explosion, equal to 25·28 vols. Hence, by the application of the foregoing equations, the per-centage composition of the gaseous mixture may be expressed as follows:—

	II.	III.	Mean.
Methyl	50·18	50·22	50·20
Hydride of Methyl . . .	49·82	49·78	49·80
	100·00	100·00	100·00

This result is confirmed by the action of alcohol in analysis No. I., and also by the determination of the specific gravity of the mixed gases, which agrees very closely with that deduced from the above numbers, as is seen from the following comparison:—

Methyl	50·20 × 1·0365	= 52·0323
Hydride of Methyl . . .	49·80 × ·5528	= 27·5294
	100·00	$\frac{79 \cdot 5617}{100} = \cdot 795617$
Specific gravity found by experiment . . .		= ·79598

The origin of the hydride of methyl in the above gaseous mixture is readily perceived, when the volatility of zincmethylium and the method of collecting the gas are taken into consideration; on opening the decomposition-tube beneath water, a copious effervescence was observed wherever the evolved gas came in contact with water; and as this effervescence was accompanied by the formation of a flocculent precipitate of oxide of zinc, it could only be caused by the presence of the vapour of zincmethylium, which, on coming in contact with water, would be instantaneously decomposed into oxide of zinc and hydride of methyl.

I have not yet endeavoured to procure the methyl free from admixture with hydride

of methyl, but have no doubt that, by collecting the gas as evolved from the decomposition-tube over mercury, and absorbing the zincmethylium vapour by dry iodine, the methyl would be left in a state of purity. It perfectly resembles in its properties, chemical and physical, the methyl procured by KOLBE from the electrolysis of acetic acid *.

Examination of the Gas β.

This gas, evolved by the action of water upon the solid and liquid products of the decomposition of iodide of methyl by zinc, proved, as might have been anticipated, to be pure hydride of methyl, derived from the decomposition of the zincmethylium with which the crystalline residue of iodide of zinc was saturated. Its eudiometrical analysis yielded the following results:—

I. In Short Eudiometer.

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0° C. and 1 metre press.
Volume of gas used (dry).	168·9	18·6° C.	3·6	750·2	118·05
Volume after action of fuming SO_3 (dry) . .	169·5	18·7	3·7	749·8	118·35

One volume of absolute alcohol, at 19° C. and 732·6 mm. pressure, absorbed ·175 vol. of this gas.

II. In Combustion Eudiometer.

	Observed volume.	Temp.	Difference of mercury level. mm	Barom. mm	Corrected vol. at 0° C. and 1 metre press.
Volume of gas used (moist)	126·7	18·9° C.	595·1	744·9	15·83
Volume after admission of O (moist)	344·9	18·9	357·7	744·8	119·64
Volume after explosion (moist)	295·9	18·9	408·7	744·5	88·44
Volume after absorption of CO_2 (dry)	260·7	18·6	446·1	744·3	72·78
Volume after admission of H (dry)	705·7	18·5	24·6	741·3	473·66
Volume after explosion (moist)	513·0	18·6	193·6	741·1	255·35

These results correspond almost exactly with those yielded by hydride of methyl, 1 vol. of which requires 2 vols. of oxygen for combustion, and generates 1 vol. of carbonic acid.

Volume of combustible gas.		Oxygen consumed.		CO_2 generated.
15·83		31·04		15·66
1	:	1·96	:	·99

* Journal of the Chemical Society, vol. ii. p. 173.

By the action of zinc upon iodide of methyl, therefore, two distinct decompositions take place, viz. 1st, the decomposition of iodide of methyl by zinc with the production of iodide of zinc and methyl,

$$\left.\begin{matrix} C_2H_3I \\ Zn \end{matrix}\right\} = \left\{\begin{matrix} C_2H_3 \\ ZnI, \end{matrix}\right.$$

and 2nd, the decomposition of iodide of methyl by zinc, with the formation of iodide of zinc and zincmethylium,

$$\left.\begin{matrix} C_2H_3I \\ 2Zn \end{matrix}\right\} = \left\{\begin{matrix} C_2H_3Zn \\ ZnI. \end{matrix}\right.$$

Zincethylium.

This body is formed under precisely the same circumstances as zincmethylium, iodide of ethyl being substituted for iodide of methyl; it is a colourless and transparent liquid, refracting light strongly, and having a peculiar penetrating odour; it is less volatile than zincmethylium, and is not so readily prepared pure, owing to its retention of a small quantity of ethyl gas in solution; its affinities are also somewhat weaker than zincmethylium, and it only takes fire in air spontaneously when exposed in considerable quantity. When allowed to absorb oxygen slowly, it forms a white amorphous oxide; it combines also directly with iodine, chlorine and bromine. In contact with water it is instantaneously decomposed into oxide of zinc and hydride of methyl,

$$\left.\begin{matrix} C_4H_5Zn \\ HO \end{matrix}\right\} = \left\{\begin{matrix} C_4H_5, H \\ ZnO. \end{matrix}\right.$$

Its formula must therefore be

$$C_4H_5Zn.$$

I reserve for a future communication the complete history of this and the following compound.

Zincamylium.

This body is generated when iodide of amyl is decomposed by zinc at the temperature of 180° C. It is a colourless and transparent liquid which emits white fumes in contact with the air, but does not spontaneously inflame; it is decomposed in contact with water into oxide of zinc and hydride of amyl,

$$\left.\begin{matrix} C_{10}H_{11}Zn \\ HO \end{matrix}\right\} = \left\{\begin{matrix} C_{10}H_{11}H \\ ZnO. \end{matrix}\right.$$

From this circumstance, and its analogy with zincmethylium, there can be no doubt that its formula is

$$C_{10}H_{11}Zn.$$

Action of Mercury upon Iodide of Methyl in presence of Light.

When iodide of methyl is exposed to sunlight in contact with metallic mercury, it

soon becomes coloured red from the separation of free iodine; after several hours' exposure this coloration disappears, and a small quantity of the yellow iodide of mercury subsides to the bottom of the liquid: after the action of sunlight for several days, the bulk of the mercury is observed to have considerably diminished, and white crystals begin to be deposited around the sides of the glass vessel: finally, after about a week's exposure, the liquid solidifies to a colourless crystalline mass: when this is digested with ether, the new compound dissolves, and is thus separated from metallic mercury, and from the small portion of iodide of mercury which is collaterally formed. Only a very small quantity of gas is evolved during the formation of the white crystalline compound. By spontaneous evaporation the ethereal solution solidifies to a mass of minute colourless crystalline scales: these, dried *in vacuo* and submitted to analysis, yielded the following numbers:—

I. ·3170 grm. dissolved in alcohol and treated with nitrate of silver, gave ·2142 grm. iodide of silver.

II. ·6205 grm. burnt with oxide of copper, gave ·0813 grm. carbonic acid, ·0505 grm. water, and ·5960 grm. protoiodide of mercury. The iodide of mercury, a small portion of which was decomposed into metallic mercury and periodide, collected as an incrustation at the front end of the combustion-tube, about a couple of inches of which had been left empty for this purpose, and projected from the furnace, the heat being so regulated that none of the iodide passed into the chloride of calcium tube, whilst none of the watery vapour condensed in the combustion-tube. When the analysis was concluded, the weight of the protoiodide of mercury, mixed with traces of periodide and metallic mercury, was determined by cutting off the part of the combustion-tube containing it, and ascertaining its weight before and after the iodide was removed. The numbers obtained agree very closely with the formula

$$C_2 H_3 Hg I,$$

which requires the following values:—

	Calculated.		Found. I.	Found. II.
2 equivs. Carbon . . .	12	3·51	——	3·57
3 equivs. Hydrogen . .	3	·88	——	·90
1 equiv. Mercury . .	200	58·51	——	96·05
1 equiv. Iodine . . .	126·84	37·10	36·56	
	341·84	100·00		100·52

This compound is therefore evidently the iodide of a new organo-metallic body, consisting of one atom of methyl and one atom of mercury, and for which I propose the name *hydrargyromethylium*: it is formed by the direct union of one atom of mercury with one atom of iodide of methyl, under the influence of light,

$$\left.\begin{matrix} C_2 H_3 I \\ Hg \end{matrix}\right\} = C_2 H_3 Hg I.$$

Iodide of hydrargyromethylium is a white solid, crystallizing in minute nacreous scales, which are insoluble in water, moderately soluble in alcohol, and very soluble in ether and iodide of methyl; by the spontaneous evaporation of these solutions the crystals are again deposited unchanged. Iodide of hydrargyromethylium is slightly volatile at ordinary temperatures, and exhales a weak but peculiarly unpleasant odour, which leaves a nauseous taste upon the palate for several days; at 100° C. the volatility is much greater, and the crystals are rapidly dissipated at this temperature when exposed to a current of air. At 143° C. it fuses and sublimes without decomposition, condensing in brilliant and extremely thin crystalline plates. In contact with the fixed alkalies and ammonia, it is converted into oxide of hydrargyromethylium, which is dissolved by excess of all these reagents; from these solutions sulphide of ammonium throws down sulphide of hydrargyromethylium as a slightly yellow flocculent precipitate of a peculiar and most insupportable odour. I have not yet further examined the reactions of this remarkable body, nor have I attempted the isolation of the hydrargyromethylium.

A corresponding compound containing amyl is formed, though with difficulty, under similar circumstances, but I have not yet succeeded in producing one containing ethyl, the iodide of this radical yielding, as I have shown *, when exposed to sunlight in contact with mercury, iodide of mercury, and a mixture of ethyl, hydride of ethyl and olefiant gases.

I have also made some preliminary experiments with other metals, and find that most of them are capable of thus entering into combination with the organic groups, methyl, ethyl, and amyl; amongst those which thus combine under the influence of light most readily, and seem to promise the most interesting results, I may mention arsenic, antimony, chromium, iron, manganese and cadmium. I hope to have the honour of laying before the Royal Society, at an early period, the results of my experiments upon these compounds.

Imperfect as our knowledge of the organo-metallic bodies may yet appear, I am unwilling to close this memoir without directing attention to some peculiarities in the habits of these compounds, which promise at least to throw some light upon their rational constitution, if they do not lead to extensive modifications of our views respecting chemical compounds in general, and especially that interesting class termed conjugate compounds.

That stanethylium, zincmethylium, hydrargyromethylium, &c. are perfectly analogous to cacodyl there can be no reasonable doubt, inasmuch as, like that body, they combine directly with the electro-negative metalloids forming true salts, from which, in most cases, and probably in all, the original group can be again separated unaltered, and therefore any view which may be taken of the new bodies must neces-

* Journal of the Chemical Society, vol. iii. p. 331.

sarily be extended to cacodyl. The discovery and isolation of this so-called organic radical by Bunsen was certainly one of the most important steps in the development of organic chemistry, and one, the influence of which upon our theoretical views of the constitution of certain classes of organic compounds, can scarcely be too highly estimated. It was impossible to consider the striking features in the behaviour of this body, without finding in them a most remarkable confirmation of the theory of organic radicals, as propounded by Berzelius and Liebig.

The formation of cacodyl, its habits, and the products of its decomposition, have for some time left no doubt of the existence of methyl ready formed in this body; and Kolbe*, in developing his views on the so-called conjugate compounds, has proposed to regard it as arsenic conjugated with two atoms of methyl ($(C_2H_3)_2As$). So long as cacodyl was an isolated example of an organo-metallic body, this view of its rational composition, harmonizing as it did so well with the facts elicited during the route of cacodyl through its various combinations and decompositions, could scarcely be contested; but now, since we have become acquainted with the properties and reactions of a considerable number of analogous bodies, circumstances arise, which I consider militate greatly against this view, if they do not render it absolutely untenable. According to the theory of conjugate radicals just alluded to, cacodyl and its congeners, so far as they are at present known, would be thus represented:—

Cacodyl	$(C_2H_3)_2As$.
Oxide of cacodyl	$(C_2H_3)_2AsO$.
Cacodylic acid	$(C_2H_3)_2AsO_3$.
Stanmethylium	$(C_2H_3)Sn$.
Stanethylium	$(C_4H_5)Sn$.
Oxide of stanethylium	$(C_4H_5)SnO$.
Stanamylium	$(C_{10}H_{11})Sn$.
Zincmethylium	$(C_2H_3)Zn$.
Zincethylium	$(C_4H_5)Zn$.
Zincamylium	$(C_{10}H_{11})Zn$.
Stibethine (Stibethyl)	$(C_4H_5)_3Sb$.
Binoxide of stibethine	$(C_4H_5)_3SbO_2$.
Oxide of stibmethylium	$(C_2H_3)_4SbO$.
Hydrargyromethylium	$(C_2H_3)Hg$.
Iodide of hydrargyromethylium	$(C_2H_3)HgI$.

It is generally admitted, that when a body becomes conjugated, its essential chemical character is not altered by the presence of the conjunct: thus for instance, the series of acids $C_nH_nO_4$, formed by the conjunction of the radicals $C_nH_{(n+1)}$ with oxalic acid, have the same neutralizing power as the original oxalic acid; and, therefore, if we assume the organo-metallic bodies above mentioned to be metals

* Journal of the Chemical Society, vol. iii. p. 372.

conjugated with various hydrocarbons, we might reasonably expect, that the chemical relations of the metal to oxygen, chlorine, sulphur, &c. would remain unchanged; a glance at the formulæ of these compounds will however suffice to show us that this is far from being the case: it is true that cacodyl forms protoxide of cacodyl and cacodylic acid, corresponding the one to a somewhat hypothetical protoxide of arsenic, which, if it exist, does not seem to possess any well-defined basic character, and the other to arsenious acid; but no compound corresponding to arsenic acid can be formed, and yet it cannot be urged that cacodylic acid is decomposed by the powerful reagents requisite to procure further oxidation, for concentrated nitric acid may be distilled from cacodylic acid without decomposition or oxidation in the slightest degree; the same anomaly presents itself even more strikingly in the case of stanethylium, which, if we are to regard it as a conjugate radical, ought to combine with oxygen in two proportions at least, to form compounds corresponding to protoxide and peroxide of tin; now stanethylium rapidly oxidizes when exposed to the air and is converted into pure protoxide, but this compound exhibits none of that powerful tendency to combine with an additional equivalent of oxygen, which is so characteristic of protoxide of tin; nay, it may even be boiled with dilute nitric acid without evincing any signs of oxidation: I have been quite unable to form any higher oxide than that described; it is only when the group is entirely broken up and the ethyl separated, that the tin can be induced to unite with another equivalent of oxygen. Stibethyl also refuses to unite with more or less than two equivalents of oxygen, sulphur, iodine, &c., and thus forms compounds, which are not at all represented amongst the combinations of the simple metal antimony.

When the formulæ of inorganic chemical compounds are considered, even a superficial observer is struck with the general symmetry of their construction; the compounds of nitrogen, phosphorus, antimony and arsenic especially exhibit the tendency of these elements to form compounds containing 3 or 5 equivs. of other elements, and it is in these proportions that their affinities are best satisfied; thus in the ternal group we have NO_3, NH_3, NI_3, NS_3, PO_3, PH_3, PCl_3, SbO_3, SbH_3, $SbCl_3$, AsO_3, AsH_3, $AsCl_3$, &c.; and in the five-atom group NO_5, NH_4O, NH_4I, PO_5, PH_4I, &c. Without offering any hypothesis regarding the cause of this symmetrical grouping of atoms, it is sufficiently evident, from the examples just given, that such a tendency or law prevails, and that, no matter what the character of the uniting atoms may be, the combining power of the attracting element, if I may be allowed the term, is always satisfied by the same number of these atoms. It was probably a glimpse of the operation of this law amongst the more complex organic groups, which led Laurent and Dumas to the enunciation of the theory of types; and had not those distinguished chemists extended their views beyond the point to which they were well supported by then existing facts,—had they not assumed, that the properties of an organic compound are dependent upon the position and not upon the nature of its single

atoms, that theory would undoubtedly have contributed to the development of the science to a still greater extent than it has already done; such an assumption could only have been made at a time when the data upon which it was founded were few and imperfect, and, as the study of the phenomena of substitution progressed, it gradually became untenable, and the fundamental principles of the electro-chemical theory again assumed their sway. The formation and examination of the organo-metallic bodies promise to assist in effecting a fusion of the two theories which have so long divided the opinions of chemists, and which have too hastily been considered irreconcileable; for, whilst it is evident that certain types of series of compounds exist, it is equally clear that the nature of the body derived from the original type is essentially dependent upon the electro-chemical character of its single atoms, and not merely upon the relative position of those atoms. Let us take, for instance, the compounds formed by zinc and antimony; by combination with 1 equiv. of oxygen the electro-positive quality of the zinc is nearly annihilated; it is only by the action of the highly oxidizing peroxide of hydrogen that the metal can be made to form a very unstable peroxide; but when zinc combines with 1 equiv. of methyl or ethyl, its positive quality, so far from being neutralized, is exalted by the addition of the positive group, and the compound now exhibits such intense affinity for the electro-negative elements as to give it the property of spontaneous inflammability. Teroxide of antimony has also little tendency to pass into a higher state of oxidation; but when its three atoms of oxygen are replaced by the electro-positive ethyl, as in stibethine, that affinity is elevated to the intense degree which is so remarkable in this body.

Taking this view of the so-called conjugate organic radicals, and regarding the oxygen, sulphur or chlorine compounds of each metal as the true molecular type of the organo-metallic bodies derived from it by the substitution of an organic group for oxygen, sulphur, &c., the anomalies above mentioned entirely disappear, and we have the following inorganic types and organo-metallic derivatives:—

Inorganic Types.		Organo-metallic Derivatives.
$As\left\{\begin{matrix} S \\ S \end{matrix}\right\}$	. . .	$As\left\{\begin{matrix} C_2H_3 \\ C_2H_3 \end{matrix}\right\}$ Cacodyl.
$As\left\{\begin{matrix} O \\ O \\ O \end{matrix}\right\}$	. . .	$As\left\{\begin{matrix} C_2H_3 \\ C_2H_3 \\ O \end{matrix}\right\}$ Oxide of Cacodyl.
$As\left\{\begin{matrix} O \\ O \\ O \\ O \\ O \end{matrix}\right\}$	. . .	$As\left\{\begin{matrix} C_2H_3 \\ C_2H_3 \\ O \\ O \\ O \end{matrix}\right\}$ Cacodylic acid.
Zn O	. . .	Zn (C_2H_3) Zincmethylium.

Inorganic Types.		Organo-metallic Derivatives.
$Zn\left\{\begin{matrix} O \\ O_x \end{matrix}\right\}$	. . .	$Zn\left\{\begin{matrix} C_2H_3 \\ O_x \end{matrix}\right\}$ Oxide of Zincmethylium.
$Sb\left\{\begin{matrix} O \\ O \\ O \end{matrix}\right\}$	. . .	$Sb\left\{\begin{matrix} C_4H_5 \\ C_4H_5 \\ C_4H_5 \end{matrix}\right\}$ Stibethine.
$Sb\left\{\begin{matrix} O \\ O \\ O \\ O \\ O \end{matrix}\right\}$	. . .	$Sb\left\{\begin{matrix} C_4H_5 \\ C_4H_5 \\ C_4H_5 \\ O \\ O \end{matrix}\right\}$ Binoxide of Stibethine.
$Sb\left\{\begin{matrix} O \\ O \\ O \\ O \\ O \end{matrix}\right\}$	. . .	$Sb\left\{\begin{matrix} C_4H_5 \\ C_4H_5 \\ C_4H_5 \\ C_4H_5 \\ O \end{matrix}\right\}$ Oxide of Stibethylium.
$Sn\ O$	. . .	$Sn\ (C_4H_5)$ Stanethylium.
$Sn\left\{\begin{matrix} O \\ O \end{matrix}\right\}$	. . .	$Sn\left\{\begin{matrix} C_4H_5 \\ O \end{matrix}\right\}$ Oxide of Stanethylium.
$Hg\left\{\begin{matrix} I \\ I \end{matrix}\right\}$	. . .	$Hg\left\{\begin{matrix} C_2H_3 \\ I \end{matrix}\right\}$ Iodide of Hydrargyromethylium.

The only compound which does not harmonize with this view is ethostibylic acid, to which Löwig assigns the formula $C_4H_5SbO_5$; but as that chemist has not yet fully investigated this compound, it is possible that further research may satisfactorily elucidate its apparently anomalous composition.

It is obvious that the establishment of this view of the constitution of the organo-metallic bodies will remove them from the class of organic radicals, and place them in the most intimate relation with ammonia and the bases of Wurtz, Hofmann and Paul Thenard; indeed, the close analogy existing between stibethine and ammonia, first suggested by Gerhardt, has been most satisfactorily demonstrated by the behaviour of stibethine with the haloid compounds of methyl and ethyl. Stibethine furnishes us, therefore, with a remarkable example of the operation of the law of symmetrical combination above alluded to, and shows, that the formation of a five-atom group from one containing three atoms, can be effected by the assimilation of two atoms, either of the same, or of opposite electro-chemical character: this remarkable circumstance suggests the following question:—Is this behaviour common also to the corresponding compounds of arsenic, phosphorus and nitrogen; and can

the position of each of the five atoms, with which these elements respectively combine, be occupied indifferently by an electro-negative or an electro-positive element? This question, so important for the advance of our knowledge of the organic bases and their congeners, cannot now long remain unanswered.

If the views which I have just ventured to suggest should be as well borne out by future researches as they are by the facts already known, they must occasion a profound change in the nomenclature of the extensive series of compounds affected by them: I have not, however, ventured to introduce this new system of nomenclature, even in the case of the new bodies described in this memoir, since hasty changes of this kind, unless absolutely necessary, are always to be deplored. In accordance with the suggested view of the constitution of the organo-metallic compounds, the following plan of nomenclature would probably be found most convenient.

Arsenic Compounds.

$(C_2H_3)_2As$	Bimethide of arsenic.
$(C_2H_3)_2AsO$	Bimethoxide of arsenic.
$(C_2H_3)_2AsO_3$	Bimetharsenic acid.
$(C_2H_3)_2AsO_3+KO$ . . .	Bimetharseniate of potash.

Zinc Compounds.

$(C_2H_3)Zn$	Methide of zinc.
$(C_4H_5)Zn$	Ethide of zinc.
$(C_{10}H_{11})Zn$	Amylide of zinc.

Tin Compounds.

$(C_2H_3)Sn$	Methide of tin.
$(C_2H_3)SnI$	Methiodide of tin.
$(C_4H_5)Sn$	Ethide of tin.
$(C_4H_5)SnO$	Ethoxide of tin.
$(C_4H_5)SnCl$	Ethochloride of tin.
$(C_4H_5)SnOSO_3$	Sulphate of ethoxide of tin.
$(C_{10}H_{11})Sn$	Amylide of tin.
$(C_{10}H_{11})SnO$	Amyloxide of tin.

Antimony Compounds.

$(C_2H_3)_3Sb$	Termethide of antimony.
$(C_2H_3)_4SbO$	Quadromethoxide of antimony.
$(C_4H_5)_3Sb$	Terethide of antimony.
$(C_4H_5)_3SbO_2$	Terethobinoxide of antimony.

Mercury Compounds.

$(C_2 H_3)Hg$	Methide of mercury.
$(C_2 H_3)HgI$	Methiodide of mercury.

In naming the new bodies described in the present paper, I have, in conformity with the nomenclature of the organic bases, adopted the principle of employing the termination "*ium*" when the body unites with one equivalent of oxygen, chlorine, sulphur, &c., like ammonium, and the terminal "*ine*" when, like ammonia, it combines with two additional atoms.

XX. *On the Arrangement of the Foliation and Cleavage of the Rocks of the North of Scotland.* *By* Daniel Sharpe, *F.R.S., V.P.G.S.*

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Introduction.

THE mineralogical characters of the rocks to which this memoir relates have been so fully described by Macculloch and other writers, that it is not necessary to enter into any details on that head; my object is to describe the usual arrangement of the layers of foliation which have given an appearance of stratification to Gneiss and Mica Schist, to show the relation of their foliation to the cleavage of the stratified slates, and to sketch out roughly the direction of the foliation and cleavage through the North of Scotland.

The most important remarks yet published upon these subjects will be found in the sixth chapter of Mr. Darwin's "Geological Observations on South America," which, in addition to his own views and observations, contains a summary of what had then been done by others: Mr. Darwin's remarks will be frequently quoted in this paper. I must also refer the reader to two papers on Slaty Cleavage, which I laid before the Geological Society in 1846 and 1848, and which are published in the third and fifth volumes of that Society's Journal.

The terms *cleavage* and *foliation* are here used nearly as they have been defined by Professor Sedgwick* and Mr. Darwin (p. 141); *cleavage* or *lamination* being applied to planes of division along which a rock may be split into thin parallel sheets independent of, and usually transverse to its stratification. *Foliation* describes the division of rocks believed to be of crystalline origin into layers of different mineral substances, whether the rock splits along those layers as in mica schist, forms a solid rock as in the granitic varieties of gneiss, or consists of distinct sheets of different materials, such as quartz, mica, chlorite, &c., which sheets may be of any thickness, from a line up to 1 or 2 feet, as may be seen both in the "laminar gneiss" and "chlorite schist series" of Macculloch. For the sake of clearness, the term *slate* is here confined to laminated rocks with a cleavage independent of their stratification, and *schist* to foliated rocks with only one set of divisional surfaces produced by the foliation. The distinctions between foliation and stratification, which Macculloch and many subsequent authors have confounded, will be easily understood when the various peculiarities of foliation have been described.

Under whatever aspect we regard them, it is difficult to draw any clear line of

* Transactions of the Geological Society, Second Series, vol. iii. p. 480.

distinction between gneiss and mica schist; considered mineralogically, they are the two extremes of a series of foliated crystalline rocks, between which every shade of variation may be found; their geological relations are the same, and the foliation of the two rocks follows exactly the same laws. Their geographical separation through Scotland has been drawn most arbitrarily; the lines which divide them on Macculloch's map have very little meaning, and there are many large tracts where the colouring might be reversed without any impropriety. The same may be said of the *chlorite schist* series of Macculloch, which, where I have seen it, is a variety belonging to the same class of rocks, but more nearly related to gneiss than to schist, in which the mineral ingredients are more numerous, and more completely separated from each other in parallel layers than is usual in gneiss or mica schist; yet there are many districts in which the separation of the ingredients of gneiss into layers is as complete as in the chlorite schist described by Macculloch*, and in which the quartz forms layers between one and two feet thick, alternating with layers of micaceous schist, or with beds of a crystalline rock without foliation or cleavage. The other varieties of schist described by Macculloch belong to the same series of rocks, and agree with gneiss or mica schist in everything but mineral composition. The difference of their constituent parts appears to have produced no alteration in the foliation of these rocks, which follows the same laws in them all; the only variation in that respect consists in the greater or less completeness of the separation of the different minerals into parallel layers; and these are but different stages of one process. For the sake of brevity, therefore, I shall use the terms gneiss and mica schist to include all the varieties of foliated rocks.

Under the name of *quartz rock*, Macculloch has classed in connection with gneiss and the various schists, two sets of rocks belonging geologically to two most distinct classes, and only resembling each other mineralogically.

The *quartz rocks* of Glenorchy and Tyndrum †, with most of the varieties composed of quartz and felspar, or quartz and mica, enumerated in Macculloch's classification of rocks, belong to gneiss, from which they differ only in simplicity of composition, and with which they are usually found; these are foliated rocks, and are here regarded as so completely part of the gneiss as to require no separate mention. But the greater part of the quartz rock which figures on the map of Scotland is an altered sandstone, of which the mineral character has been changed by plutonic action; it is a sedimentary deposit in which the stratification is usually distinctly visible. The difference between these two quartz rocks is so obvious, that they may be distinguished even in reading Macculloch's descriptions. The latter forms no part of the subject of this paper, being a stratified, and not a foliated rock; but it was necessary to mention it, lest any of the following remarks on the subject of foliation should have been supposed to apply to it ‡.

* Western Isles, vol. ii. p. 284. † Macculloch, Transactions of the Geological Society, vol. ii. p. 478.

‡ I have entered more fully into this subject in a paper laid before the Geological Society.

Contortions of the Foliation.

When an observer enters any district of gneiss or schist in search of order in the arrangement of their folia, his first impression will be that of despair; so numerous are the convolutions of both rocks, varying in importance from curves which have determined the outline of mountain ridges to folds no larger than those in the drapery of a curtain, and so complicated and involved do they appear, that it seems hopeless to seek for any general plan among them. These contortions have often been figured *, but no representation has ever shown a tithe of their complication; they are the most numerous along the dip of the foliation, but they often occur on the line of strike also; and in spots where the rock is only exposed to a small extent, it is often impossible to judge of the prevalent direction of the foliation.

But where large sections are well seen, a more extended observation will show that even where the gneiss is most convoluted, there is still a general direction of the folia or layers of which it is composed, which may be caught by disregarding the minor folds, and fixing the attention only on the larger curves formed by a combination of many of the smaller flexures. This will sometimes be more readily seen from a distance, at which the minuter complications are lost sight of. The simpler lines of curve thus obtained may perhaps still be in themselves complicated; but by repeating the process of simplification and disregarding the secondary flexures, a series of lines of dip are at last obtained, sufficiently simple to admit of classification and of representation in a reduced drawing.

The observations of the direction of the strike must be corrected by repetition; and where the lines of strike are most curved, the gneiss will often be found rising into elliptical bosses, the longer axes of which are all parallel and give the general direction of the strike of the district.

Gneiss and mica schists are not equally contorted throughout; the convolutions are usually most complicated where the inclination of the folia is slightest; where their dip is steeper the rock is usually less contorted, and where the foliation is perpendicular it usually follows true planes without any distortion either on the dip or strike. The granitic varieties of gneiss, in which the foliation is slightly marked, are, I believe, more common where the angle of inclination is slight; and the separation of the minerals is usually well marked where the gneiss is either perpendicular or highly inclined. Thus observations taken where the foliation is perpendicular, admit of much greater accuracy than can be obtained where it is slightly inclined.

Arrangement of the Foliation in Arches.

If any extensive region of gneiss or schist is crossed in a direction transverse to the strike, and the larger and bolder curves noted to the exclusion of the minor con-

* Macculloch, Western Isles, Plate 12, and frontispiece; also Geological Transactions, vol. ii. Plate 31 and 32.

volutions, the foliation will usually be found to be arranged in great arches, in which the inclination of the folia increases as they are more distant from the centre until they reach the perpendicular; the whole thus forming parts of a flat arch bounded by two perpendiculars, a theoretical restitution of which is shown in figure 2.

Beyond the perpendiculars which form the boundaries of the arch there are usually comparatively narrow bands, in which the dip of the folia is irregular; these are succeeded by other perpendicular planes, usually parallel to the first, which form the boundaries of other arches standing on each side of the first, but which are often incomplete. As the planes and curved surfaces are extended in the direction of the strike, each arch here represented is the section of a semi-cylinder, which is often of great length.

The perpendicular planes of foliation frequently run along important ridges of hills, which has brought them into notice; while the lower and flatter central portions, in which the general arrangement is disguised by the contortions, have attracted less attention. Hence several geologists have mentioned "the fan-like or radiating structure in the metamorphic schists of the Alps, in which the folia in the central crests are vertical, and on the two flanks inclined inwards*;" but it has been overlooked, that the fan-like form is produced by combining portions of two adjoining arches: an inspection of the sections, fig. 1 to 5, will convince every one that the arch is the figure of real importance in the arrangement, and the *fan-shaped* form only catches the eye when seen without the rest of the figure.

Analogy between Foliation and Cleavage.

If the arrangement of the foliation which has been just described be compared with that of the cleavage planes of the true slates which I have explained elsewhere†, the two phenomena will be found to correspond in all their main features; both forming great arches bounded by vertical planes, with the peculiarity that in both the dip of the divisional surfaces is least at the central axis, and gradually increases towards the boundaries till it becomes perpendicular. The difference between the two lies in the convolution of the layers of gneiss and schist, which contrasts strongly with the regularity of the flat parallel surfaces produced in slate by the cleavage; yet the difference is rather of degree than of principle; for in Devonshire, where the cleavage forms an unbroken arch with a diameter of sixty miles, and where we can judge of its real characters better than in more disturbed districts, the central part undulates in low curves‡, which are evidently analogous to the convolutions of the central portions of the arches of foliation.

Mr. Darwin has stated, that when in various parts of South America he met with laminated and foliated rocks either alternating with or near to one another, the cleavage and foliation were generally parallel; and from this circumstance, and an

* Darwin, South America, p. 164, where Von Buch and Studer are quoted.

† Journal of the Geological Society, vol. iii. p. 92.

‡ Journal of the Geological Society, vol. iii. p. 95.

occasional tendency in stratified slates to exhibit an incipient mineralogical change along the planes of their cleavage, he infers "that foliation and cleavage are parts of the same process: in cleavage, there being only an incipient separation of the constituent minerals; in foliation, a much more complete separation and crystallization*."

Along the southern border of the Highlands, a band of slate, in which the stratification and cleavage planes are equally visible, rests on foliated mica schist, affording an excellent opportunity of testing these views. The result of an examination of that district, fully bears out Mr. DARWIN's opinion: where the two formations come together, the cleavage and foliation are not only conformable at the junction, but combine to form an arch in common. I have selected four sections to illustrate this conformity under different circumstances: at Dunkeld, and on the east side of Loch Lomond, fig. 1 and 3, an arch formed principally of the foliation of the mica schist, is completed by the addition of the cleavage of part of the slate on its southern flank. In Strath Earn and the lower part of Glen Shee, fig. 4 and 5, the foliation of the mica schist, besides forming a complete arch, supplies part of the imperfect portion of another arch to the south, in which it is combined conformably with the cleavage planes of the slates: a similar conformity between the foliation and cleavage is found throughout; but the stratification of the slate, indicated in the sections by the dotted lines, is unconformable to the foliation of the mica schist, and is of no account in the question under consideration†.

An additional proof that one cause has produced both the foliation and cleavage, is found in the continuation of the same divisional planes through clay slate and mica schist from Loch Katterin to Glen Shee; these are alternately planes of cleavage in the slate and of foliation in the schist; the boundary line between the formations being deflected, while the divisional planes keep a direct course and pass from one rock to the other several times without changing their direction or their inclination. When Professor SEDGWICK announced the persistence of cleavage planes through beds of different mineral character, it was at once admitted that the uniform cleavage of the various beds must be regarded as one process. There is the same reason for connecting together foliation and cleavage, now that the foliation of the schist is shown to be continuous with the cleavage of the slates. For these reasons the two phenomena will now be described together.

Geographical Arrangement of the Foliation and Cleavage.

It is obvious, from what has been stated of the greater distinctness and freedom from contortion of the foliation when vertical, that in studying its arrangements over any district, it will be best to begin by determining the boundary lines of the

* Geological Observations on South America, p. 155 and 165.

† Not to encumber this paper with details foreign to its principal object, I reserve an account of the stratified rocks of the Highland border for a communication to the Geological Society.

arches along which the foliation is perpendicular. In fact it is not always possible to distinguish the central axis of the arches, as the slight inclination and numerous waves of the foliation are most confusing to the observer near the crown of the arch. In some places the correspondence between the central axis of the foliation and the watershed of the district will be an assistance, but in many cases the axis of the arch of foliation will be most easily ascertained by dividing the distance between the two vertical boundaries. For these reasons, in describing the geographical arrangement of the foliation and cleavage through the Highlands, I shall point out in most detail the lines on which their planes are perpendicular.

On the Map which accompanies this paper, the direction of the vertical planes is marked by double lines as far as I observed them or find them recorded; these are continued in double lines of dots to show the supposed continuation of the observed portions: the central axes are indicated by a single thick line, continued by dots in the same manner over the unobserved portions of country: the fainter single lines represent the strike of the foliation over the intervening spaces, the lines being closest where the dip is the steepest. Similar lines are used both for the foliation of the gneiss and schist, and for the cleavage of the slates, the colours on the Map indicating the rock being a sufficient distinction. It must be understood that this map is but a first rough attempt to lay down in a geographical form a set of phenomena which few observers have thought worth recording*, and which require long and patient examination before it can be completed.

Commencing on the south of the Highlands, the first perpendicular line will be found about four miles north of the Highland border, following a direction not quite uniform, but on the whole about N. 60° E., and running partly through clay slate and partly through mica schist, as follows; on the east side of Loch Lomond the cleavage of the slate is perpendicular at the farm of Cashel, half-way between Rowardennan and Balmaha, striking N. 30° E. I did not trace the course of this line to the west coast, but proceeding eastward, I observed it between Loch Chon and Loch Ard striking N. 65° E., and again a few hundred yards above the foot of Loch Ketterin striking N. 45° E., in both cases through slate. There are obviously some faults in this district which have broken the line and thrown its eastern end to the southward. In Strath Earn the boundary of the slate trends somewhat southward of its previous course; but this has not altered the direction of the perpendicular plane, which runs N. 60° E. through the mica schist one and a half mile south of St. Fillan's. The line follows nearly the same direction through Perthshire without further disturbance, running N. 45° E. through the clay slate on the north side of

* MACCULLOCH often mentions the strike of the gneiss in his "Western Isles," but rarely gives the dip. A few observations on the point occur in Nicol's useful Guide to the Geology of Scotland. The only consecutive series of observations which I have met with, are laid down on the geological maps of Sutherland and Banffshire by the late Mr. CUNNINGHAM, published in the forty-sixth and fifty-seventh numbers of the Journal of the Highland Society for September 1839 and June 1842: these have been of great service to me.

Dunkeld; through mica schist at the Brig-o-Cally with a strike of N. 50° E., beyond which spot I did not follow it. Thus along the Highland border the foliation of the mica schist and the cleavage of the slate are both vertical, along lines so closely corresponding, that they may be considered as continuous, which is perhaps the strongest evidence that can be adduced to show the identity of the causes which produced the two phenomena.

To the south of the perpendicular which has been traced above, the cleavage hangs over to the south, with a steep dip of about N.N.W., thus forming the commencement of another arch, which is broken off abruptly along the line of junction of the clay slate and Old Red Sandstone, in the manner shown at the southern extremity of the sections, fig. 1 to 5. But on the north side, the cleavage, and then the foliation dip about S.S.E. towards the perpendicular for a space of five or six miles, first at a high angle, and then at a lower inclination accompanied with many waving contortions; the dip then changes to about N.N.W. for a similar space of five or six miles, and again increases till it reaches the perpendicular on a line nearly parallel to the first, the whole forming an arch varying from ten to twelve miles in width, composed partly of cleavage and partly of foliation: the central axis of this arch runs along the high ridge of hills on the south side of Loch Tay.

The perpendicular plane which forms the northern boundary of the arch just described is broken and often irregular in its direction, and entirely confined to the gneiss or mica schist, but as I crossed it at distant intervals I can only map its course approximatively. The first point observed on the west was in Strath Fillan, about one and a half mile south of Tyndrum, where a vertical foliation runs about N. 50° E. through a hard and very quartzose gneiss: the same perpendicular plane runs N. 45° E. through Ben Lawers; N. 60° E. at the ridge between Strath Tay and Glen Tummel, a little to the east of Schehallion; crosses the pass of Killicrankie in a direction very little north of east, and runs N. 35° E. across Glen Shee, about a mile below the Spittal;—a line drawn N. 50° E. from the western point observed in Strath Fillan to Killicrankie passes through the intermediate spots named, and gives N. 50° E. as the general bearing of this vertical plane of foliation, leaving out of account the observation in Glen Shee, where the plane seems to have been thrown to the south by some local disturbance.

If this line were continued westward from Tyndrum, it would pass near Inverary, but some eruptive masses of porphyry on the west side of Loch Fyne have destroyed the regularity of the foliation in that neighbourhood; the vertical line, however, reappears further west and crosses Knapdale, where it attracted MACCULLOCH's attention: the same author also describes the foliation of the schists meeting in a roof-like form down the middle of Cantyre. Thus he has pointed out both the central axis of the arch just described, as well as its northern border, without observing the arched arrangement of the foliation *.

* Western Isles, vol. ii. p. 287.

The next arch which crosses the Highlands, with a diameter of between twenty-five and thirty miles, runs for the most part through gneiss: its southern boundary coincides with the northern boundary of the arch just described; I only observed on two points the perpendicular foliation which forms its northern limit: two perpendicular planes, about a mile apart bearing N. 40° E., cross the valley of the Spey, near the junction of the road from Loch Laggan with that which runs down Strath Spey, from which spot they run on through Corbuie into the line of the Monaglea mountains; I observed the perpendicular again in those mountains four miles N.W. of Kingussie, where the regularity of its course was destroyed by the intrusion of a neighbouring boss of granite, and the direction of the foliation varied; but as this spot bears N. 40° E. from that at which the perpendicular crossed the Spey, we may conclude that to be the direction of the line sought for. The axis of this great arch runs for some distance along the central ridge of the Grampians. The great granitic masses of Ben Cruachan and Ben Muick Dhui, with several minor bosses of granite, have broken through the middle of the arch, in each instance disturbing the regularity of the foliation for some distance around; thus in Glen Feshie, on the western side of the Ben Muick Dhui range, the foliation strikes for some distance nearly north, yet the general features of an arch are preserved, of which the crown coincides with the ridge separating the waters of the Feshie and Geauley.

I have not examined the district to the west of the Ben Muick Dhui range, but this deficiency is partly supplied by a paper of Mr. CUNNINGHAM's on Banffshire*. In the map which accompanies that memoir, the cleavage is marked as vertical with a direction of N. 45° E. in the slates on the south of the town of Banff, and again with the same direction at Kinairdy, south of Aberchirder, while throughout nearly all the rest of the county the dip is represented as inclining to the S.E. at various angles. It appears from these data that though the direction of the foliation and cleavage is nearly the same on the east and west of the granite of Ben Muick Dhui, the whole arch on the eastern side is thrown much to the north of that on the western side of the granite; for the perpendicular boundary of the arch near Banff is about on the line of the centre of the arch at the Grampians: as this is the greatest irregularity met with in these phenomena, I regret that I had not time to work out the details connected with it; but having everywhere found that the regularity of the foliation and cleavage is disturbed in the neighbourhood of granite, I cannot hesitate in ascribing this deflection to the influence of the granite which occupies so large a part of Aberdeenshire, and of which the northerly direction of the foliation in Glen Feshie is the commencement.

I can give but a meagre account of the next line of perpendicular foliation, which runs nearly parallel to that last described at a distance of ten miles farther north, forming the northern boundary of an arch of the usual character, and of that diameter. The perpendicular in question runs N. 35° E. through Coryaraick, cross-

* Journal of the Highland Society, No. 57, p. 447.

ing the top of the pass taken by the old military road from Fort Augustus: this line is nearly parallel to the line of Lochs through the Great Glen, and about six miles south of them. I observed the cleavage of the clay slate vertical on two lines on the south bank of Loch Leven, one less than a mile east of the inn at Ballahulish, striking N. 30° E., the other, three and a half miles west of the same inn, striking N. 40° E.: though these lines are separated from the perpendicular seen at Coryaraick by the granite of Ben Nevis, they obviously belong to the same boundary.

The perpendicular line just mentioned forms the southern edge of a larger arch of gneiss, of which the northern boundary running through the centre of Ross-shire consists of several vertical lines separated by tortuous foliation, which together form a band several miles wide. The most southerly spot on which I crossed this band, was on the road from Fort William to Arasaig, on which traverse the gneiss is well exposed in a most instructive section: a line of vertical gneiss runs due north along each of the high ridges of hills between Loch Eil-head and Loch Sheil, and between these the foliation is highly inclined and much contorted, forming with the perpendiculars several of the fan-like arrangements before alluded to. Westward of Glen Finnen also the gneiss is vertical on several lines, gradually changing in direction as we go westward from N. 5° E. to N. 20° E., which last is the prevailing strike in that district. In taking a broad view of the phenomena, we must regard these lines of vertical foliation as constituting one band.

The next traverse made was from Fort Augustus to Glenelg along Glen Morrison and Glen Shiel: several lines of perpendicular gneiss cross this road in the high region which separates the waters of the Shiel and the Clunie, viz. one at the head of Loch Clunie, striking N. 30° E.; again at the Clunie inn on the top of the pass with the same strike, and another a mile down Glen Shiel, striking north: the prevailing direction is here N. 30° E.

On Loch Linchart the gneiss is vertical, both at the head of the Loch and at the village of Garve, with a direction of N. 45° E., which would carry the latter line through Ben Wyvis. I did not visit the country north of this place, but in Mr. CUNNINGHAM's map of Sutherland the gneiss is laid down as vertical, with a strike of N. 45° E. at two places on the north of the Kyle of Sutherland, beyond which I can only carry the line on conjecturally.

A band drawn N. 30° E. would pass through all the above-mentioned spots in Ross-shire at which the foliation is vertical; taken in connection with a line drawn from Ballahulish to Coryaraick, and continued eastward in the same direction, it incloses an arch, which at its southern extremity is not fifteen miles across, but which widens to twenty-five miles on the north-east: the Great Glen with its unrivalled chain of Lochs runs nearly parallel to the southern border of this area, halfway between its perpendicular boundary and the central axis of the arch: the granite of Loch Garry has broken through the centre, disturbing the regularity of the foliation for several miles around.

The perpendicular band which has been traced through Ross-shire is the most

northerly line of vertical foliation belonging to the great system of north-easterly strike which occupies so large a part of Scotland; for to the north-west of that band there is only half an arch broken off nearly along its central axis, at a line which may be drawn about N. 25° E. from the head of Loch Maree to Loch Eribol in Sutherland, and which divides the gneiss of Scotland into two districts of very unequal size, distinguished by different directions of their foliation. The gneiss immediately to the eastward of the last-mentioned line dips about E. 15° S. at angles rarely exceeding 15° or 20°: this dip gradually increases as we proceed eastward, till it reaches the perpendicular on the line just traced through the middle of Ross-shire, thus forming half an arch only twenty miles wide at its southern, and nearly thirty miles wide at its northern extremity, of which the western half is entirely wanting. But on the western side of the line between Loch Eribol and Loch Maree-head all the gneiss of the main land and that of the island of Lewis strikes towards the north-west; thus the axes of elevation of the two districts run nearly at right angles to one another. Mr. Cunningham, who has drawn attention to the line commencing at Loch Eribol as far as it runs through Sutherland, regarded the gneiss to the east of that line as of a more recent formation than that to the westward of it, stating that the eastern gneiss often overlies the stratified quartz rock and limestone, which never occurs to the westward of the line in question*.

Although the superposition of the gneiss to the quartz rock and red sandstone in other parts of the same line has been mentioned by Macculloch†, we may require further evidence before accepting so remarkable a statement; there are so many districts in the Highlands where secondary and truly stratified quartz rock has been confounded with gneiss that a mistake of the kind need create no surprise. It is however worth remarking, that the alteration of the Old Red Sandstone into quartz rock has taken place principally along or to the east of the line which separates the two districts of gneiss, from Loch Eribol to Loch Maree-head and Loch Carron; while the Old Red Sandstone resting on the gneiss to the west of that line is unaltered.

The want of parallelism between the lines of perpendicular foliation and cleavage, forming the boundaries of the great arches of gneiss, schist and slate which traverse Scotland, is an unexpected phenomenon deserving particular attention. The most southerly of these arches, that including Loch Tay, which consists principally of mica schist and clay slate, widens slightly in its course westward; but all the other arches, traversing districts consisting for the most part of gneiss, widen considerably as they proceed eastward with uniform regularity. The line from Loch Eribol to

Loch Maree, along the supposed axis of an arch, runs about . . .	N. 25° E.
The perpendicular through Ross-shire, runs about	N. 30° E.
That through Coryaraick, runs about	N. 35° E.
That through Corbuie and the Monaghlea Mountains, runs about .	N. 40° E.
And that through Ben Lawers, runs about	N. 50° E.

* Geognostical Account of Sutherland, p. 96 to 100.

† Western Isles, vol. ii. p. 94, and Plate 31, fig. 2.

If these lines were continued towards the south-west, they would converge between Lough Foyle and Lough Swilly in the middle of the great mica schist district of the North of Ireland.

As we cannot doubt that the lines forming the opposite boundaries of each arch of elevation are to be regarded as of contemporary formation, the divergence above pointed out militates strongly against M. Elie de Beaumont's favourite theory of the parallelism of contemporary mountain chains. Perhaps the explanation of the divergence is, that the area enclosed between each pair of lines is an ellipse of which we only see a portion in Scotland: this is rendered probable by the frequent occurrence on the surface of gneiss of elliptical elevations on a small scale.

The section given on Plate XXIII. fig. 1, is drawn across the Highlands in a direction transverse to the strike of the foliation; it shows at one view, as far as its scale admits, the dip of the divisional surfaces of the foliation and cleavage and the succession of arches which have been pointed out. It commences at Kyle Rhea Ferry in Ross-shire, on the shore opposite to the Isle of Skye, and is traced from there to the east-south-east, in as straight a line as my observations admitted of, through Fort Augustus to the Highland border below Dunkeld, being a length of about ninety miles. This is as long a line as can be drawn through the district of gneiss and schist, and is as free from the disturbing masses of granite and porphyry as any section which could have been chosen. So constant is the direction of the foliation, that any other line across the Highlands parallel to this would exhibit all the same principal features, differing only in the local disturbances. It is impossible on a small scale to represent the contortions of the gneiss, and the lines here given are only intended to represent the larger features of the phenomena, so that the section is hardly more than a diagram expressing general results. I have added below it a more theoretical diagram, fig. 2, in which the forms of the arches are completed, which are indicated by the lines of the true section, and the outline of the country added to show the relation borne by the position of the mountains and principal valleys to the inclination of the foliation and cleavage.

I have but little to add respecting the district of gneiss lying to the west of the line from Loch Eribol to the head of Loch Maree, and which should probably be continued to Loch Carron; throughout which the strike, as far as has been observed, is about north-west. The gneiss is only seen at intervals, between which it is covered up by Old Red Sandstone, so that we cannot see any continuous section, nor obtain a good idea of the relations of the different masses to each other. There seem to be a succession of arches of moderate diameter, of which we see the perpendicular walls standing out in relief, while the central parts between being lower are covered by the sandstone. It will be seen on the Map that the gneiss is vertical, both at Cape Wrath, at Sandwood Bay five miles south of that Cape, again vertical at Loch Inchard, on both sides of Storr Point, and on the north side of Loch Maree; all with a strike between N. 45° W. and N. 25° W. I found the same direction in the

gneiss of the Isle of Lewis as far as I examined it: there is a vertical plane crossing the island N. 45° W. from Broad Bay near Stornaway to Barvas, on the south side of which is a broad arch, which I only followed partially; its central axis is seen at Callernish striking N. 20° W., so that its other boundary may be looked for towards the southern end of Lewis.

Let us now quit these dry details, and turn to the connection between the direction of the foliation of the rocks and the physical features of the country, which will bring out many points of great interest. The most rugged and elevated hills are usually found either along the lines of vertical foliation or where the dip nearly approaches the perpendicular. Thus the southern perpendicular plane crosses Ben Voirlich, and runs close to Ben Ledi and Ben Venue: the next perpendicular described runs through Ben Lawers; the next through the Monaghlea Mountains; another through Coryaraick, and the northern line through Ben Wyvis: and besides crossing these, which are among the highest mountains in Scotland formed of gneiss and mica schist, the perpendiculars run along many high ridges, which will be seen most readily by reference to the Map.

The central axis of each of the great arches described, is also frequently occupied by hills of considerable elevation, but less rugged than those on the boundary lines, of which the hills on the south side of Loch Tay and the Grampians are the most striking examples.

The lowest ground frequently occurs about half-way between the perpendicular walls of the arches and their central axes, and all the great valleys which run parallel to the strike of the foliation occupy this position; as instances may be quoted the line of Loch Tay, Loch Dochart, and the upper reach of Loch Tyne; Strath Spey, from Laggan downwards; the valley of the Findhorn, and the Great Glen; all of which occupy lines intermediate between the centre and the boundary of their respective areas. These great valleys are not cracks analogous to the so-called valleys of elevation of the stratified districts; they seem to be lines of less elevation left at the original upheaval of the areas, and due to some unequal operation of the upheaving force. The annexed diagram shows a frequent form of outline of the country included in each arch of the foliation; but there are also several cases

where there is a valley on only one side of the central axis without any corresponding depression on the other side.

Perhaps the most striking evidence of the connection of the physical geography with the foliation of the rocks, will be found in observing on the Map how many of

the valleys and lakes of the Highlands follow lines which, if produced, would converge on the North of Ireland, near the points of convergence of the perpendiculars already described.

It must not however be supposed that all the great physical features of the Highlands are connected with the foliation of the gneiss and schists; there have been many disturbing forces at work at various later periods, which have broken up the surface into hills and valleys with very different directions. The most important of these are connected with the outbursts of granite, porphyry and similar rocks, far more numerous than are represented on any of our maps, which have broken through the gneiss and schists, and deranged the regularity of their foliation. Many districts also may have been disturbed by agents which have not shown themselves on the surface; of these, the neighbourhood of Loch Lomond is a remarkable instance. But as the gneiss appears to be the most ancient of the formations now visible in Scotland, so also the physical features connected with the foliation of the gneiss must be regarded as the earliest of which we can take cognizance, and we must refer to later periods those which appear to be independent of that phenomenon.

Foliation different from Stratification.

In the chapter already quoted, Mr. Darwin has combated the opinion prevailing among geologists "with respect to the origin of the folia of quartz, mica, felspar and other minerals composing the metamorphic schists, that the constituent parts of each were separately deposited as sediment, and then metamorphosed*." Nevertheless that opinion still appears to hold its ground, and has been lately re-asserted in a publication of high authority†.

The remarks already made on the analogy between foliation and cleavage confirm Mr. Darwin's view, that "foliation and cleavage are parts of the same process;" for on no other supposition can we explain the conformity between the two where seen in contact, and their being combined in the same arch of elevation. Now as cleavage is almost always transverse to the bedding and obviously a change produced in the beds after their deposition, it follows that foliation also is distinct from bedding or sedimentary stratification.

But besides the argument to be drawn from the analogy of cleavage and foliation, a direct comparison of these with the usual disposition of the beds in stratified rocks of sedimentary origin, will eqnally serve to distinguish true stratification from those phenomena.

We have seen that the arrangement of the foliation of gneiss and schist is in large flattish arches, in which the dip is slight near the central axis, and gradually in-

* Geological Observations on South America, p. 165.

† Sir C. Lyell's Manual of Elementary Geology, London, 1851, where the arguments for the sedimentary origin and subsequent metamorphism of gneiss, mica schist, &c. are produced at great length, and their structure is represented as *stratification* and nowise related to cleavage, pp. 467 to 481.

creases in inclination as we recede from the centre till it becomes vertical at the same distance on each side of the centre; this arrangement is precisely analogous to that of the cleavage of slates which I first described in 1846, and have found in all subsequent observations to be the usual disposition of the planes of cleavage. But this is very rarely the position of the beds of an elevated district; when stratified rocks have been raised into an arch or a dome, the steepest inclination of the beds is usually found near the axis or point of elevation, and the inclination diminishes as the distance from the disturbing force increases. The contrary will undoubtedly be sometimes found; yet it is very seldom that we see the distant beds dip at a higher angle than those nearer to the axis of disturbance. But this, which is a rare exception in planes of stratification, is almost the universal rule with cleavage and foliation.

The contrast just pointed out is so general, that when an observer finds it difficult to distinguish between planes of cleavage and stratification, as sometimes happens where either or both sets of planes are obscure, he will generally be right in referring to bedding those planes whose inclination diminishes in receding from the axis of elevation, and to cleavage those which become steeper at a distance from the axis. And a similar empirical rule will equally assist us to distinguish foliation from stratification.

The contortions of gneiss and mica slate are also far more complex than are ever found in the most disturbed strata, and are such as could only be produced in matter in a state of at least semi-fluidity; for they are not accompanied with any fractures across the layers of rock, such as are found in beds which have been bent after their deposition. The materials of the foliated rocks seem to have been in a state sufficiently fluid to allow the mineral ingredients to separate freely and arrange themselves according to their chemical or crystalline affinities, and while that process was going on to have been subjected to enormous pressure along certain axes of elevation, which has influenced the crystallizing action in so far as to have determined the direction of the parallel layers of different minerals, and has also raised up those layers into the great arches now seen and caused the contortions of certain portions of them.

Finally, not only are the arches of foliation quite different from those of elevated strata, but the general arrangement of those arches is such as has never yet been found in any beds of undoubted sedimentary origin; for it must be recollected that the arches represented in the section, Plate XXIII. fig. 1, run across Scotland side by side, exhibiting traces of a symmetry which was probably perfect before it was disturbed by the eruptions of granite and porphyry.

Gneiss and Mica Schist improperly termed Metamorphic.

The term *Metamorphic Rocks* has been applied to gneiss, mica schist, &c., under the supposition that they still maintained their original bedding, with a crystalline character superadded to it; when that theory is abandoned, there will be no longer

any propriety in retaining the name which implies it. The materials of gneiss and mica schist may have previously existed in some other form; but that alone is not enough to induce us to call them metamorphic; for the name might be applied for the same reason to granite and syenite, or even to lava, all of which have probably undergone various mutations. It would be better to restrict the term *metamorphic* to rocks of a sedimentary origin which have undergone changes affecting only their mineralogical character without losing the evidence of their bedding. It will also add to the clearness of our language if we confine the term *slate* to laminated rocks of stratified origin, and call the fissile foliated rocks *schists*; in which manner the two words have been used in this paper.

Appendix.—Among the attempts made to explain the cause of the lamination of slaty rocks, reference has often been made to the interesting experiments of Mr. R. W. Fox, since repeated by Mr. Hunt, upon the changes produced by long-continued voltaic electricity upon masses of clay, bricks, &c.*. The results produced have a sufficient analogy, both to lamination and foliation, to rouse our attention to the subject, without coming near enough to the great features described in this paper to allow us to regard them as a solution of the problem. In the mass of clay operated on by Mr. Hunt (fig. 12, p. 451), one side has assumed a laminated and the other a foliated structure, the latter being curved and more contorted than the former, as is usually the case in similar structures in the earth; but we do not see the regularity which is so striking in laminated rocks, and the two structures, instead of harmonizing as in nature, appear in a sort of opposition, being produced on opposite sides of the mass in an unconformable position.

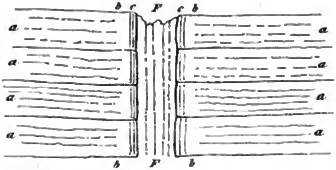
The changes produced in the brick (51) and plaster of Paris (49), p. 453, are more analogous to what may sometimes be seen in rocks traversed by fissures or mineral veins than to slaty cleavage. In these experiments the mass acted upon was found to be laminated on the side nearest the zinc plate of the battery, and considerably indurated on the copper side.

In examining the coast of Portugal near Cascaes, I met with several parallel perpendicular fissures between 10 and 30 feet wide, which run N.W. from the coast about a mile west of Cascaes towards the granite hills of Cintra, through beds of limestone and sandstone of the ages of our lower greensand. Where these fissures cut through the cliff they may be well examined, and the following notes were made respecting one of them. The cliff consists of thick beds of sandstone, in which all the minor traces of the bedding are obliterated for a breadth of 2 or 3 feet on each side of the fissure, only the great divisions remaining visible, and for about 6 inches on each side of the fissure the sandstone is laminated, splitting in perpendicular layers parallel to the walls of the fissure. The crack itself is filled by vertical layers of sandy

* Memoirs of the Geological Survey, vol. i. p. 451 and 453.

clay lining each side, and a mass of calcareous spar down the centre. The annexed diagram, fig. 6, will give an idea of the appearance on the cliff.

Fig. 6.



a a. Beds of sandstone, unaltered.

b b. Hardened portion of the sandstone, deprived of the minor traces of stratification.

c c. Sandstone close to the walls of the fissure vertically laminated.

F F. The fissure filled with calcareous spar and clay.

The changes which have taken place near the fissure are identical with those described by Mr. Hunt (49 and 51), the mass of the rock being laminated on the one part and rendered more solid on the other, which we may safely conclude to have been produced by galvanic currents passing between the mass filling up the fissure and the body of the rock on each side; but it would be too bold a speculation which should attempt to explain the regular cleavage of whole formations of slate by comparing it to the trifling change described above.

Explanation of the Map and Sections, Plates XXIII., XXIV.

As the object of the Map is to represent the direction of the foliation and cleavage, the geological features which have no reference to that subject have been kept as little prominent as possible, and only as many colours employed as were indispensable. The whole of the foliated rocks are painted of one yellow colour, which thus includes gneiss, mica schist, chlorite schist, hornblende schist, and the gneissose quartz rock. The stratified slates are coloured purple. The brown colour includes the Old Red Sandstone, the stratified quartz rock and the limestones associated with them. The more modern formations, which cover a very small space in the Highlands, are omitted. Granite, syenite and the older porphyries are coloured pink, and the more modern trap-rocks red. Thus two colours, the yellow and purple, represent the rocks affected by the foliation and cleavage; the pink includes all the plutonic rocks which have broken through and disturbed the foliation and cleavage; the brown and red represent the rocks which overlie and conceal the rocks affected by those phenomena.

The *strike* or direction of the foliation and cleavage across the surface is indicated by black lines; when these are double the foliation or cleavage is vertical; a single

thicker line represents the central axis of an arch of foliation. The dotted lines show the direction supposed to be followed, in unexamined districts, by the lines just explained. All the lines are laid down on the Map with more continuity and regularity than really exist; this error can only be fully corrected by a minute examination of the whole country.

As the object of the sections (figs. 1, 3, 4 and 5) is to show the direction of the dip of the foliation and cleavage, continuous lines have been used to represent them, while the dotted lines mark the stratification. The contortions of the foliation are indicated conventionally, as is explained at p. 447, since it would not be possible to represent their real complication on so small a scale. In fig. 2 the upper line is an ideal completion of the curves indicated by the dips of the foliation and cleavage seen in fig. 1, and the lower line is a rough sketch of the outline of the country; but it is not intended to be inferred that the rocks ever reached up to the upper line. In all the sections the scale of height considerably exceeds that of length, but the angles of dip are preserved as nearly as the observations taken on a large scale over broad tracts of country admit of.

Note.—The *strike* or direction of the foliation, cleavage and bedding on the plane of the earth's surface was taken with a pocket-compass; the corrections for the variation of the needle being derived from the directions on the Admiralty Charts; viz. for the

Forth	25·30
Aberdeen	26·30
Banff	27
Loch Ryan	27·30
Loch Eil	28·20
Isle of Lewis	30·94

The angles of dip were taken with a pocket-clinometer, which is sufficiently accurate for the object required, as the irregularity of the surfaces to be measured leaves all such observations liable to errors of 2° or 3°.

XXX. *On the Change of Refrangibility of Light. By* G. G. STOKES, *M.A., F.R.S., Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

Received May 11,—Read May 27, 1852.

1. THE following researches originated in a consideration of the very remarkable phenomenon discovered by SIR JOHN HERSCHEL in a solution of sulphate of quinine, and described by him in two papers printed in the Philosophical Transactions for 1845, entitled 'On a Case of Superficial Colour presented by a Homogeneous Liquid internally colourless,' and 'On the Epipolic Dispersion of Light.' The solution of quinine, though it appears to be perfectly transparent and colourless, like water, when viewed by transmitted light, exhibits nevertheless in certain aspects, and under certain incidences of the light, a beautiful celestial blue colour. It appears from the experiments of Sir JOHN HERSCHEL that the blue colour comes only from a stratum of fluid of small but finite thickness adjacent to the surface by which the light enters. After passing through this stratum, the incident light, though not sensibly enfeebled nor coloured, has lost the power of producing the same effect, and therefore may be considered as in some way or other qualitatively different from the original light. The dispersion which takes place near the surface of this liquid is called by Sir JOHN HERSCHEL *epipolic*, and he applies the term *epipolized* to a beam of light which, having been transmitted through a quiniferous solution, has been thereby rendered incapable of further undergoing epipolic dispersion. In one experiment, in which sun-light was used, a feeble blue gleam was observed to extend to nearly half an inch from the surface. As regards the dispersed light itself, when analysed by a prism it was found to consist of rays extending over a great range of refrangibility: the less refrangible extremity of the spectrum was however wanting. On being analysed by a tourmaline, it showed no signs of polarization. A special experiment showed that the dispersed light was perhaps incapable, at any rate not peculiarly susceptible, of being again dispersed.

2. In a paper 'On the Decomposition and Dispersion of Light within Solid and Fluid Bodies,' read before the Royal Society of Edinburgh in 1846, and printed in the 16th volume of their Transactions, as well as in the Philosophical Magazine for June 1848, Sir DAVID BREWSTER notices these results of Sir JOHN HERSCHEL's, and states the conclusions, in some respects different, at which he had arrived by operating in a different way. The phenomenon of internal dispersion had been discovered by him some years before, and is briefly noticed in a paper read before the Royal Society

of Edinburgh in 1833*. It is described at length, as exhibited in the particular case of fluor-spar, in a paper communicated to the British Association at Newcastle in 1838†. In Sir DAVID BREWSTER'S experiments the sun's light was condensed by a lens, and so admitted into the solid or fluid to be examined; which afforded peculiar facilities for the study of the phenomena. On examining in this way a solution of sulphate of quinine, it was found that light was dispersed, not merely close to the surface, but at a long distance within the fluid: and Sir DAVID BREWSTER was led to conclude that the dispersion produced by sulphate of quinine was only a particular case of the general phenomenon of internal dispersion. On analysing the blue beam by a rhomb of calcareous spar, it was found that a considerable portion of it, consisting chiefly of the less refrangible rays, was polarized in the plane of reflexion, while the more refrangible of its rays, constituting an intensely blue beam, had a different polarization.

3. On repeating some of Sir JOHN HERSCHEL'S experiments, I was immediately satisfied of the reality of the phenomenon, notwithstanding its mysterious nature, that is to say, that an epipolized beam of light is in some way or other qualitatively different from the light originally incident on the fluid. On making the observation in the manner of Sir DAVID BREWSTER, it seemed no less evident that the phenomenon belonged to the class of internal dispersion‡. Nevertheless, the singular phenomenon discovered by Sir JOHN HERSCHEL manifested itself even in this mode of observation. If indeed the vessel containing the solution were so placed that the image of the sun in the focus of the lens lay a little way inside the fluid, the phenomenon was masked, because the increase of intensity due to an increase of concentration in approaching the focus made up for the decrease of intensity due to passing out of the blue band. But when the vessel was moved so that the focus of the lens fell either further inside the fluid or else outside the vessel, the narrow blue band adjacent to the surface was seen *as well as* the blue beam which shot far into the fluid. Light which has been "epipolized" by transmission through a moderate thickness of the solution is indeed capable of undergoing further dispersion, but not *epipolic* dispersion, *if that term be restricted* to the dispersion by which the narrow blue band is produced. It was no doubt of great importance to assign to the phenomenon its true place as a member of the class of phenomena of internal dispersion. Nevertheless the mystery was by no means cleared up; rather, we were prepared to expect something of the same sort

* Edinburgh Transactions, vol. xii. p. 542.

† Eighth Report.—Transactions of the Sections, p. 10.

‡ By this, I merely mean that, to take a particular example, the exhibition of a blue light by a solution of sulphate of quinine appeared to be a phenomenon of the same nature as the exhibition of a red light by a solution of the green colouring matter of leaves, although the latter does not manifest the same singular concentration as the former in the neighbourhood of the surface by which the light enters; and the latter had already been observed by Sir DAVID BREWSTER, and the phenomenon designated as *internal dispersion*. I make this remark because Sir DAVID BREWSTER has applied this same term to another class of phenomena which are totally different.

in other instances of internal dispersion. In fact, the mystery consisted, not in the narrowness of the stratum from which most of the blue light came, but in the circumstance that it was possible for light, by passing across such a stratum, to be deprived of the power of producing the same effect again, without, apparently, being altered in any other respect.

4. To one who regards light as a subtle and mysterious agent, of which the laws indeed are in a good measure known to us, but respecting the nature of which we are utterly ignorant, the phenomenon might seem merely to make another striking addition to the modes of decomposition with which we were already acquainted. But in the mind of one who regards the theory of undulations as being for light what the theory of universal gravitation is for the motions of the heavenly bodies, it was calculated to excite a much more lively interest. Whatever difficulty there might be in explaining how the effect was produced, we ought at least to be able to say what the effect was that had been produced; wherein, for example, epipolized light differed from light which had not undergone that modification.

In speculating on the nature of the phenomenon, there is one point which deserves especial attention. Although the passage through a thickness of fluid amounting to a small fraction of an inch is sufficient to purge the incident light from those rays which are capable of producing epipolic dispersion, the dispersed rays themselves traverse many inches of the fluid with perfect freedom. It appears therefore that the rays producing dispersion are in some way or other of a different nature from the dispersed rays produced. Now, according to the undulatory theory, the nature of light is defined by two things, its period of vibration, and its state of polarization. To the former corresponds its refrangibility, and, so far as the eye is a judge of colour, its colour *. To a change, then, either in the refrangibility or in the state of polarization we are to look for an explanation of the phenomenon.

5. Regarding it at first as an axiom that the dispersed light of any given refrangibility could only have arisen from light of the same refrangibility contained in the incident beam, I was led to look in the direction of polarization for the required change in the nature of the light. Since a fluid has no axes, circular polarization is

* It has been maintained by some philosophers of the first eminence that light of definite refrangibility may still be compound, and though no longer decomposable by prismatic refraction might still be so by other means. I am not now speaking of compositions and resolutions depending upon polarization. It has even been suggested by the advocates of the undulatory theory, that possibly a difference of properties in lights of the same refrangibility might correspond to a difference in the law of vibration, and that lights of given refrangibility may differ in tint, just as musical notes of given pitch differ in quality. Were it not for the strong conviction I felt that light of definite refrangibility is in the strict sense of the word homogeneous, I should probably have been led to look in this direction for an explanation of the remarkable phenomena presented by a solution of sulphate of quinine. It would lead me too far from the subject of the present paper to explain the grounds of this conviction. I will only observe that I have not overlooked the remarkable effect of absorbing media in causing apparent changes of colour in a pure spectrum; but this I believe to be a subjective phenomenon, depending upon contrast.

the only kind which can here come into play. As some fluids are doubly refracting, transmitting right-handed and left-handed circularly polarized light with different velocities, so, it might be, this fluid was doubly absorbing, absorbing say right-handed circularly polarized light of certain refrangibilities with great energy, and freely transmitting left-handed. The right-handed light, absorbed, in the sense of withdrawn from the incident beam, might have been more strictly speaking scattered, and thereby depolarized. The common light so produced would be equivalent to two streams, of equal intensity, one of right-handed, and the other of left-handed circularly polarized light. Of these the latter would be freely transmitted, while the former would be scattered anew, and so on. Yet this hypothesis, sufficiently improbable already, was not enough. New suppositions were still required, to account for the circumstance that an epipolized beam, when subjected to prismatic analysis with a low magnifying power, exhibited no bands of absorption in the region to which, as regards their refrangibility, the dispersed rays principally belong; so that altogether this theory bore not the slightest semblance of truth.

6. I found myself thus fairly driven to suppose that the change of nature consisted in a change of refrangibility. From the time of NEWTON it had been believed that light retains its refrangibility through all the modifications which it may undergo. Nevertheless it seemed to me less improbable that the refrangibility should have changed, than that the undulatory theory should have been found at fault. And when I reflected onthe extreme simplicity of the whole explanation if only this one supposition be admitted, I could not help feeling a strong expectation that it would turn out to be true. In fact, we have only to suppose that the invisible rays beyond the extreme violet give rise by internal dispersion to others which fall within the limits of refrangibility between which the retina of the human eye is affected, and the explanation is obvious. The narrowness of the blue band observed by Sir JOHN HERSCHEL would merely indicate that the fluid, though highly transparent with regard to the visible rays, was nearly opake with regard to the invisible. According to the law of continuity, the passage from almost perfect transparency to a high degree of opacity would not take place abruptly; and thus rays of intermediate refrangibilities might produce the blue gleam noticed by Sir JOHN HERSCHEL, or the blue cylinder, or rather cone, observed by Sir DAVID BREWSTER. We should thus, too, have an immediate explanation of a remarkable circumstance connected with the blue band, namely that it can hardly be seen by strong candle-light, though readily seen by even weak daylight. For candle-light, as is well known, is deficient in the chemical rays situated beyond the extreme violet.

7. My first experiments were made with coloured glasses. A test tube was about half filled with a solution consisting of disulphate of quinine dissolved in 200 times its weight of water acidulated with sulphuric acid. The tube, having been first covered with black paper, with the exception of a hole by which the light might enter, was placed in a vertical position in front of a window, the hole being turned

towards the light. On looking down from above, in a direction nearly parallel to the surface of the glass, a blue arc was well seen, extending only a very short distance into the fluid, and situated immediately behind the hole. As this arc, though extremely distinct, was not of course what could be called brilliant, I did not at first venture, for the experiment I had in view, to use any but pale glasses. Having no direct means of determining which were opake with regard to the invisible rays situated beyond the extreme violet, I sought among a collection of orange, yellow, and brown glasses, which, from transmitting mainly the less refrangible rays, seemed the most likely to absorb the chemical rays. I presently found a pale smoke-coloured glass, which, when placed immediately in front of the hole, prevented the formation of the blue arc, although when placed immediately in front of the eye it transmitted a large proportion of the light of which the arc consisted. The colour of the arc was of course modified, and rendered more nearly white.

On trying other pale glasses, I found one of a puce colour, which, when placed in front of the hole, allowed the arc to be formed, though it absorbed it when placed in front of the eye. A yellow, and likewise a yellowish green glass allowed the arc to be seen in both positions; but its colour was decidedly different according as the glass was placed in front of the hole or in front of the eye. The breadth, too, of the arc was differently affected by different coloured glasses placed in front of the hole, some causing the light to be more, and others less concentrated towards the surface of the test tube than when the incident light was unimpeded.

8. The sun's light was next reflected horizontally into a darkened room, and allowed to pass through a hole in a vertical board which was placed in the window. The hole contained a lens of rather short focus. On placing a test tube containing the solution, in a vertical position, in front of the lens, at such a distance that the focus lay some way inside the fluid, the narrow blue band described by Sir John Herschel and the blue beam mentioned by Sir David Brewster were seen independently of each other. On trying different coloured glasses, which were placed, first in front of the fluid, and then in front of the eye, it was found that the blue beam, as had previously proved to be the case with the narrow band, was for the most part differently affected according as the glass was placed so as to intercept the incident or the dispersed light. Moreover, the long blue beam and the narrow band did not behave in the same manner under the action of the same coloured glass.

9. To my own mind these experiments were conclusive as to the fact of a change of refrangibility. Admitting that the effect of a coloured glass is simply to stop a certain fraction of the incident light, that fraction being a function of the refrangibility, it is plain that the results can be explained in no other way. It must be confessed however that these results are merely an extension of that which precisely constitutes the peculiarity of the phenomenon. For, take the case of the narrow blue band formed by ordinary daylight. Imagine a glass vessel with parallel sides to be filled with a portion of the solution, and placed so as to intercept, first the incident,

and then the dispersed light. In the first position the light incident on the fluid under examination would be "epipolized" by transmission through the fluid contained in the vessel, and therefore the blue band would be cut off, whereas when the vessel was held in front of the eye the blue band would be freely transmitted. Hence the effects of the coloured glasses are analogous to, but less striking than, the effect of a stratum of the solution of sulphate of quinine in the imaginary experiment above described. There is to be sure one important difference in the two cases, namely, that in the case of the stratum of fluid the epipolic dispersion which is prevented in the fluid under examination is produced near the first surface of the stratum, whereas no such dispersion is produced, or at any rate necessarily produced, in the coloured glasses. Whatever the reader may think of the results obtained with coloured glasses, the next experiment it is presumed will be deemed conclusive.

10. The board in the window containing the lens having been replaced by a pair of boards adapted to form a vertical slit, the sun's light was reflected horizontally through the slit, and transmitted through three Munich prisms placed one after the other close to it. A tolerably pure spectrum was thus formed at the distance of some feet from the slit. A test tube containing the solution was then placed vertically a little beyond the extreme red of the spectrum, and afterwards gradually moved horizontally through the colours. Throughout nearly the whole of the visible spectrum the light passed through the fluid as it would have done through so much water; but on arriving nearly at the violet extremity a ghost-like gleam of pale blue light shot right across the tube. On continuing to move the tube, the blue light at first increased in intensity and afterwards gradually died away. It did not however cease to appear until the tube had been moved far beyond the violet extremity of the spectrum visible on a screen. Before disappearing, the blue light was observed to be confined to an excessively thin stratum of fluid adjacent to the surface by which the light entered, whereas when it first appeared, namely when the tube was placed a little short of the extreme violet, the blue light had extended completely across it. It was certainly a curious sight to see the tube instantaneously lighted up when plunged into the invisible rays: it was literally *darkness visible*. Altogether the phenomenon had something of an unearthly appearance.

11. Since the fluid is so intensely opake with regard to rays of extreme refrangibility, it might be expected, that, though it appears transparent and colourless when examined merely by viewing a white object through it, it would yet exhibit a very sensible absorbing action with regard to the extreme violet rays when subjected to prismatic analysis. To try whether such were really the case, I reflected the sun's light horizontally through a slit, at which was placed a test tube filled with the liquid, and analysed the line of light by a prism, the eye being defended by a deep blue glass. I was barely able to make out the fixed line H in Plate XXV., that is, the less refrangible band of the pair, although in similar circumstances I can generally see about as far beyond the more refrangible band as it is beyond H. However, to make the result

more decisive by using a greater thickness, as well as to render the observation strictly differential, I placed a tumbler filled with water behind the slit, the blue glass before it, and then viewed the slit through the prism. I saw as far as usual into the violet. The water was then poured out and replaced by the solution of sulphate of quinine, which, when viewed by transmitted light, appeared as transparent as the water which it had replaced. When the tumbler was now placed behind the slit, the blue beam of dispersed light was observed to extend quite across it, a distance of about three inches, and would evidently have gone much further. On viewing the slit through the prism, the spectrum was found to be cut off about half-way between the fixed lines G and H. The termination was pretty definite, which indicates that, at least for that part of the spectrum, the absorbing energy of the fluid rapidly increased with the refrangibility of the light; there was, however, an evident diminution of intensity produced by the fluid, extending from the termination of the spectrum to near G.

12. There could no longer be any doubt, either as to the fact of a change of refrangibility, or as to the explanation thereby of the remarkable phenomenon exhibited by sulphate of quinine. Epipolized light is merely light which has been purged of the invisible, or at most feebly illuminating rays more refrangible than the violet; and the term itself, which in fact was only adopted provisionally by Sir John Herschel, and which has now served its purpose, may henceforth be discarded, especially as it is calculated to convey a false impression respecting the cause of the phenomenon. It remained to examine other instances of internal dispersion, of which, according to Sir David Brewster's observations, the dispersion produced by sulphate of quinine is only a particular case; to endeavour to make out the laws according to which a change of refrangibility takes place; and, if possible, to account for these laws on mechanical principles.

13. In giving an account of my further experiments, I think it best to describe in detail the phenomena observed in some of the more remarkable instances of internal dispersion before attempting to draw any general conclusions. It will save repetition to explain in the first instance the methods of observation employed, which on the whole may very fairly be divided into four, though occasionally it was convenient to employ intermediate methods, or a combination of two of them. Of course I frequently availed myself of Sir David Brewster's method of observation, in which the effect of the incident light is studied as a whole; but the methods here referred to relate to an investigation of the separate offices of the portions of light of different degrees of refrangibility which are found in the incident beam. As my researches proceeded, new methods of observation suggested themselves, but these will be described in their place.

Methods of Observation employed.

First Method.—The sun's light was reflected horizontally through a small lens, which was fixed in a hole in a vertical board. The cone of emergent rays was

allowed to enter the solid or fluid examined. A coloured glass or other absorbing medium was then placed, first so as to intercept the incident rays, and then between the substance examined and the eye. For shortness' sake these positions will be designated as *the first* and *the second*. Sometimes a coloured glass was allowed to remain in front of the hole, and a second glass was added, first in front of the hole and then in front of the eye.

Second Method.—The sun's light, reflected as before, was transmitted through a series of three or four Munich prisms placed one immediately after the other, and each nearly in the position of minimum deviation. It was then transmitted through a small lens in a board close to the last prism, and so allowed to enter the body to be examined, which was generally placed so that the first surface coincided, or nearly so, with the focus of the lens. The diameter of the lens was much smaller than the breadth or height of the prisms, so that the lens was completely filled with white light, the component parts of which however entered in different directions. Regarding the image of the sun in the focus of the small lens as a point, we may conceive the light incident on the body under examination as consisting of a series of cones, corresponding to different refrangibilities, the axes of which lay in a horizontal plane and intersected in the centre of the lens, the vertices being arranged in a horizontal line near the surface of the body examined.

Third Method.—The sun's light was reflected horizontally through a vertical slit, and received on the prisms, which were arranged as before, but placed at the distance of several feet from the slit. A large lens of rather long focus was placed immediately after the last prism, with its plane perpendicular, or nearly so, to the beam of light which had passed through the prisms, and with its centre about the middle of this beam. The body examined was placed at the distance of the image of the slit, or nearly so.

Fourth Method.—Everything being arranged as in the third method, a board with a small lens of short focus was placed at the distance of the image of the slit, or between that and the image of the sun, which was a little nearer to the prisms, inasmuch as the focal length of the large lens commonly employed, though much smaller, was not incomparably smaller than the distance of the lens from the slit. A second slit was generally added immediately in front of the small lens. The body examined was placed at the focus of the small lens. The dispersed light was viewed from above, and analysed by a prism, being refracted sideways.

The object of these several arrangements will appear in the course of the paper. The prisms employed consisted, three of them of flint glass and one of crown. The refracting angles of the former were about 43°, 33°, and 24°, and that of the latter about 45°. The refracting faces of the smallest of the prisms (the flint of 43°) were 1·35 inch high and 1·60 long. The small lens used was one or other of a pair of which the apertures were 0·34 inch and 0·22 inch, and the focal lengths 0·75 inch and 0·50 inch. The focal length of the large lens generally used was about twelve inches.

Once or twice a lens was tried which had a focal length about three times as great, but the light proved too faint for most purposes. In the third method it was sometimes convenient to employ a lens of only $6\frac{1}{2}$ inches focal length, but the 12-inch lens was employed in the fourth method, except on a few occasions, when the lens of 36 inches focal length was used. With the 12-inch lens the length of the spectrum from the fixed line B to H was usually about an inch and a quarter.

It will be convenient for the purposes of this paper to employ certain terms in a particular sense, but as some of these terms relate to phenomena which have not yet been described, it will be well previously to relate in detail what was observed in one remarkable instance of internal dispersion.

Solution of Sulphate of Quinine.

14. The effects of some pale coloured glasses in the case of this fluid have already been mentioned. But there is one glass of which the effect is still more striking. It is well known that a deep cobalt blue glass is highly transparent with regard to the chemical rays. Accordingly I found that a blue glass, so deep that only the brighter objects in a room could be seen through it, produced but very little effect when placed so as to intercept the light incident on the fluid. When placed immediately in front of the eye, at first everything disappeared except the light reflected from the convexities of the glass tube; but when the eye became a little accustomed to the darkness it was possible to make out the existence of the band. The contrast between the effects of this glass and of the pale brown glass already mentioned was most striking.

15. When the fluid was examined by the second method, the dispersed light was found to consist of two beams, separated from each other at their entrance into the fluid, that is, at the vertical surface of separation of the fluid and the containing vessel, and afterwards still further separated by divergence. Of course each beam must have been made up of a series of cones having their axes diverging from the centre of the lens, and their vertices situated at its focus. The first beam, or that which was produced by light of less refrangibility, consisted of the brighter colours of the spectrum in their natural order. It had a discontinuous, sparkling appearance, and was plainly due merely to motes which were suspended in the fluid. On being viewed from above through a NICOL's prism, it was found to consist chiefly of light polarized in the plane of reflexion. Taken as a whole, it served as a fiducial line to which to refer the position of the second beam, and thereby judge of the refrangibility of the rays by which it was produced.

This second beam was a good deal the brighter of the two. Its colour was a beautiful sky-blue, which was nearly the same throughout, but just about its first border, that is, where it arose from the least refrangible of those rays which were capable of producing it, the colour was less pure. It had a perfectly continuous appearance. When viewed from above through a doubly refracting achromatic prism of quartz,

which allowed a direct comparison of the two images, it offered no traces of polarization. It was produced by light polarized in a vertical or horizontal plane as well as by common light, and in that case, as well as in the former, manifested no traces of polarization *.

The short distance that the more refrangible rays were able to penetrate into the fluid might readily be perceived in this experiment, but the second method of observation was not adapted to bring out this part of the phenomenon.

16. On examining the fluid by the third method, the result was very striking, although of course only what might have been anticipated. The principal fixed lines of the violet, and of the chemical parts of the spectrum beyond, were seen with beautiful distinctness as dark planes interrupting an otherwise perfectly continuous mass of blue light. To see any particular fixed line with most distinctness, it was of course necessary to hold the eye in the corresponding plane, when the dark plane was foreshortened into a dark line. From the red end of the spectrum, as far as the line G, or thereabouts, the light passed freely through the fluid, or at least was only reflected here and there from motes held in mechanical suspension. About G the dispersion just commenced to be sensible, and there were traces of that line seen as a dark plane interrupting a mass of continuous but excessively faint light. For some distance further on the dispersed light remained so faint that it might have been passed over if not specially looked for. It was about half-way between G and H, or a little before, that it first became so strong as to arrest attention, and a little further on it became very conspicuous, the tint meanwhile changing to a pale sky-blue. The light was very copious about the two broad bands of the group H, and for some distance from H towards G. Some of the fixed lines less refrangible than H were very plain, and beyond H a good number were visible, which will presently be further described. The whole system of fixed lines thus visible as interruptions in the dispersed light had a resolvable appearance; but with a very narrow slit and a lens of long focus at the prisms the light would have been too faint for convenient observation.

The dispersed light about G, and for some distance further on, was so very faint that I might have overlooked it had it not arrested my attention when observing by the fourth method; indeed, I have sometimes specially looked for it in the third arrangement without having been able to see it. Practically speaking, the dispersion might be said to commence about half-way between G and H.

* These two results, namely, that the blue beam which constitutes the greater part of the light dispersed by a solution of sulphate of quinine is unpolarized, or according to his expression possesses a *quaquaversus* polarization, and that that still remains the case when the incident light is polarized, have been already announced by Sir DAVID BREWSTER, who appears to have been led to attend to the polarization of the light from Sir JOHN HERSCHEL's observation, that the blue light arising from epipolic dispersion in a solution of sulphate of quinine was unpolarized. It seemed important however to repeat the observation on the blue beam obtained in a state of isolation.

17. On refracting the whole system sideways through a prism of moderate angle held in front of the eye, the fixed lines became confused, and the finer ones disappeared. The edges of the broad bands H were tinged with prismatic colours, like the edges of two slips of black velvet placed on a sheet of pale blue paper, and viewed through a prism. This experiment exhibits the compound character of the dispersed light, notwithstanding the perfect homogeneity of the incident light.

18. The third method of observation is well adapted to bring into view the variation in the absorbing energy of the medium corresponding to a variation in the refrangibility of the incident rays. When the eye is placed vertically over the vessel containing the solution, so that the dark planes corresponding to the fixed lines of the spectrum are projected into dark lines, of which the length is not exaggerated by obliquity, the boundary of the dispersed light is projected into a curve, which serves to represent to the eye the relation between the absorbing power of the medium and the refrangibility of the incident light. This curve is not exactly that which Sir John Herschel has treated of in the theory of absorption, and considered as the type of the absorbing medium to which it is applied, but nevertheless it serves much the same purpose. It is true, that, independently of any change in the absorbing energy of the medium, an increasing faintness in the dispersed light would produce to a certain extent an approximation of the curve to its axis; but practically, in the case of sulphate of quinine, as well as in a great many others, the appearance is such as to leave no doubt as to the existence of a most intense absorbing energy on the part of the medium with respect to rays of very high refrangibilities*.

In the case of a solution of sulphate of quinine of the strength of one part of the disulphate to 200 parts of acidulated water, it has been already stated that a portion of the rays which are capable of producing dispersed light passed across a thickness of 3 inches. On forming a pure spectrum, the fixed line H was traced about an inch into the fluid. On passing from H towards G, the distance that the incident rays penetrated into the fluid increased with great rapidity, while on passing in the contrary direction it diminished no less rapidly, so that from a point situated at no great distance beyond H to where the light ceased, the dispersion was confined to the immediate neighbourhood of the surface. When the solution was diluted so as to be only one-tenth of the former strength, a conspicuous fixed line, or rather band of sensible breadth, situated in the first group of fixed lines beyond H, was observed to penetrate about an inch into the fluid. On passing onwards from the band abovementioned in the direction of the more refrangible rays, the distance that the incident rays penetrated into the fluid rapidly decreased, and thus the rapid increase in the absorbing energy of the fluid was brought into view in a part of the spectrum in

* I should here remark, that, after the researches described in this paper had far advanced, I met accidentally with a passage in the Comptes Rendus, tom. xvii. p. 883, in which M. Ed. Becquerel mentions a solution of acid sulphate of quinine as a medium eminently remarkable for its absorbing power with respect to the rays more refrangible than H.

which, with the stronger solution, it could not be so conveniently made out, inasmuch as the posterior surface of the space from which the dispersed light came almost confounded itself with the anterior surface of the fluid.

The high degree of opacity with regard to rays of great refrangibility which the addition of so small a proportion of sulphate of quinine is sufficient to produce in water is certainly very remarkable; nevertheless it is only what I have constantly observed while following out these researches.

19. In observing by the fourth method, the part of the spectrum to which the incident light belonged was determined sometimes by the colour, sometimes by means of the fixed lines of the spectrum. It almost always happened that there were motes enough suspended in the fluid to cause a portion of the dispersed beam to consist merely of light which had undergone ordinary reflexion. When the whole dispersed beam was analysed by a prism, the beam which consisted of light reflected from motes was separated from the rest; it was in general easily recognised by its sparkling appearance, but at any rate was known by its consisting almost wholly of light polarized in the plane of incidence, whereas the truly dispersed light was unpolarized. It consisted of course of light of definite refrangibility, the same as that of the incident light, and thus served as a fiducial line to which to refer by estimation the refrangibilities of the component parts of the dispersed light. Of course this part of the observation was possible only when the incident rays belonged to the visible part of the spectrum.

On moving the lens horizontally through the colours of the spectrum, in a direction from the red to the violet, it was found that the dispersion was first perceptible in the blue. When the dispersed light was separated by a prism from the light reflected from motes, it was found to consist of an exceedingly small quantity of red; further on some yellow began to enter into its composition; further still, perhaps about the junction of the blue and indigo, the dispersed beam began to grow brighter, and was found on analysis to contain some green in addition to the former colours. In the indigo it got still brighter, and when viewed as a whole was somewhat greenish. Further still it became something of a pale slaty blue, and was found on analysis to contain some indigo, or at least highly refrangible blue. On proceeding further the dispersed light became first of a deeper blue and then, a little short of the fixed line H, whiter. At a considerable distance beyond H the dispersed light was if anything a shade more nearly white.

By this method of observation the dispersion can be detected earlier in the spectrum than by the third method, and moreover the change in the colour of the dispersed light is much more easily perceived; indeed the most striking part of this change takes place while the dispersed light is so very faint that it can hardly be seen in observing by the third method; moreover, even in the bright part of the dispersed beam, it is not at all easy by the latter method to make out the change of tint corresponding to a change in the refrangibility of the incident rays, because the tint

changes so gradually and so slightly that the eye glides from one part of the dispersed beam to another without noticing any change.

20. It has been already mentioned that the blue beam of dispersed light seen in a solution of sulphate of quinine was produced whether the incident light was polarized in or perpendicularly to the plane of reflexion, or more properly plane of dispersion, that is, the plane containing the incident ray and that dispersed ray which enters the eye. A question naturally presents itself, whether the intensity of the dispersed light is strictly the same in the two cases. By combining a lens of rather short focus and a doubly refracting prism with the four prisms, I satisfied myself that the difference of intensity, if there were any, was not great, but the experiment presented some practical difficulties. However, the result of the following experiment appeared to be as decisive as a negative result could well be.

The arrangement being the same as in the third method, but the lens in front of the prisms having a focal length of only 6·5 inches, the incident light was polarized in a vertical plane previously to passing through the slit, by transmission through a pile of plates. The two beams of light were seen as usual in the fluid, namely, the blue beam due to internal dispersion, and the fainter coloured beam due to motes. The former of these, which was quite separate from the latter, exhibited the principal fixed lines belonging to the highly refrangible part of the spectrum. A plate of selenite was then interposed immediately in front of the vessel, so as to modify the polarization of the light entering the fluid. This plate was obtained by an irregular natural cleavage, and was cemented with Canada balsam between two discs of glass. When examined by polarized light it exhibited a succession of beautiful and varied tints, according to the various thicknesses of the different parts. Now when the plate was moved about in front of the vessel, without altering its perpendicularity to the incident light, different portions of the beam due to motes were observed to disappear and reappear, or at least to become faint and then bright again, so that a person ignorant of the cause, and not looking at the disc, might have supposed that the observer had been holding in front of the vessel a piece of dirty glass, having the dirt laid on in patches; but in whatever manner the disc was moved in its own plane without rotation, or turned round an axis perpendicular to its plane, not the slightest perceptible change was produced in any part of the blue beam.

Explanation of Terms.

21. In all the experiments described in this paper in which a spectrum was formed for the sake of examining the separate action of portions of light of different refrangibilities, the length of the spectrum was horizontal, so that the fixed lines were vertical. Nevertheless it will be convenient, for the sake of shortness, to use the prepositions *above* and *below* to signify respectively *on the more refrangible side of* and *on the less refrangible side of.*

The principal fixed lines of the visible spectrum will be denoted by letters in ac-

cordance with Fraunhofer's admirable map. These lines are now too well known to need description.

The only map of the fixed lines of the chemical spectrum which I had for a good while after these researches were commenced is Professor Draper's, which will be found in the twenty-second volume of the Philosophical Magazine (1843). Of course this map cannot be compared for accuracy of detail with Fraunhofer's map of the visible spectrum, nor does it profess to give more than some of the most conspicuous lines selected from among a great multitude. The suppression of so many lines, without any representation by shading of their general effect, renders it difficult to identify those which are laid down, at least if I may judge from my own observations; besides, Professor Draper's spectrum was so much purer than the one with which I found it most convenient to work, that the two are not comparable with each other.

22. I have made a sketch of the fixed lines from H to the end, which accompanies this paper. The fixed lines of the visible spectrum are so well known that I thought it unnecessary to begin before H. A solution of sulphate of quinine is a very good medium for showing the lines, but a yellow glass, which will be mentioned presently, is quite as good, or rather better. The map represents the spectrum as seen with the lens of 12 inches focal length in front of the prisms. The breadth of the slit was not always quite the same: it may be estimated at about the $\frac{1}{20}$th of an inch. The map contains 32 fixed lines or bands more refrangible than H, which is the utmost that I have been able on different occasions to see with this lens, though with a lens of longer focus and a narrower slit the number of fixed lines which might be counted was, as might be expected, a good deal larger. As I have not yet identified these lines, except in certain cases, with those which had previously been represented by means of photographic impressions, I have thought it advisable not to attempt an identification, but to attach letters to the more conspicuous lines in my map without reference to former maps. As the capitals L, M, N, O, P have already been appropriated to designate certain fixed lines, I have made use of the small letters *l*, *m*, *n*, *o*, *p*, to prevent confusion.

In drawing the map, I have endeavoured to preserve the character of the lines with respect to blackness or faintness, sharpness or diffuseness. The distances were not laid down by measurement, except here and there, and they are not, I fear, quite so accurate as might be desired; still, I feel assured that no one viewing the actual object would feel any difficulty in identifying the lines with those in my map, provided the circumstances under which his spectrum was formed at all approached to those under which mine was seen when the arrangement as to focal length of the lens, &c. was that most convenient for general purposes.

The more conspicuous lines in the part of the spectrum represented in the map may conveniently be arranged in five groups, which I will call the groups H, *l*, *m*, *n*, *p*. The group H consists chiefly of the well known pair of bands of which the first contains Fraunhofer's line H; the second band I have marked *k*, in accordance with

Professor DRAPER's map. The most conspicuous object in the next group consists of a broad dark band, *l*. This band is between once and twice as broad as H, and is darker in the less refrangible half than in the other. With a lens of 3 feet focal length and a narrow slit it was resolved into lines, which is probably the reason why it is altogether omitted in Professor DRAPER's map, while the first three lines of the group (if I do not mistake as to the identification) are represented, forming his group L. Under the circumstances to which the accompanying map corresponds, the band *l* appears as a very striking object, perhaps, with the exception of the bands H, *k*, the most conspicuous in the whole spectrum. With a still lower power it appears as a very black and conspicuous line. A double line beyond *l* completes the group *l*, after which comes another remarkable group *m*, consisting of five lines or bands. Of these the first is rather shady, though sharply cut off on its more refrangible side, but the others, and especially I think the second and third, are particularly dark and well-defined. I have marked the middle line *m*, not because it is more conspicuous than its neighbours, but on account of its central situation. After a very faint group, consisting apparently of four lines, comes another very conspicuous group *n*, consisting of two pairs of dark bands followed by another pair of bands which are broad and very dark. The first of these is a good deal broader than the second, but is not so broad as the band H; the second is followed by a fine line. This is as far as it is easy to see; but when the sunshine is clear, and the arrangements are made with a little care, a group of six lines is seen much further on. Of these, the first two are only moderately dark, and the first is rather diffuse; they stand off a little from the others, and are a little closer together than the other four. Of the latter, the first, marked *o*, is very strong, considering the faintness of the light which it interrupts; the second and third are faint, and difficult to see; the fourth, marked *p*, is black like the first, and a good deal broader. The line *p* was situated, by measurement, as far beyond H as H beyond *b*. Once or twice in the height of summer, and under the most favourable circumstances, I have observed two broad dusky bands still further on. The first of these had the appearance of being resolvable into two. The excessively faint light seen beyond the second seemed to end rather abruptly at the distance represented by the border of the accompanying plate, as if there were there the edge of another dark band beyond which nothing could be seen. In order to see the dusky bands last mentioned, and even to see the group *p* to most advantage, it was necessary to allow the central part of the beam incident on the prisms to pass through them close to their edges, so that evidently a great deal of light was lost by passing by the prisms altogether. This circumstance, combined with others which I have observed, convinces me that the great obstacle to seeing the fixed lines in this part of the spectrum consists in the opacity of glass. Were glass as transparent with respect to the invisible rays of very high refrangibility as it is with respect to the rays belonging to the visible spectrum, I know not how much further I might have been able to see.

I have endeavoured to identify the fixed lines in my map with the fixed lines represented in M. SILBERMANN's map of the chemical spectrum, with a copy of which my friend Professor THOMSON has kindly furnished me. I am still uncertain respecting the identification. M. SILBERMANN's map is so very much more detailed than my own, and must have been made with so much purer a spectrum, that the two systems of lines are not directly comparable.

23. From the difficulty of identification some persons might be disposed to imagine that the chemical rays, and those which produced the blue light in a solution of quinine, were of a different nature, and had each a system of fixed lines of its own. For my own part, I was too well acquainted with the Protean character of fixed lines to regard the difficulty of identification as any valid argument in support of such a view. And that this difficulty arose from nothing more than the different degrees of purity of the spectra is now put past dispute, for my friend Mr. KINGSLEY of Sidney Sussex College, to whom I recently showed some of the experiments mentioned in this paper, has kindly taken for me some photographs of spectra having nearly the same degree of extent and purity as those with which I worked, and these show the fixed lines just as they appeared in a solution of sulphate of quinine and in other media*.

24. The position of a point in the spectrum which does not coincide with one of the principal fixed lines, will be denoted by referring it to two of those lines, in a manner which will be most easily explained by an example. Thus $\frac{1}{2}$GH, G$\frac{1}{2}$H, GH$\frac{1}{2}$ will be used to denote respectively a point situated at a distance below G equal to half the interval from G to H, a point midway between G and H, and a point situated at the same distance above H. In using this notation, the letters denoting fixed lines will be written in the order of refrangibility, and the fraction expressing the part of the interval between these lines, which must be conceived to be measured off in order to reach the point whose position it is required to express, will be written before, between, or after the letters, according as the measurement is to be taken from the first line in the negative direction, from the first line in the positive direction, or from the second line in the positive direction, the positive direction being that of increasing refrangibility.

25. From the experiments already described, it appears that the beam of dispersed light which was observed in the experiments of Sir DAVID BREWSTER consisted of two very distinct portions, one arising merely from light reflected from motes, and the other having a far more remarkable origin. It will be convenient to have names for these two kinds of dispersion, and I shall accordingly call them respectively *false internal dispersion* and *true internal dispersion*, or simply *false dispersion* and *true dispersion* when the context sufficiently shows that internal dispersion is spoken of. When dispersion is mentioned without qualification, it is to be understood of true dispersion. Now that it appears that the mere reflexion of light from solid particles held in mechanical suspension has nothing to do with that remarkable kind of internal

* See note A at the end.

dispersion which is characterized by the "*quaquaversus* polarization," the phenomenon of false dispersion ceases to be of much interest in an optical point of view; while on the other hand the phenomenon of true dispersion, which had always been very remarkable, is now calculated to excite a great additional interest. It will be convenient to mention here the principal characters by which true and false dispersion may be distinguished, although it will be anticipating in some measure the results of observations yet to be described.

26. In true dispersion the dispersed light has a perfectly continuous appearance. In false dispersion, on the other hand, it has generally more or less of a sparkling appearance, and on close inspection is either wholly resolved into bright specks, or so far resolved as to leave on the mind the impression that if the resolution be not complete it is only for want of a sufficient magnifying power.

In true dispersion the dispersed light is perfectly unpolarized. In false dispersion, on the contrary, at a proper inclination the light is almost perfectly polarized in the plane of reflexion.

In false dispersion, which is merely a phenomenon of reflexion, the dispersed light has of course the same refrangibility as the incident light. In true dispersion heterogeneous dispersed light arises from a homogeneous beam incident on the body by which the dispersion is produced.

27. In those bodies, whether solid or liquid, which possess in a high degree the power of internal dispersion, the colour thence arising may be seen by exposing the body to ordinary daylight, looking at it in such a direction that the regularly reflected light does not enter the eye, and excluding transmitted light by placing a piece of black cloth or velvet behind, or by some similar contrivance. It has been usual to speak of the colour so exhibited as displayed by reflexion. As however the cause now appears to be so very different from ordinary reflexion, it seems objectionable to continue to use that term without qualification, and I shall accordingly speak of the phenomenon as *dispersive reflexion**. Thus dispersive reflexion is nothing more than internal dispersion considered as viewed in a particular way.

28. The tint exhibited by dispersive reflexion is modified in a peculiar manner by the absorbing power of the medium. In the first place, the light which enters the eye in a given direction is made up of portions which have been dispersed by particles situated at different distances from the surface at which the light emerges. The word *particle* is here used as synonymous, not with *molecule*, but with *differential element*. If we consider any particular particle, the light which it sends into the eye has had to traverse the medium, first in reaching the particle, and then in proceeding towards the eye. On account of the change of refrangibility which takes place in dispersion, the effect of the absorption of the medium is different for the two portions of the whole path within the medium, so that this effect may be regarded as a function of

* I confess I do not like this term. I am almost inclined to coin a word, and call the appearance *fluorescence*, from fluor-spar, as the analogous term *opalescence* is derived from the name of a mineral.

two independent variables, namely, the lengths of the path before and after dispersion; whereas, had the light been merely reflected from coloured particles held in suspension, the effect of absorption would have been a function of only one independent variable, namely, the length of the entire path within the medium.

29. When false dispersion abounds in a fluid, it may be detected at once by the eye, without having recourse to any of the characters already mentioned whereby it may be distinguished from true dispersion. When a fluid is free from false dispersion it appears perfectly clear, when viewed by transmitted light, although it may be highly coloured, and may even possess to such an extent the property of exhibiting true internal dispersion as to display, when properly viewed, a copious dispersive reflexion. On the contrary, when false dispersion abounds, the fluid, if not plainly muddy, has at least a sort of opalescent appearance when viewed by transmitted light, which, after a little experience, the eye in most cases readily recognises. In viewing the phenomenon of dispersive reflexion, as exhibited in a fluid, it might be supposed that the fluid was water, or else some clear though coloured liquid, holding in suspension a water colour in a state of extreme subdivision. But on holding the fluid before the eye, so as to view it by transmitted light, or rather view a bright well-defined object through it, the illusion is instantly dispelled. The reason of this difference appears to admit of easy explanation, and will be noticed further on.

30. Light will be spoken of in this paper as *active* when it is considered in its capacity of producing other light by internal dispersion. A medium will be said to be *sensitive* when it is capable of exhibiting dispersed light under the influence of light (visible or invisible) incident upon it. In the contrary case it will be called *insensible*.

I shall now return to the description of the appearances exhibited by some of the media most remarkable for their sensibility.

Decoction of the Bark of the Horse-Chestnut (Æsculus hippocastanum).

31. In Sir JOHN HERSCHEL's second paper it is stated that esculine possesses in perfection the peculiar properties which had been found to belong to quinine. Having tried without success to procure the former alkaloid, I was content to let this substance pass, till I found how admirably a mere decoction or infusion of the bark of the tree answered for all purposes of observation.

This medium is even more sensitive than a solution of sulphate of quinine, and disperses like it a blue light. The description of the mode of dispersion in the latter medium will apply in almost all points to the former: the principal difference consists in the circumstance that in the horse-chestnut solution the dispersion begins earlier in the spectrum than in the solution of quinine. In a solution of sulphate of quinine of convenient strength, we have seen that the dispersion came on at about G½H, the excessively faint dispersion which was exhibited earlier being left out of consideration, whereas in a decoction of the bark of the horse-chestnut, diluted so as to be

of a convenient strength, it came on a little before G. This explains the reason of an observation of Sir DAVID BREWSTER's, who has remarked that "a beam of light that has passed through the esculine solution disperses blue light, but not copiously, when transmitted through the quinine solution; but the beam that has passed through quinine is copiously dispersed when transmitted through esculine*."

Green Fluor-Spar from Alston Moor.

32. It is well known that some specimens of fluor-spar exhibit a sort of double colour. In particular, a variety found at Alston Moor, which is green when seen by transmitted light, appears when viewed in a certain manner of a beautiful deep blue. This blue colour seems to have been considered by Sir JOHN HERSCHEL as merely superficial. It has been shown however by Sir DAVID BREWSTER to arise from light dispersed in the interior of the crystal, and to have no particular relation to the surface.

The crystal with which the following observations were made was of a fine but not intense green when viewed by transmitted light. On viewing a pure spectrum through it, there was found to be a dark band of absorption in the red. This band was narrow, and by no means intense. The crystal exhibited a copious deep blue by dispersive reflexion.

33. On admitting into the crystal a cone of sunlight formed by a lens of short focus, and then analysing the dispersed beam, it was found to consist of a very little red followed by a dark interval, then green, faintly fringed below with less refrangible colours down perhaps to the orange, then blue, or bluish-green, followed by a great deal of indigo or violet. Independently of the gap in the red, the spectrum was not quite continuous, for a band of bluish-green, not very broad, was separated by dusky bands from the green below and the indigo above. The separate red band and the two dusky bands were all so faint as to be difficult to see.

The dispersed beam was readily proved to be truly dispersed, for it was unpolarized, and a pale brown glass cut it off when placed in the first position, although it transmitted it in a great measure when placed in the second.

34. When the crystal was examined by the third method, the general result closely resembled that produced by sulphate of quinine. The dispersion commenced about half-way between G and H, and continued from thence onwards far beyond H. It was strongest about H. The fixed lines were seen with beautiful distinctness as dark planes in the crystal. The groups H, *l*, *m* were quite evident, and *n* might be seen without difficulty. I have even seen some of the fixed lines of the group *p*. The tint of the dispersed light appeared as nearly as possible uniform throughout. The distance to which this light could be traced from the surface, did not at all diminish so rapidly in this crystal, with an increase in the refrangibility of the incident light,

* Philosophical Magazine, vol. xxxii. (June 1848), p. 406.

as it had done in the case of a solution of sulphate of quinine. Indeed, it was difficult to say how far the decrease in the depth to which the incident rays could be traced, by means of the dispersed light which they produced, was due merely to the increasing faintness of the light, and how far it indicated a real increase in the absorbing energy of the crystal; whereas in the case of sulphate of quinine the appearance presented unequivocally indicated a very rapid increase of absorbing power.

35. On examining the crystal by the second method, the general appearance was the same as in the case of sulphate of quinine, but the beam of falsely dispersed light was absent. In addition to the copious beam of deep blue light dispersed by the most refrangible rays, there was however a faint beam of red or reddish light dispersed by rays of low refrangibility. This beam was too faint to be seen by the third method of examination. It will be remembered that the prismatic analysis of the transmitted light gave a band of absorption in the red. Another crystal of a pale colour, which did not give a similar band of absorption in the red, exhibited nothing but the blue beam of dispersed light when examined by the second method.

36. On examining the crystal by the fourth method, the extreme red proved inactive. The activity commenced about the most refrangible limit of the red transmitted by a deep blue glass, when the dispersed light was red, but extremely faint. On moving the lens onwards through the spectrum, the dispersed light rapidly became brighter, and then died away. When at its brightest, although even then it was almost too faint for prismatic examination, it appeared to consist of not quite homogeneous light a little lower in refrangibility than the active light. For a considerable distance further on there was no sensible dispersion produced. The dispersed light became again perceptible when the active light belonged to the greenish yellow, or not till the blue, according to the intensity of the incident light. As the lens moved on the dispersed light remained faint for a considerable time. It was first reddish and then brownish, with a refrangibility answering to its colour. When the active light was at $G\frac{1}{2}H$, or thereabouts, the dispersed light rapidly grew much brighter, and became of a fine blue. On analysis it was found to consist of rays the refrangibility of which ranged within wide limits. The red rays were, however, almost wholly wanting, while the rays belonging to the more refrangible part of the spectrum resulting from the analysis of the dispersed beam were particularly copious. The most refrangible limit of the dispersed light did not quite reach in refrangibility the active light. The dispersed light was most copious when the active light belonged to the neighbourhood of H. As the lens moved on the dispersed light grew less bright, and gradually died away.

Solution of Guaiacum in Alcohol.

37. This is one of the media mentioned by Sir DAVID BREWSTER, who remarks that it "disperses, by the stratum chiefly near its surface, a beautiful violet light."

When this fluid is examined by the third or fourth method, it is found to exhibit a copious internal dispersion, which begins to be conspicuous much lower down in the spectrum than in the cases already described. In observing by the third method, the true dispersion appeared to commence about the end of the green, the dispersed light being reddish-brown. By the fourth method the dispersion could be traced as low down as D$\frac{1}{3}$ *b*, the dispersed light being reddish. As the lens moved onwards, in a direction from the red to the violet, the more refrangible colours entered in succession into the dispersed beam, and it became successively brownish, yellowish, greenish, and bluish. In whatever part of the spectrum the lens might be, it was found that the most refrangible part of the dispersed beam was of lower refrangibility than the active light. This could be easily determined by means of the beam of falsely dispersed light, which was always visible so long as the active light belonged to the visible part of the spectrum.

38. With the third arrangement the fixed lines were seen as before by means of the dispersed light, but in this fluid they could be seen much lower down in the spectrum than in the solution of sulphate of quinine. The group H was seen on a greenish ground. About the group *l* the ground was still greenish, but the dispersed light was not very copious. The beautiful violet light mentioned by Sir David Brewster is produced only by rays of extremely high refrangibility, and is found to extend from the beginning of the group *m* to the end of the group *n*, and even further. This part of the dispersion is best seen with a rather dilute solution.

39. In a solution of guaiacum, just as in the solution of sulphate of quinine, the absorbing power of the medium increases very rapidly with the refrangibility of the light. This is shown by the rapid decrease in the distance from the surface to which the dispersed light can be traced. The reason why the violet dispersed light is confined to a very thin stratum adjacent to the surface by which the light enters, is simply that the medium is so nearly opake with regard to the invisible rays beyond the extreme violet that all such rays are absorbed by the time the light has passed through a very thin stratum of the fluid.

40. If the solution be strong the colour is of considerable depth. In all such cases it is necessary to take the precaution, mentioned by Sir David Brewster, of transmitting the incident beam as near as possible to the upper surface, so as just to graze it. The absorption of the medium would otherwise modify the tint of the dispersed beam.

41. The solutions of quinine and guaiacum present a striking contrast with respect to the change of tint of the dispersed beam. In the former solution the change is but slight, if we except that part of the dispersion which is very faint; whereas in the latter, the prismatic colour which makes the nearest match to the composite tint of the dispersed beam runs through nearly the entire spectrum, as the refrangibility of the active light changes from that of the green rays to that of invisible rays situated far beyond the extreme violet.

Tincture of Turmeric.

42. This fluid is very sensitive, and exhibits a pretty copious dispersive reflexion of a greenish light. In its mode of internal dispersion it strongly resembles a solution of guaiacum, but the final tint of the dispersed light does not correspond to so high a mean refrangibility. When the fluid was examined by the third method, the true dispersion appeared to commence about *b*. The absorbing power was so great for the rays of high refrangibility, that from a little above F (in the case of tincture not diluted with alcohol) to the end the dispersed light seemed to be confined to the mere surface. By the fourth method the dispersion was as usual traced a little lower down in the spectrum. When the dispersed beam was first perceived it was nearly homogeneous, and its refrangibility was only a very little less than that of the active light. As the refrangibility of the active light increased, new colours, in the order of their refrangibility, entered into the dispersed beam, which became more and more composite, while at the same time its upper limit became distinctly separated from the beam of falsely dispersed light, which, when the whole dispersed beam was analysed by a prism, was always found in advance of the other. The tint of the dispersed beam passed from orange through yellow to yellowish green, which was its final tint. Tincture of turmeric is well adapted for exhibiting the fixed lines in the invisible part of the spectrum, though perhaps not quite so well as a solution of sulphate of quinine.

Alcoholic Extract from the Seeds of the Datura Stramonium.

43. This fluid, which I was led to try in consequence of Sir DAVID BREWSTER'S paper, proved to be remarkably sensitive, and exhibited a copious dispersive reflexion of a pale but lively green. The general phenomena are so nearly the same as in a solution of sulphate of quinine that there is no need of a separate description. The principal difference consists in the tint, which is green instead of blue. In the present case, however, the fluid, in addition to its dispersion of green, dispersed a red beam under the influence of certain red rays. As the lens employed in the fourth method of examination was moved from the extreme red onwards, the light was at first inactive, but when the lens reached a certain point of the spectrum, a red beam of truly dispersed light suddenly appeared, which disappeared with almost equal suddenness as the lens moved on. In this mode of observation the refrangibility of the dispersed could hardly be distinguished from that of the active light; but on combining the first and third methods, by removing the lens, placing the vessel truly in focus, and holding a blue glass alternately in front of the vessel and in front of the eye, I satisfied myself that the truly dispersed beam, taken as a whole, was of lower refrangibility than the light by which it was produced. The utility of the blue glass depended upon the circumstance that the upper extremity of the extreme red which it transmitted nearly coincided with the point of the spectrum at which the red beam occurred. This red beam was doubtless due to the presence of a small quantity of

chlorophyll, or one of its modifications. The light transmitted by the fluid exhibited on prismatic analysis the absorption band in the red which is so characteristic of that substance.

The colour of the solution was a pale brownish yellow; it would no doubt have been still paler, and perhaps nearly colourless, had the sensitive principle to which the green dispersion was due been present in equal quantity but in a state of purity. As it was, the fluid was pale enough to exhibit well, when poured into a test tube and held in front of a window, a narrow arc on the side of the incident light, like sulphate of quinine, only in this case the arc was green instead of blue.

Frequency of the occurrence of true internal dispersion having the same general character as that which takes place in the cases above described.

44. If we except the red dispersed beam produced by red rays in the crystal of fluor-spar and in the stramonium extract, a strong similarity may be observed in the mode of internal dispersion which takes place in the cases hitherto described. As the refrangibility of the incident light continually increases, the rays are at first inactive. At a certain point of the spectrum, varying according to circumstances, the true dispersion begins to be sensible, but is faint at first. After remaining faint for some distance it presently becomes more copious. It remains very conspicuous through the whole of the violet and beyond, and then gradually dies away. It consists at first of light of comparatively low refrangibility, and then new colours in the order of their refrangibility enter into it. Frequently the greater part of the change of prismatic composition takes place while the dispersed light is very faint, so that practically speaking we may almost say that the tint is uniform. Sometimes, when the dispersion just commences, the dispersed light is nearly homogeneous, and has a refrangibility so nearly equal to that of the active light that the beams due to true and false dispersion can hardly be separated.

45. Now this, so far as I have observed, is much the commonest kind of true internal dispersion, although sometimes the phenomenon presents very striking singularities. In the paper in which Sir DAVID BREWSTER first announced the discovery of internal dispersion, he remarks "that it is a phenomenon which occurs almost always in vegetable solutions, and almost never in chemical ones or in coloured glasses*." For my own part, I have rarely met with a vegetable solution which did not exhibit more or less the phenomenon of *true* internal dispersion. Its existence may in general be easily detected in the following manner. The sun's light being reflected horizontally through a lens, a deep blue glass is left in such a position as to intercept the light incident on the vessel containing the fluid, which is placed at the focus of the lens. A pale brown glass of the proper kind is then placed so as to intercept, first the incident, and then the dispersed light. A vessel with flat sides filled with a solution of sulphate of quinine would be better, and then the placing of the

* Edinburgh Transactions, vol. xii. p. 542.

medium in the second position might be dispensed with, the medium being sensibly transparent. Sometimes it is useful to have recourse to analysis through a doubly refracting prism, or a rhomb of calcareous spar. In this way true internal dispersion may often be detected in a fluid which is actually muddy, in which case, were the effect of the incident light observed as a whole, the true would be masked by the enormous quantity of false dispersion which such a medium would offer.

46. The fluids obtained by treating the leaves and other parts of plants with alcohol or hot water are almost always sensitive, so far as I have observed. The solutions in water presently ferment, and are frequently highly sensitive in the early stages of fermentation; they are usually more or less sensitive in all stages. Different kinds of fungus furnish very sensitive solutions. When aqueous solutions become muddy by decomposition, other clear and often highly sensitive liquids may be obtained from them by various chemical processes. Port and sherry are decidedly sensitive. In such cases the fluid is a mixture of several substances, of which some may be sensitive and others insensible. When vegetable substances are isolated they are frequently insensible, or else so very slightly sensitive when examined under great concentration of the highly refrangible rays, that it is quite impossible to say whether the sensibility thus exhibited may not be due to some impurity: thus, several solutions containing sugar, salicine, morphine, or strychnine were found to be insensible. A solution of veratrine in alcohol proved to be sensitive in a pretty high degree, dispersing internally a bluish light. Sir David Brewster has remarked that a solution of sulphate of strychnine in alcohol dispersed light *after it had stood for some days*. This observation I have verified with reference to *true* dispersion, which the solution exhibits, though not very copiously, after it has been made some time. There can be little doubt that the sensitive principle in this case is not strychnine, but some product of its decomposition. I now come to some instances of internal dispersion which are far more striking.

Solution of Leaf-Green in Alcohol.

47. It was in this very remarkable fluid that the phenomenon of internal dispersion was first discovered by Sir David Brewster, while engaged in researches relating to absorption. The character of the internal dispersion of a solution of leaf-green is no less remarkable than the character of its absorption. On account of the close connexion which seems to exist between the two phenomena, it will be requisite first to say a few words about the latter.

When green leaves are treated with alcohol, a fluid is obtained which is of a beautiful emerald-green in moderate thicknesses, but red in great thicknesses, and which has a very remarkable effect on the spectrum. A good number of the following observations on the internal dispersion of leaf-green were made with a solution obtained from the leaves of the common nettle, by first boiling them in water and then treating them with cold alcohol, the leaves having previously been partially

dried by pressing them between sheets of blotting paper. Nettle was chosen partly because it stands boiling without losing its green colour, and partly for other reasons. My object in boiling the leaves was to obtain the green colouring matter more nearly in a state of isolation, but it seems to have the additional advantage of giving a solution less liable to decomposition. Indeed, this fluid seemed disposed to remain permanently unchanged when kept in the dark; but a small portion of it which was exposed to strong light had its colour rapidly discharged.

48. When fresh leaves are left in contact with alcohol in the dark, or in only weak light, the colour of the fluid changes by degrees, and it seems to approximate (making allowance for impurities) to a type which is nearly represented by the fluid obtained in this manner from laurel leaves, or that obtained by treating with alcohol tea leaves from which a good deal of brown colouring matter has first been extracted by water. This type was rather ideal than actual, being derived from a comparison of different cases, until it seemed to be realized in the case of a fluid obtained by re-dissolving in alcohol a crust which had formed itself at the bottom of a test tube containing leaf-green. The principle to which the peculiar absorption and internal dispersion of such a fluid seems due may be called modified leaf-green. The fluid itself is not green but olive-coloured, becoming red at great thicknesses.

49. When solutions of leaf-green, and of its various modifications, are examined in different thicknesses by the light of a candle, there are five bands of absorption which may be observed in the spectrum. These will be called, in the order of their refrangibility, Nos. 1, 2, 3, 4 and 5, the bright bands below the respective dark bands being also numbered in the same manner. Of the dark bands, Nos. 1, 2, 3 and 5, are the first four in Sir David Brewster's plate*. No. 4 is mentioned in the memoir, but not represented in the plate, which corresponds to a thickness not sufficient to bring out this band. The last band in the plate could not be seen without strong light. The dark bands Nos. 1 and 2 are situated in the red, No. 3 about the yellow or greenish yellow, No. 4 in the green, and No 5 early in the blue. Of these, No. 1 is in small thicknesses by far the most intense, and it may be readily seen even in a very dilute solution; it might apparently be used as a chemical test of chlorophyll, or one of its modifications. The test would be of very easy application, since it would be sufficient to hold a test tube with the liquid at arm's length before a candle at a little distance, and view the linear image of the flame through a prism applied to the eye.

50. Fresh and modified leaf-green differ much in the order in which the bright bands are absorbed, and in the degree to which the dark bands are developed before they cease to be visible by the absorption of the part of the spectrum in which they are situated. In the green fluid, the dark band No. 5 is not usually seen, because the spectrum is there cut off, unless a very small thickness be used. With a moderate thickness, Nos. 2 and 3, especially the former, are well seen, and No. 1 is very intense. As the absorption goes on, the bright bands Nos. 2 and 3 are absorbed,

* Edinburgh Transactions, vol. xii.

and there is left the red band No. 1, and a double green band, consisting of the bright bands Nos. 4 and 5, separated by the dark band No. 4, which by this time has come out. In modified leaf-green, the dark bands Nos. 4 and 5 are much more conspicuous than in the green fluid, but No. 3 is wanting, or all but wanting. With a thickness by which the absorption is well developed, the conspicuous bright bands are in this case Nos. 1 and 3, and next to them No. 2, whereas in the green fluid Nos. 2 and 3 were quickly absorbed, or at least the whole of No. 2, and the greater part of No. 3.

51. It seems worthy of remark, that, especially in the case of the green fluid, the absorbing power alters with the refrangibility of the light at a very different rate on the two sides of the intense dark band No. 1. This might be inferred from the order in which the bright bands disappear; but it was rendered visible to the eye by the following easy experiment. A narrow test tube was partly filled with a solution of leaf-green, and then a few drops of alcohol were added, which remained at the top, and there diluted the solution. The tube was then held before a candle, and the linear image of the flame was viewed through a prism. In the under part the dark band No. 1 was broad, the bright band No. 2 being narrow, and almost obliterated, but in the upper part the dark band No. 1 was very narrow. Now on tracing upwards the sides of this dark band, it was found that the less refrangible side was almost straight, and the diminution in the breadth of the band was produced by the encroachment of the bright band No. 2. Speaking approximately, we may say that in proceeding from the extreme red onwards, at a certain point of the spectrum the fluid passes abruptly from transparent to opake, and then gradually becomes almost transparent again.

52. It may here be remarked, that although the absorption produced by leaf-green is best studied in a solution, its leading characters may be observed very well by merely placing a green leaf behind a slit, as near as possible to the flame of a candle, and then viewing the slit through a prism.

53. After this digression relating to the absorption of leaf-green, it is time to come to its internal dispersion. And first, when a cone of white light coming from the sun is admitted horizontally into the fluid, as close as possible to its upper surface, and the beautiful red beam of dispersed light is analysed by a prism, the spectrum is found to consist of a bright red band of a certain breadth, followed by a dark interval, and then a much broader green band not near so brilliant. There is usually but little false dispersion, and what there is may be almost entirely got rid of by analysing the beam by a NICOL's prism, so as to view it by light polarized in a plane perpendicular to the plane of dispersion. Now on raising the vessel without removing the prism from the eye, it was found that a dark band, which was in fact the absorption band No. 1, appeared almost exactly in the middle of the bright red band. On continuing to raise the vessel, so as to make the dispersed rays pass through a still greater thickness of the medium before reaching the eye, the dark

band increased in width, and when the red beam was almost absorbed, the part that was left consisted of two cones of red, one at each side of the dark band, which by this time had become broad. The whole appearance seemed to indicate that the bright red beam of dispersed light had a very intimate connexion with the intense absorption band No. 1.

54. Among coloured glasses, there is one combination which produces a very striking effect. When a deep blue glass is placed in the first position, the dispersed light, if the solution be at all strong, is confined to a very thin stratum adjacent to the surface, and is best seen by placing the vessel so that the surface of the fluid at which the light enters is situated at a little distance on either side of the focus of the lens, when there is seen a bright circle of a most beautiful crimson colour. It might be supposed that the red of which this circle mainly consists was nothing but the extreme red transmitted by the blue glass. But it is readily shown that such is not the case. For in the first place, the fluid transmits pretty freely the red transmitted by the blue glass, whereas the red light found in this circle is almost confined to the surface of the fluid. Again, it was found that a pale brown glass, which transmitted freely the extreme red, almost entirely cut off the bright circle, when placed in the first position without removing the blue glass, although it freely transmitted it when placed in the second position. It appears, therefore, that the bright circle is due, not to the red, but to the highly refrangible rays transmitted by the blue glass.

55. When a solution of leaf-green was examined by the third method, the appearance as seen from the outside was very singular. The fixed lines in all the more refrangible part of the spectrum were seen as interruptions in a bright red ground verging to crimson. The beauty and purity of the tint, and the strange contrast which it presented to the colours belonging to that part of the spectrum, were very striking. About H the tint began to verge towards brown, and the fixed lines beyond H were seen on a brownish red ground. That the ground on which the fixed lines of somewhat less refrangibility were seen was rather crimson than red, arose, no doubt, from the mixture of a little blue or violet light due to false dispersion, and to the scattering which took place at the surface of the glass.

56. On looking down from above, the places of the more conspicuous bands of absorption were indicated by dark teeth, with their points turned towards the incident light, interrupting the dispersed light. It is to be understood that the light was transmitted as close as possible to the upper surface, so that the absorption by which these teeth were formed took place *before* dispersion. In this way the places of the absorption bands Nos. 1, 2 and 4, were perfectly evident. No. 3, it will be remembered, was by no means conspicuous. When the solution is of convenient strength, the absorption is so rapid beyond the bright band No. 5, that the dispersion is confined to a thin stratum close to the surface by which the light enters, and therefore no dark tooth would be seen corresponding to the dark band No. 5.

57. On following the active light through the spectrum, in the direction of increasing refrangibility, the dispersion was found to commence with a bright, but narrow tail of pure red light, which shot right across the vessel. The light by which this tail was produced belonged to the more refrangible part of the extreme red band which is transmitted by a moderate thickness of the fluid. The activity of the incident light commenced almost abruptly: the same, it will be remembered, was the case with the absorbing power of the medium. After the tail of red light came the intense absorption band No. 1, where the dispersed light was confined to the immediate neighbourhood of the surface by which the active light entered. At this place a very bright band of dispersed light was visible on looking at the vessel from the outside. In this part of the spectrum the active and the dispersed light were both red; but that dispersion was accompanied by a change of refrangibility, was shown by the effect of absorbing media. Thus the long red tail and the bright band adjacent to the surface were differently affected by a blue glass, according as it was held in the first or the second position; and the bright band, though much enfeebled, was still plainly visible through a considerable thickness of the fluid, although a stratum having a thickness of only a very small fraction of an inch was sufficient to absorb the rays by which the band was produced. Although the dispersion continued throughout the whole of the visible spectrum and beyond, it was comparatively feeble in the brightest part of the spectrum. It became pretty copious again in the neighbourhood of the dark band No. 4, and remained copious throughout the blue and violet. In the green, the dispersed light was red, slightly verging towards orange, and in the blue and violet it was red verging a little towards brown.

58. It may seem superfluous, after what precedes, to bring forward any further proof of the reality of a change of refrangibility. Nevertheless the following experiment, which was in fact performed at an early stage of these researches, may not be deemed wholly unworthy of notice, as not involving the use either of absorbing media or of false dispersion.

A small narrow triangle of white paper was stuck on to the outside of the vessel containing the leaf-green, in such a manner that its axis was vertical, and its vertex, which was uppermost, was situated at the height of the middle of the spectrum. A narrow vertical slit was then placed at the distance of the image of the first slit, where the fixed lines were formed, and moved sideways till the light immediately beside the fixed line G passed through it. The vessel was then placed a few inches behind the slit, and moved sideways till the riband-shaped beam of homogeneous light, which passed through the second slit, was incident on the vertex of the triangle. On looking at the vessel from the front, as nearly as was convenient in the direction of the incident light, there appeared a bright vertical bar corresponding to a section of the incident beam. This bar was of two colours, namely, red in the upper half, where the light fell on the fluid, and indigo in the under half, where it fell on the paper. On refracting the whole system sideways,

through a prism of moderate angle applied to the eye, the objects appeared in the following order as regards refrangibility. First came the upper half of the bright bar, which was only a very little widened by refraction, so that it consisted of red light which was approximately homogeneous. Next came the triangle, with its vertex a little rounded, and its edges tinged with prismatic colours. The vertex, which had formerly coincided with the bright bar, now lay a little to one side of its upper half. The triangle was of course seen by means of the diffused light of the room, which was not perfectly dark, and therefore its refrangibility must have corresponded to the brightest part of the spectrum, or nearly so. Lastly came the under half of the bright bar, which was much more refracted than the triangle, so as to be shifted almost completely off it. The paper triangle was far too close to the first surface of the fluid to allow of attributing the dislocation of the bright bar to any error depending upon parallax; but to prevent all possible doubts on this score, I took care to refract the system both right and left, and the result was the same in the two cases. The conclusion is therefore inevitable, that the indigo light which had changed its colour by dispersion from leaf-green had changed its refrangibility at the same time.

59. In viewing a solution of leaf-green in a pure spectrum, I noticed a phenomenon which further indicates the close connexion which seems to exist between the absorption and internal dispersion of this fluid. On holding the eye vertically over the fluid, and looking down at the dispersed light through a red glass, I observed five minima of illumination, having for the most part the shape of teeth with their bases situated at the surface by which the light entered, and their points turned inwards. These minima occupied positions intermediate between the bands of absorption, so far at least as the positions of the latter were indicated by dark teeth pointing in the contrary direction. The first minimum was situated a little beyond the intense absorption band No. 1, and corresponded in position to the bright band No. 2. The second was situated a little further on. The maximum intervening between this and the third was but slight, so that the second and third together formed pretty nearly one broad minimum. The third and fourth were situated one at each side of the dark band No. 4, so as to correspond in position to the bright bands Nos. 4 and 5. The fifth was situated a little way beyond the bright band No. 5. This last minimum was not tooth-shaped, inasmuch as it occurred at a part of the spectrum where the dispersed light was almost confined to the surface of the fluid. These minima are best seen when the solution is rather weak. They may be perceived without using a red glass, though not so easily as with its assistance. With a stronger solution it was observed that the first minimum ran obliquely into the dark tooth corresponding to the absorption band No. 1.

60. The reason of the occurrence of these minima appears to be simply this, that the more copiously dispersed light is produced, the more rapidly the incident light is used up in producing it, so that minima of activity correspond to points of the

spectrum at which the incident light penetrates to comparatively great distances into the fluid before it is absorbed. The oblique position observed in the first minimum is readily explained by considering that the illumination at any point of the field of view depends conjointly upon the activity of the incident light, which is a function of its refrangibility, and upon the fraction of the incident light left unabsorbed, which last is a function both of the refrangibility and of the distance from the first surface.

61. It seems worthy of remark, that while the quantity of dispersed light is liable to fluctuations having an evident relation to the bands of absorption which occur throughout the spectrum, the quality of the light dispersed, as regards its refrangibility, appears rather to have reference to the intense absorption band No. 1.

Extract from blue leaves of the Mercurialis perennis.

62. The juice of this plant has the property of turning blue by exposure to the air. Some leaves and stalks which had turned blue were treated with alcohol, and a green fluid was thus obtained much resembling in colour the ordinary solutions of leaf-green, but I think of a rather bluer green than usual. In its mode of absorption, too, it much resembled ordinary solutions of leaf-green, to which substance no doubt the greater part of its colour was due. Its internal dispersion however was very peculiar, for it dispersed a copious orange in place of a blood red like the extracts from fresh green leaves in general, those of the *Mercurialis perennis* included. On analysis the dispersed beam was found to consist chiefly of a red band, similar to that which occurs in solutions of leaf-green, and of a yellow or orange and yellow band, a good deal brighter than the former, from which it was separated by an intervening dark band. When the fluid was examined by the second method, it was found that the yellow dispersion was produced principally by the brightest part of the spectrum. After a considerable time the fluid lost its fine green colour, as is very often the case with solutions of leaf-green, and became yellowish brown, but the red and yellow dispersions still continued.

When the fluid was examined by the fourth method, it was found that the red rays dispersed a red, just as in a solution of leaf-green. The additional dispersion which was so conspicuous in this fluid began almost abruptly about the fixed line D. When it was first observed, the refrangibility of the orange dispersed light could hardly, if at all, be separated from that of the active light. As the lens moved on, the orange beam rapidly grew brighter, and yellow entered into it; and now it was easy to see that the beam of falsely dispersed light lay at its more refrangible limit. The orange and yellow dispersed beam was brightest at about $D\frac{2}{3}E$; but though it decreased in intensity it could be traced far beyond that point, in fact, throughout the spectrum.

63. I have generally found that when a copious dispersion commences almost abruptly at a certain point of the spectrum, it is followed by a band of absorption in

the transmitted light. This law did not seem applicable to the orange dispersion exhibited by the solution just mentioned; but then it is to be remembered that the solution contained a quantity of chlorophyll, which produces absorption bands with such energy that it would naturally mask the bands which might be due to another colouring principle with which it was mixed. To try whether the law would be obeyed if the chlorophyll were got rid of, I boiled in water some portions of the root and young shoots which had turned blue, chlorophyll being insoluble in water. The solution thus obtained was red, in small thicknesses pink, and dispersed copiously a yellow or rather orange light. On subjecting the fluid to prismatic analysis, a band of absorption was seen at the place expected. Since aqueous solutions of this nature are liable to decomposition, frequently decomposing before sunlight can be obtained by which to examine them, the red solution was concentrated by evaporation and purified by alcohol, in which the orange-dispersing principle is soluble, as had already appeared from the properties of the alcoholic solution. The alcoholic solution thus obtained remained unchanged, at least for a long time, and had the further advantage over the aqueous solution of presenting the sensitive principle more nearly in a state of isolation, though it was still contaminated by some principle which dispersed a whitish light under the influence of rays of high refrangibility.

64. The blue colouring matter may be readily extracted by cold water, but is decomposed by boiling. The blue solution dispersed an orange light like the other, but the dispersed light could not be nearly so well seen, just as would be the case were the red orange-dispersing fluid mixed with an insensible blue fluid of a much deeper colour, so that the mixture of the two would be blue. And in fact when the blue fluid was changed to red by boiling the colour became far less intense.

Archil and Litmus.

65. It is stated by Sir David Brewster that a very remarkable example of internal dispersion, which had been pointed out to him by Mr. Schunk, is exhibited in an alkaline or in an alcoholic solution of a resinous powder produced from orcine by contact with the oxygen of the air. Not being able readily to procure a specimen of orcine, I tried archil, and obtained from it and litmus some very remarkable solutions.

In the fluid state in which archil is sold, the colour is much too deep for convenient optical examination. When a small quantity of archil is diluted with a great deal of water, the diluted fluid is very sensitive. It is red by transmission, or in small thicknesses purple, but exhibits by dispersive reflexion a pretty copious but rather sombre green.

66. When the fluid was examined by different methods, it was found to disperse a little red, some orange, and a great deal of green. The red dispersion was so slight, that in observing by the third method it appeared doubtful whether there was any except false dispersion. It commenced in the red, when the active and dispersed lights had the same refrangibility, or nearly so. The orange dispersion commenced

about the fixed line D, the dispersed light being at first nearly homogeneous, and of the same refrangibility as the active light. On proceeding onwards in the spectrum, in observing by the fourth method, the orange beam became brighter, and yellow entered into it, but no colour beyond that, so that the orange and yellow beam was left behind by the beam of falsely dispersed light, from which it was separated by a perfectly dark interval. The green dispersion began about *b*, or a little beyond, coming on almost abruptly. The manner of its commencement was best observed by the fourth method, by holding a prism to the eye while the lens was moved through the spectrum. In this way it was found that on arriving at the point of the spectrum above mentioned, a gleam of green light shot across the dark space which before separated the beam of falsely dispersed light from the orange beam of truly dispersed light. As the lens moved on, the green dispersed light grew brighter, but its more refrangible limit did not seem to pass, or at least much to pass, the refrangibility it had at first; so that the green beam of truly dispersed light was almost immediately left behind by the beam of falsely dispersed light. The former, on being left behind, soon died away.

67. We might suppose either that the red, orange and green dispersions are due to the same sensitive principle, or that they are produced by three distinct sensitive principles mixed together in the solution. The latter would appear the more probable supposition, to judge by the apparent want of connexion between the three dispersions. This view is strongly confirmed by the following results. Some ether was poured on archil in the fluid state, and after being gently moved about and allowed to stand, a little was withdrawn without agitation. A purplish rose-coloured fluid was thus obtained, which was highly sensitive, exhibiting the orange and green dispersions but not the red. The orange dispersion was far more copious, in proportion to the whole quantity of dispersed light, than had been the case with archil diluted with water.

Some archil was violently agitated with ether, and after subsidence the ether was withdrawn. This ethereal solution was much deeper in colour than the former, and exhibited the red dispersion in addition to the orange and green. On adding a small quantity of water, and agitating, a separation, or at least partial separation, of the sensitive principles took place; for the upper fluid exhibited the orange dispersion abundantly, but none of the red, and little or none of the green, while the under fluid exhibited the green and red dispersions with little, if any, of the orange. The upper fluid exhibited a pretty copious dispersive reflexion of reddish orange, and the under fluid a remarkably copious reflexion of a fine green. A similar separation, more or less perfect, took place in other cases, the dispersion of orange bearing to that of green a greater ratio in the ether than in the water. Some of the green-dispersing fluids thus obtained were most remarkable on account of the extraordinary copiousness of the reflected green, and the strange contrast which it presented to the transmitted tint, which was a purplish red.

The red dispersion in the second ethereal solution, though decided, was by no means copious. In the case of archil merely diluted with water, it had been so slight that its existence might have been considered doubtful. It might be supposed that the first solution was not sufficiently concentrated to exhibit the red dispersion, in which case the red and green dispersions might have been due to the same sensitive principle. But an ethereal extract from dried archil, which was plainly concentrated enough, did not exhibit the red dispersion, although it did exhibit the orange and green dispersions. None of the sensitive principles appear to constitute the chief part of the colouring matter of this dye-stuff.

68. When some of these ethereal solutions were examined by the third method, with a lens of shorter focus than usual, the appearance was very singular. At the less refrangible end of the spectrum the incident light was quite inactive; and then, on reaching a certain point, a copious dispersion of orange commenced abruptly. This continued with no particular change for some distance further on, when it passed abruptly into green. The fourth method showed however that the former dispersion continued, and was only masked, in the third method of observation, by a new and more powerful dispersion of green which then commenced. And in fact when the green-dispersing principle was separated, or partially separated, by water, the orange dispersion was seen to continue where before it appeared to have been exchanged for green.

69. I ought here to mention that a similar separation did not take place on the addition of water only to an ethereal extract from archil previously dried. The condition which determined the separation in the first case appeared to be the presence of a small quantity of ammonia, which would evaporate on drying the archil. And in fact when a small quantity of ammonia was added to the extract from dried archil, a partial separation was effected. I do not here enter into the question whether one of the sensitive principles may be obtained from the other, whether, for example, a chemical combination of the orange-dispersing principle with ammonia might disperse a green, or a green with a little orange. A solution containing a mixture of the same substance in two different states of chemical combination, both compounds being sensitive, is not the less justly regarded as containing two distinct sensitive principles.

70. The preceding results are mentioned, not for their own sake, but merely for the sake of the method of examination employed. The results indeed are so imperfect as to be worthless on their own account. A complete optico-chemical examination of archil and litmus would itself alone furnish a subject for research of no small extent; but it belongs rather to chemistry than to general physics. It is quite possible that internal dispersion may turn out of importance as a chemical test. The dispersing such a tint, and the having the dispersed light produced by light of such a refrangibility, form together a double character of so peculiar a nature that it enables us, so to speak, *to see* a sensitive principle in a solution containing many sub-

stances, some of them, perhaps, coloured, so that the colour of the solution may be very different from what it would be if the sensitive principle were present alone.

71. The law mentioned at the beginning of art. 63 did not seem very applicable to archil when the fluid was merely diluted with water. But when the orange-dipersing and green-dispersing principles were obtained, as it would appear, more nearly in a state of isolation, by means of ether and water, the law was found to be obeyed. Thus, when the ethereal solution which exhibited the orange dispersion and little else was examined by the third method, the disperson was found to commence with a tail of light followed by a dark tooth, indicating the position of a band of absorption. When the light transmitted by a certain thickness of this fluid was subjected to prismatic examination, it was found to consist of red followed by some orange, when the spectrum was cut off with unusual abruptness. After a broad dark interval came the most refrangible colours faintly appearing. Those solutions which exhibited a copious dispersion of green gave, in addition to a band obliterating the yellow, a very distinct band separating the green from the blue. A similar band, but by no means distinct, might be seen in archil merely diluted; and it is particularly to be observed that this band, which occurred a little above the point of the spectrum where the green dispersion commenced, became more conspicuons when the green-dispersing principle was present more nearly in a state of isolation.

72. Two portions of litmus were treated, one with ether and the other with alcohol, which were allowed to remain in contact with the solid. Both extracts, but especially the latter, were highly sensitive, exhibiting dispersions of orange and green similar to archil, and due apparently to the same sensitive principles. The ethereal extract dispersed chiefly orange, while the alcoholic extract dispersed orange and green in nearly equal quantities. The latter extract exhibited a remarkably copious dispersive reflexion of a colour nearly that of mud, and was altogether one of the strangest looking fluids that I have met with. On viewing it in such a manner that no transmitted light entered the eye, one might almost have supposed that it was muddy water taken from a pool on a road. But when the bottle containing it was held between the eye and a window the fluid was found to be perfectly clear, and of a beautiful purple colour.

Canary Glass.

73. Among media which possess the property of internal dispersion in a high degree, Sir DAVID BREWSTER mentions a yellow Bohemian glass, which dispersed a brilliant green light. This led me to seek for such a glass, and it proved to be pretty common in ornamental bottles and other articles. The colour of the glass by transmitted light is a pale yellow. Its ornamental character depends in a great measure upon the internal dispersion, which occasions a beautiful and unusual appearance in the articles made of it. The commercial name of the glass is canary glass. The following observations were made with a small bottle of English manufacture.

74. When the sun's light was admitted without decomposition the dispersed beam

was yellowish green. The dispersion was so copious that when a large lens was used the dispersed beam approached to dazzling. The prismatic composition of this beam was extremely remarkable. The beam was found on analysis to consist of five bright bands, which were equal in breadth and equidistant, or at least very nearly so, and were separated by narrow dark bands. The first bright band was red, the second reddish orange, the third yellowish green, the fourth and fifth green. I have very frequently observed dark bands, or at least minima, in the spectrum resulting from the prismatic analysis of dispersed beams, but I have not met with any example so remarkable as this, except in a class of compounds which the properties of canary glass led me to examine.

75. On analysing a beam of sun-light transmitted through a certain thickness of the glass, there was found to be a dusky absorption band a little below F, another less distinct at F$\frac{1}{2}$G, and the spectrum was cut off a little below G.

76. When the glass was examined by the third method, the dispersion was found to commence abruptly about the fixed line *b*. It remained remarkably copious throughout the whole of the visible spectrum and far beyond, with the exception of a band beginning a little above F, and having its centre at about F$\frac{1}{3}$G, where there was a remarkable minimum of activity. This band, it will be observed, was situated between the bands of absorption already mentioned. The tint of the dispersed light appeared to be uniform throughout, except perhaps where the dispersion was just commencing. This was the best medium I have met with for showing the fixed lines of extreme refrangibility, though some others were nearly as good.

77. On examining the glass by the fourth method, it was found that the dispersion commenced nearly where the dispersed light ended, that is, the lowest refrangibility of the rays capable of being dispersed was nearly the same as the highest refrangibility of the rays constituting the dispersed beam exhibited by white light as a whole. The dispersion appeared indeed to commence a little earlier, at about the refrangibility of the fourth dark band in the spectrum of the entire dispersed beam. When the small prism was held to the eye with one hand, while the small lens in the board was gradually moved with the other, in a direction from the red to the violet, through the part of the spectrum where the dispersion commenced, it was found that the region of the first four bands was lighted up almost simultaneously, the whole field of view having been previously dark. When the lens was moved a very little further on the dispersed beam with its five bands was formed complete. Indeed the whole five appeared almost simultaneously. Speaking approximately, and in fact with almost perfect accuracy, we may say that if white light be conceived to be decomposed into two portions, the first containing rays of all refrangibilities up to that of the fixed line *b*, or thereabouts, and the second containing rays of all greater refrangibilities, the dispersed light produced by white light as a whole belongs exclusively to the first portion; and yet, were the bottle illuminated by the first portion alone, no dispersion whatsoever would be produced, whereas were it illuminated by the second portion

alone, which contains not a ray having the same refrangibility as any one of the dispersed rays, the dispersion would be exhibited in full perfection.

Common Colourless Glasses.

78. Sir David Brewster states that he has met with many specimens, both of colourless plate and colourless flint glasses, which disperse a beautiful green light. All the colourless glasses which I have examined dispersed light internally to a greater or less extent, with the exception of some few specimens belonging to Dr. Faraday's experiments. A beautiful green seems to be the commonest tint of the dispersed beam, and this I have found in wine glasses, decanters, apothecaries' bottles, pieces of unannealed glass, &c.; also in many specimens of plate and crown glass. The green was generally of a finer tint than that dispersed by the canary glass, but was not near so copious. On analysis it was found to consist usually of red and green separated by a dark band, or rather a minimum of brightness. Those specimens which were examined by the third and fourth methods were found to exhibit a little false dispersion, produced chiefly in the brightest part of the spectrum, but the greater part was true dispersion. This dispersion was produced chiefly by a rather narrow band, comprising the fixed line G, where there appeared to be a remarkable maximum of sensibility. The line G lay a little above the lower limit of the band. Below the band dispersion also took place, though not near so copiously, and there appeared to be another maximum of sensibility some way further down in the spectrum; but above the band dispersion almost entirely ceased of a sudden; a very unusual circumstance when the active and the dispersed light are well separated in refrangibility. The position of the band in the spectrum, and the distribution of the illumination in it, which are very peculiar, were the same in all the specimens which were sufficiently sensitive to admit of being examined by the third method, but the tint of the dispersed light was not quite the same.

79. Orange-coloured glasses are frequently met with which reflect from one side, or rather scatter in all directions, a copious light of a bluish-green colour, quite different from the transmitted tint. In such cases the body of the glass is colourless, and the colouring matter is contained in a very thin layer on one face of the plate. The bluish green tint is seen when the colourless face is next the eye. As this phenomenon was supposed by Sir John Herschel to offer some analogy with the reflected tints of fluor-spar and a solution of sulphate of quinine, I was the more desirous of determining the nature of the dispersion. It proved on examination to be nothing but false dispersion, so that the appearance might be conceived to be produced by an excessively fine bluish-green powder contained in a clear orange stratum, or in the colourless part of the glass immediately contiguous to the coloured stratum. The phenomenon has therefore no relation to the tints of fluor-spar or sulphate of quinine. It is true that the very same glass which displayed a superficial reflexion of bluish green, when examined by condensed sun-light exhibited also, in its colour-

less part, a little true dispersion, just as another colourless glass would do. But this has plainly nothing to do with the peculiar reflexion which attracts notice in such a glass.

Observations on the preceding results.

80. There is one law relating to internal dispersion which appears to be universal, namely, that when the refrangibility of light is changed by dispersion it is *always lowered*. I have examined a great many media besides those which have been mentioned, and I have not met with a single exception to this rule. Once or twice, in observing by the fourth method, there appeared at first sight to be some dispersed light produced when the small lens was placed beyond the extreme red. But on further examination I satisfied myself that this was due merely to the light scattered at the surfaces of the large prisms and lens, which thus acted the part of a self-luminous body, emitting a light of sufficient intensity to affect a very sensitive medium.

81. Consider light of given refrangibility incident on a given medium. Let some numerical quantity be taken for a measure of the refrangibility, suppose the refractive index in some standard substance. Let the refrangibilities of the incident and dispersed light be laid down along a straight line AX (fig. 2) taken for the axis of abscissæ; let AM represent the refrangibility of the incident light, and draw a curve of which the ordinates shall represent the intensities of the component parts of the truly dispersed beam. According to the law above stated, no part of the curve is ever found to the right of the point M; but in other respects its form admits of great latitude. Sometimes the curve progresses with tolerable uniformity, sometimes it presents several maxima and minima, or even appears to consist of distinct portions. Sometimes it is well separated from M, as in fig. 2; sometimes it approaches so near to M that the most refrangible portion of the truly dispersed beam is confounded with the beam due to false dispersion.

82. Let $f(x)$ be the ordinate of the curve corresponding to the abscissa x, a the abscissa of the point M. Since $f(x)$ is equal to zero when x exceeds a, the curve must reach the axis at the point M at latest, unless we suppose the function capable of altering abruptly, as is represented in fig. 3. I do not think that such an abrupt alteration, properly understood, is necessarily in contradiction with the law of continuity. For the sake of illustration, let us consider the phenomenon of total internal reflexion. Let P be a point in air situated at the distance z from an infinite plane separating air from glass. Conceive light having an intensity equal to unity, and coming from an infinitely distant point, to be incident internally on this plane at an angle $\gamma+\theta$, where γ is the angle of total internal reflexion. The intensity at P is commonly, and for most purposes correctly, considered as altering abruptly with θ, having, so long as θ is negative, a finite value which does not vanish with θ, but being equal to zero when θ is positive. The mode in which the law of continuity is in this case obeyed is worthy of notice. In the analytical expression for the vibration, when θ passes from negative to positive, the coordinate z passes from under a

circular function into an exponential with a negative index, containing in its denominator λ, the length of a wave of light. As θ increases through zero, the expression for the vibration alters continuously; but if z be large compared with λ it decreases with extreme rapidity when θ becomes positive. On account of the excessive smallness of λ, it is sufficient for most purposes to consider the intensity as a function of θ which vanishes abruptly; and indeed it would be hardly correct to consider it otherwise. For the use of the term *intensity* implies that we are considering light as usual, whereas those phenomena which require us to take into account the disturbance in the second medium which exists when the angle of incidence exceeds that of total internal reflexion, lead us to consider the nature as well as the magnitude of that disturbance, which no longer consists of a series of plane waves constituting light as usual. It is in some similar sense that I mean to say that we may suppose the function $f(x)$, which expresses the intensity of the truly dispersed light, to alter abruptly, without thereby implying any violation in the law of continuity. In observing by the fourth method, the portion of the spectrum operated on, though it may be small, is necessarily finite, and in some cases no separation could be made out between the beams of truly and falsely dispersed light. Hence I cannot undertake to say from observation, whether the variation of $f(x)$ be always continuous, though sometimes very rapid, or be in some cases actually abrupt. I think, however, that observation rather favours the former supposition, a supposition which, independently of observation, seems by far the more likely.

83. Although the law mentioned in Art. 80 is the only one which I have been able to discover, relating to the connexion between the intensity and the refrangibility of the component parts of the dispersed beam, which appears to be always obeyed, and which admits of màthematical expression, there are some other circumstances usually attending the phenomenon which deserve notice.

When dispersion commences almost abruptly on arriving at a certain point of the spectrum, the dispersed beam is very frequently almost homogeneous at first, and of the same refrangibility as the active light. If the dispersed beam, when first perceived, be decidedly heterogeneous, its refrangibility extends almost, if not quite, to that of the active light, so that it is difficult, if not impossible, to separate the beams of truly and falsely dispersed light. On the other hand, when dispersion comes on gradually, it is generally found that the refrangibility of even the most refrangible part of the dispersed beam does not come up to that of the active light.

Thus in the cases of the red dispersion exhibited by a solution of leaf-green, and of the orange dispersions exhibited by solutions obtained from archil and from the *Mercurialis perennis*, the dispersed light was at first nearly homogeneous, and of the same refrangibility as the active light. In the case of the green dispersions shown by a solution obtained from archil, and by canary glass, the dispersed light was heterogeneous from the first; but still, when it first commenced, a portion of it had nearly the same refrangibility as the active light. In a solution of sulphate of quinine

the dispersion came on gradually, being perceptible when the active light belonged to the middle of the spectrum; and in this case the dispersed light consisted of colours of low refrangibility. The bright part of the dispersion however came on pretty rapidly, when the active light approached the extreme limit of the visible spectrum, and accordingly the dispersed beam consisted in that case chiefly of light of high refrangibility.

84. The mode of absorption of any medium may very conveniently be represented by a curve, as has been done by Sir JOHN HERSCHEL. To represent geometrically in a similar manner the mode of internal dispersion, would require a curved surface. Let the refrangibility of light be measured as before, and suppose for simplicity's sake the intensity of the incident light to be independent of the refrangibility, so that dy may be taken to represent the quantity of incident light of which the refrangibility lies between y and $y+dy$. Considering the effect of this portion of the incident light by itself, let x be the refrangibility of any portion of the dispersed light, and $zdxdy$ the quantity of dispersed light of which the refrangibility lies between x and $x+dx$. Then the curved surface, of which the coordinates are x, y, z, will represent the nature of the internal dispersion of the medium. We must suppose the intensity of the incident light referred to some standard independent of the eye, since the illuminating power of the rays beyond the violet, and even of the extreme violet, is utterly disproportionate to the effect which in these phenomena they produce.

From the nature of the case, the ordinate z of the surface can never be negative. The law mentioned in Art. 80 may be expressed by saying, that if we draw through the axis of z a plane bisecting the angle between the axes of x and y, at all points on the side of this plane towards x positive, the curved surface confounds itself with the plane of xy.

85. Let us consider the form of this surface in two or three instances of internal dispersion. For facility of explanation, suppose the plane of xy horizontal, let x be measured to the right, y forwards, and z upwards. Let a line drawn in the plane of xy through the origin, and bisecting the angle between the axes of x and y, be called for shortness the line L. In all cases the surface rises above the plane of xy only to the left of the line L.

In the case of a solution of leaf-green, the surface consists as it were of two mountain ranges running in a direction parallel to the axis of y, or nearly so. The first range, if prolonged, would meet the axis of x at a point corresponding to the place of the dark band No. 1 in the red, or nearly so. The second would meet it somewhere in the place corresponding to the green. The green range is much broader than the red, but very much lower, and is comparatively insignificant. The ridge of the red range is by no means uniform, but presents a succession of maxima and minima. The range commences at the end nearest to the axis of x with a very high peak, by far the highest in the whole surface. In following the ridge forwards,

five minima or passes may be observed, with hills intervening. The ordinates y of the first four of these minima correspond to the refrangibilities of the bright bands Nos. 2, 3, 4 and 5. The last minimum lies a little further on. Whether similar minima exist in the green range is not decided by observation, on account of the faintness of the green dispersed light.

In the case of canary glass, the surface consists of five portions like mountain ranges running parallel to the axis of y, and having abscissæ belonging to the red, reddish orange, yellowish green, green, and more refrangible green, respectively. These ranges do not all start from the immediate neighbourhood of the line L, but on the side towards the axis of x end almost in cliffs, at points at which the ordinate y is nearly equal to the abscissa of the fifth range, perhaps a little less. Thus the first three ranges are well separated from the line L. The ranges are intersected by a sort of valley running parallel to the axis of x, and having for its ordinate y the refrangibility of F$\frac{1}{2}$G. With the exception of the minima which occur where the ranges are intersected by this valley, the ridges run on very uniformly, and it is only very gradually that the ranges die away.

The form of the surface which expresses the internal dispersion of a solution of sulphate of quinine, may be gathered from the description of that medium. In this case the surface resembles a rising country, not intersected by any remarkable mountain ranges or valleys.

Fig. 4 is a rude representation of the internal dispersion in a solution of leaf-green. The curves represented in the figure must be supposed to be turned through 90° about the lines on which they stand, and will then represent sections of the surface already described, made by vertical planes parallel to the axis of x. OL is the straight line bisecting the angle xOy. The figure is merely intended to assist the reader in forming a clear conception of the general nature of the phenomena, and must not be trusted for details. No attempt is made to represent the several maxima and minima in the intensity of the red beam of dispersed light. In any such figure, if we suppose homogeneous light to be incident on the medium, and wish to lay down the place of the falsely dispersed beam, we have only to draw a straight line parallel to the axis of x, through the point in the axis of y which corresponds to the refrangibility of the incident light, and find where this line cuts the straight line OL which bisects the angle xOy.

On the cause of the clearness of fluids, notwithstanding a copious internal dispersion which they may exhibit.

86. It has been already remarked, that though water holding a water colour in suspension makes an admirable imitation of a highly sensitive fluid, when the latter is viewed by dispersive reflexion alone, the two fluids have a totally different appearance when viewed by transmitted light. The cause of this difference appears to be plain enough. The light due to internal dispersion emanates from each portion of

the fluid which is under the influence of the active light, and emanates apparently in all directions alike. I have not attempted to determine experimentally whether the intensity is strictly the same in all directions. The experiment would be very difficult, especially for directions nearly coinciding with that of the active light, because in that case the light which was really due to internal dispersion would be mixed up with the glare which is always found in the neighbourhood of light of dazzling brightness. However, I have seen nothing which led me to suppose that the intensity was different in different directions. We may express the results of observation extremely well, by saying that the fluid or solid medium is self-luminous so long as it is under the influence of the active light.

Accordingly, when a bright object, such as the sky, or the flame of a candle, is viewed through a highly sensitive fluid, the regularly transmitted light is accompanied by some side light due to internal dispersion. The latter, however, emanating in all directions alike from the influenced particles, is too faint, when contrasted with the regularly transmitted light, to make any sensible impression on the eye. But when a fluid, itself insensible, holds in suspension a great number of solid particles of finite size, the light reflected from such particles is reinforced, in directions nearly coinciding with that of the incident light, by a great quantity of diffracted light, so that a bright object viewed through such a fluid is surrounded by a sort of nebulous haze, giving the fluid a milky appearance.

Washed Papers.

87. In a paper "On the Action of the Rays of the Solar Spectrum on Vegetable Colours," Sir JOHN HERSCHEL mentions a peculiarity which he had observed in paper washed with tincture of turmeric, which consists in its being illuminated, when a pure spectrum is thrown on it, to a much greater distance at the violet end than is the case with mere white paper*. This phenomenon was attributed by Sir JOHN to a peculiarity in its reflecting power, and was considered as a proof of the visibility of the ultra-violet rays. The colour of the prolongation of the spectrum was yellowish green. Sir JOHN appears to have been in doubt whether the greenish yellow colour was to be attributed to the mixture of the true colour of the ultra-violet rays with the yellow of the paper due to diffused light, or to the real colour of the ultra-violet rays themselves, which on that supposition would have been incorrectly termed "lavender."

88. The fact of the change of refrangibility of light having been established, there could be little doubt that the true cause of the extraordinary prolongation of the spectrum on paper washed with tincture of turmeric, was very different from what Sir JOHN HERSCHEL had supposed, and that it was due to a change of refrangibility in the incident light, which was produced by the medium in a solid state. Tincture of turmeric has already been mentioned as a medium which possesses in a high

* Philosophical Transactions for 1842, p. 194.

degree the property of internal dispersion. It was the observation of Sir John Herschel's already mentioned, which led me to try this medium. But it is by no means essential that a sensitive substance should be in solution, or in the state of a transparent solid, in order that the change of refrangibility which it produces should admit of being established by direct experiment, although of course the mode of observation must be changed.

89. A piece of paper was prepared by pouring some tincture of turmeric on it, and allowing it to dry. In this way the part which was deeply coloured by turmeric was in juxtaposition with the part which remained white, which was convenient in contrasting the effects of the two portions. The sun's light being reflected horizontally into a darkened room through a vertical slit, the paper was placed in a pure spectrum formed in the usual manner. On the coloured part the fixed lines were seen with the utmost facility far beyond the line H, on a yellowish ground. The colours too of all the more highly refrangible part of the spectrum were totally changed. From the red end, as far as the line F, or thereabouts, there was no material change of colour; but a little further on a very perceptible reddish tinge came on, which was quite decided at F$\frac{1}{2}$G, where it was mixed with the proper colour of that part of the spectrum. About G$\frac{4}{7}$H the colour became yellowish. The reality of a change of refrangibility was easily proved by refracting the spectrum on the screen by a prism applied to the eye. When the refraction took place in a plane parallel to the fixed lines, they were seen distinctly throughout the spectrum; but when it took place in a plane perpendicular to the former, the fixed lines in the less refrangible part of the spectrum, and as far as F, were distinctly seen; but in the rest of the spectrum they were more or less confused, or even wholly obliterated, according to their original strength, the refracting angle and dispersive power of the prism, and its distance from the paper. With a prism of small angle the edges of the broad bands H were seen tinged with prismatic colours.

90. The change of refrangibility was further shown by the following observation. The paper was placed in the pure spectrum in such a manner that the line of junction of the coloured and uncoloured parts ran lengthways through the spectrum, so that the same fixed line was seen partly on the coloured and partly on the uncoloured portion. On viewing the whole through a prism of moderate angle applied to the eye, and so held as to refract the system in a direction perpendicular to the fixed lines, the line F was seen uninterrupted, but G was dislocated, the portion formed on the yellow part of the paper being a good deal less refracted than that formed on the white. The latter was indeed faintly prolonged into the yellow part of the paper, so that on this part G was seen double; but the image which was by far the more intense of the two was less refracted than that formed on the white paper. The whole appearance was such as to create a strong suspicion of some illusion, as if some other group of fixed lines formed on the yellow part of the paper had been mistaken for G, though certainly no reason appears why such a group should not have had its coun-

terpart on the white part. However, to remove all doubts, I refracted the system in the direction of the fixed lines, and then turned the prism round the axis of the eye through 90°, when the plane of refraction was situated as before. At first the two portions of the line G were of course seen in the same straight line; and the perfect continuity with which, as the prism turned round, the appearance changed into what had been first seen, left not the shadow of a doubt as to the reality of the dislocation.

91. The cause of the whole appearance is plain enough. The light coming from the illuminated part of the yellow paper consisted, in the neighbourhood of G, of two portions; the first, indigo light, which had been scattered in the ordinary way; the second and larger portion, heterogeneous light having a mean refrangibility a good deal less than that of G, which had arisen from homogeneous light of higher refrangibility. The absence of the first occasioned the faint prolongation of the more refracted part of the line G; the absence of the second gave rise to the less refracted part.

92. The broad bands H were seen faintly but quite distinctly on the white paper. On refracting them sideways by a prism of moderate angle held to the eye, they became confused, and tinged with prismatic colours. The confused images of these bands, seen in the white and coloured parts, were nearly continuous. It thus appears that the visibility of the bands H on the white paper was due to a change of refrangibility which that substance had produced in violet light of extreme refrangibility.

93. Effects similar to those produced by paper coloured by tincture of turmeric are also produced by turmeric powder, or even by the root merely broken across. Notwithstanding the roughness of the latter, the bands H and fixed lines far beyond are seen with the utmost facility.

94. These phenomena are much better observed by covering the slit with a deep blue glass, which absorbs all the bright part of the spectrum, while it freely transmits the violet and invisible rays, which are mainly efficient in this class of phenomena. In this way fixed lines may be seen on common white paper far beyond H. These lines may be seen without the use of the blue glass, by allowing the bright colours to pass by the edge of the paper, and receiving on it only the extreme violet and invisible rays.

95. Paper coloured by turmeric having exhibited so well the sensibility of that substance, I was induced to try various other washed papers, in fact, papers washed with most of the fluids with which I had made experiments. I found almost always that sensitive solutions gave rise to sensitive papers, exhibiting a change of refrangibility of the same character as that shown by the solution. Besides the turmeric paper, the two most remarkable were paper washed with a pretty strong solution of sulphate of quinine, and paper washed with the extract from the seeds of the *Datura stramonium*. I should here observe, that it was not till long after the time when these experiments were made that I was acquainted with the high sensibility of a decoction of the bark of the horse-chestnut. The former of the papers just mentioned ex-

hibited the fixed lines of the invisible rays on a blue, and the latter on a green ground. The dispersion produced by the quinine paper was not exhibited so early in the spectrum as in the case of turmeric, nor was it so copious in the extreme violet rays, and for some distance further on, but the quinine paper seemed superior to the other for showing the fixed lines of extreme refrangibility. With the turmeric paper the group *n* was plain enough, but with the quinine paper I have seen some fixed lines of the group *p*. The stramonium paper was, on the whole, I think superior to the quinine paper in point of the copiousness of the dispersed light, but seemed hardly equal to it for showing the fixed lines of extreme refrangibility. However, it is likely that paper washed with a solution of the sensitive principle in a state of purity would have been quite equal to the quinine paper in this respect.

96. A washed paper is a little more convenient for use than a solution, but, as might be expected, it does not show the fixed lines with quite as much delicacy, nor is it quite so good for tracing the spectrum to the utmost limits to which it can be traced with the substance employed.

97. The sensibility of fresh leaf-green could not be made out on a washed paper by this mode of observation, but the sensibility of the substance extracted by alcohol from black tea, from which the brown colouring matter had been removed by hot water, was plainly exhibited by the redness which it produced in the highly refrangible part of the spectrum.

98. Paper washed with a solution of guaiacum seemed an exception to the general rule; but this is not to be wondered at, since a paper prepared in this manner is turned green when exposed to the light, and it is difficult to prevent some degree of discoloration. That the fluid state is not essential to the exhibition of the sensibility of this substance, was however plainly shown by the high degree of sensibility of the solid resin from which the solution was made. In this case the bands H were seen on a greenish ground. The dispersion of a fine blue light under the influence of rays of still higher refrangibility was hardly, or not at all, exhibited by the solid resin.

99. Shell-lac, common resin, glue, are all highly sensitive. The ground on which the fixed lines in the neighbourhood of H are seen is brown in the case of shell-lac, and greenish in the case of resin and glue. The sensibility of glue is evidently not due to gelatine, for isinglass is almost, if not quite, insensible. These are merely a few instances of sensibility: I shall defer further mention of the subject till I have described a better mode of observation. I will merely observe for the present, that several washed papers proved not greatly inferior to turmeric paper for showing the fixed lines about and beyond H.

Effect of refracting a Narrow Spectrum in a Vertical Plane.

100. In the arrangement last described, when a short slit is used, the spectrum received on the washed paper or other substance is of course narrow, so that the fixed lines formed on the paper are but short, and may roughly be regarded as mere

points. If, now, the whole be viewed through a prism, so as to be refracted in a vertical plane, the effect is very striking. For facility of explanation suppose the red to be to the left, and the rays to be refracted upwards, so that to the observer the image is thrown downwards. The original spectrum on the screen is decomposed by the prism held to the eye into two spectra, which diverge from each other. The first of these runs obliquely downwards from left to right, and contains the natural colours of the spectrum from red to violet. It consists of light which has been scattered in the ordinary way by the substance on which the primary spectrum is received, and the cause of its obliquity is evident. The second spectrum is horizontal, that is to say, it approximates to the form of a long rectangle having its longer sides horizontal. Of course it would be theoretically possible to render the vertical sides the longer, but when the whole arrangement of the apparatus is such as to be convenient for observation, the horizontal sides are much longer than the others. In this second spectrum the colours run *horizontally*, that is to say, the lines of equal colour are horizontal. The interruptions of the primary spectrum corresponding to fixed lines, almost reduced to points, are now elongated, so that in this strangely formed spectrum the principal fixed lines of the solar spectrum are seen running *across* the colours.

101. It will be convenient to have a name for the second of the two spectra above mentioned. As the term *secondary spectrum* is already appropriated to something altogether different, I shall call it the *derived spectrum*. The first of the diverging spectra may be called the *primitive spectrum*, while the original spectrum, considered as not yet decomposed by the prism held to the eye, may be called, for distinction, as in fact it has been already called, *primary*.

102. In accordance with the law enunciated in Art. 80, it is found that the derived spectrum appears *always on one and the same side* of the primitive, being *less refracted*.

103. The brilliancy of the derived spectrum, its extent, both vertically and horizontally, the colours of which it mainly consists, the distribution of its illumination in a horizontal direction, all depend upon the nature of the substance upon which the primary spectrum is received. As a general rule, it may be stated that it starts from the neighbourhood of the brightest part of the primitive spectrum, and extends from thence onwards to a good distance beyond the extreme violet; and that with a given substance its colour is pretty uniform, that is, does not much change in passing from one vertical section to another. Sometimes the derived spectrum remains very bright up to its junction with the primitive, or at least till it gets so near that the superior brilliancy of the primitive spectrum prevents all observation on the derived; sometimes it remains dull to a considerable distance from the primitive spectrum, and then, opposite a highly refrangible part of the primitive spectrum, a strong illumination comes on in the derived, lasts for some distance, and afterwards gradually dies away. Many of the results mentioned in this paragraph are better observed by a somewhat different method, which will shortly be described.

104. It has been already stated that the bands H were distinctly seen on common white paper, the substance usually employed as a screen in experiments on the spectrum, but that this was due to a change of refrangibility produced in the extreme violet rays. These same bands have been seen on paper in the experiments of others, though of course their visibility was not attributed to its true cause. By the method of observation described in Art. 100, or still better, by a method not yet explained, it may be seen that the change of refrangibility produced by white paper is by no means confined to the extreme violet rays, and those still more refrangible, but extends from about the middle of the spectrum to a good distance beyond the extreme violet. The distance to which the illumination can be traced by means of light merely scattered in the ordinary way, may be seen by examining the primitive spectrum. In the primitive spectrum formed on white paper and other white substances, I have not been able to trace the illumination beyond the edge of the broad band H, which accords very well with the illuminating power of the extreme violet when received directly into the eye.

Illuminating Power of the Rays of high Refrangibility.

105. The prolongation of the spectrum seen on turmeric paper was brought forward by Sir John Herschel as a proof of the visibility of the ultra-violet rays, or rather as a confirmation of other experiments which had led him to the same conclusion. Of course, the experiment with turmeric must now be regarded as having no bearing on the question; but from the way in which Sir John speaks of it, it would appear that he thought the other experiments not so conclusive as to be independent of the confirmation which they received from this. The experiment with the distorted spectrum, indeed, must now be put out of account, because in this experiment, as I have been informed by Sir John Herschel, the light was only thrown on a screen. Accordingly, the question of the visibility of these rays may be regarded as open to further investigation.

While engaged in some of the experiments described in Art. 89, I had occasion to form a pure spectrum in air in a well-darkened room, the slit itself by which the sun's rays entered being covered by a deep blue glass, so that no great quantity of light entered even at this quarter. Now, if ever, it would appear that the ultra-violet rays ought to be seen by receiving them directly into the eye; for the blue glass was so transparent with regard to these rays that the fixed lines far beyond H were seen with facility, even on substances, such as white paper, which stand low in the scale of sensibility; and the length of the spectrum from B to H was about an inch and a quarter, so that when the extreme violet rays entered the pupil, supposed to be held near the pure spectrum, not only the extreme red rays transmitted by the blue glass, but even the brighter part of the transmitted blue and violet rays fell altogether outside it. However, on holding the eye a few inches in front of the pure spectrum, so as to see the fixed lines distinctly, the bands H were indeed seen with

great facility; but I was not able to make out fixed lines beyond the end of the group l, that is, about the end of FRAUNHOFER's map. However, the eyes of different individuals may differ much in their power of being affected by the highly refrangible rays. It must be confessed, that on looking in the direction of the prisms, a good deal of blue light was seen, consisting of light which had been scattered at the surfaces of the prisms and lens. This light, though far from dazzling, was sufficient to prevent the eye from seeing excessively faint objects, even though they might be well defined. For want of a heliostat, I did not attempt an experiment I was meditating for securing a more perfect isolation of the ultra-violet rays *.

However, it seems to me to be a point of small importance, so far as regards its bearing on other physical questions, whether the illuminating power of these rays is absolutely null or only excessively feeble. It is quite certain, that if not absolutely null, their illuminating power is at least utterly disproportionate to the effect which they produce in the phenomena to which the present paper relates, and indeed that is true even of the violet rays. By *illuminating power*, I mean of course, power of producing the sensation of light when received directly into the eye; for by giving rise to light of lower refrangibility, they are able to illuminate strongly an object on which they fall.

Mode of Observation specially applicable to Opake Bodies.

106. In some of the experiments already described, the change of refrangibility was exhibited, which was produced by washed papers and solid bodies. There exists, however, a mode of observation far preferable to those which have already been explained as applicable to such cases, and which may even in some instances be employed with advantage in the examination of transparent bodies. In the experiment described in Art. 100, the primitive spectrum is pure, but the derived spectrum impure, on account of the finite length of the slit. Were the slit reduced to a point, it is true that the derived spectrum would become pure like the primitive, but then the quantity of light would be so small that the primary spectrum would hardly bear prismatic analysis. It is well, once for all, to examine a few sensitive opake substances in a very pure spectrum, because then the exhibition of fixed lines running across the colours in the derived spectrum removes even the shadow of a doubt as to the reality of the change of refrangibility of the incident light. Besides this, the only theoretical advantage in having the primitive spectrum very pure is, that it might be expected to enable us to detect any very rapid fluctuations in the colour or intensity of the dispersed light. Of course, I am now speaking only with reference to experiments in which the observer is employing the spectrum to examine some substance, not employing the substance to examine the spectrum. But practically, I have not found any advantage on this account; for abrupt, or almost abrupt changes in the colour or intensity of the dispersed light hardly ever, if ever, occur,

* See note B.

except when the active and the dispersed light have very nearly the same refrangibility. But such changes could not be observed even with a pure primitive spectrum, because in the place where they occur the primitive and derived spectra overlap; and independently of this, the brilliancy of the primitive spectrum would prevent all exact observation of the derived. It is true, that in the case of chlorophyll, or some of its modifications, changes of intensity having apparently somewhat the same nature were observed when the active and the dispersed light were widely separated in refrangibility. But the sensibility of this substance is difficult, if not impossible, to observe in the case of a washed paper or a green leaf, except by one of the methods not yet described, so that it is not to be expected that such fluctuations could be made out. Besides, it is to be remembered that the fluctuations observed in the case of solutions of chlorophyll, were fluctuations in the rate at which dispersed light was produced, not fluctuations in the sum total of the dispersed light produced by the time the active light was exhausted. Fluctuations of the former kind by no means imply fluctuations of the latter; and indeed, the circumstance, that maxima of activity in the solution correspond to minima of transparency, would seem to show that the total quantity of light dispersed, considered as a function of the refrangibility of the active light, is not subject to these fluctuations, or at least not to anything like the same extent. Now the total quantity of red light dispersed by a green leaf, or by a paper washed with a solution of chlorophyll, must depend upon the sensibility of this substance and upon its transparency conjointly, and therefore it is likely enough that such maxima and minima would not be observed, even were the dispersed light much stronger than it is.

107. Suppose now the slit by which the light enters to be placed in a horizontal instead of a vertical position, so as to lie in the plane of refraction. Corresponding to light of any given refrangibility, the image of the slit formed after refraction through the prisms and lens will now be a narrow parallelogram, which may be regarded as a horizontal line. The series of these lines, succeeding one another in a horizontal direction, and consequently overlapping, forms the spectrum incident on the body examined. This spectrum is now no longer pure, but only approximately so, a point, however, which, as we have seen, is not of much consequence. But by this trifling sacrifice two very great advantages are gained. The first is increase of illumination. When the slit is vertical, the spectrum received on the body occupies a rectangle having for breadth the length of the image of the slit; but when it is horizontal, the same, or very nearly the same quantity of light is concentrated into a rectangle having the same length as before (the length of the image of the slit being disregarded compared with that of the spectrum), but having for its breadth only the length of the image of a line drawn across the slit. Hence the intensity of the incident light is increased in the ratio of the breadth to the length of the slit. The second advantage is purity in the derived spectrum, a point of much consequence, because sometimes the composition of this spectrum presents very remarkable

peculiarities. If the slit be not too long, the spectrum formed in air is still sufficiently pure to allow us to make out in a general way what are the refrangibilities of those portions of the incident light which are most efficient in producing dispersed light; and this is nearly all that can be done even when the spectrum is very pure.

108. The method of observation which has just been described is that which latterly I have almost exclusively employed in examining opake substances. As it will be convenient to have a name for it, I shall speak of examining a substance in *a linear spectrum*. In examining substances which are only slightly sensitive, it is often highly advantageous to cover the slit with a blue glass.

109. Fig. 5 is intended to represent the usual appearance of the primary linear spectrum, and of the primitive and derived spectra. XY is the primary spectrum, as seen by the naked eye, RV, ST are the primitive and derived spectra into which it is separated by the prism held to the eye. The direction of the shading in RV is intended to represent the composition of this spectrum, which may be regarded as consisting of an infinite number of images of the slit arranged obliquely in the order of their refrangibility. The direction of the shading in ST is that of the lines of the same colour and same refrangibility. Of course the figure does not represent the amount of vertical displacement of the primary spectrum when viewed through the prism held to the eye.

110. There is another mode of observation which I have occasionally found convenient when the object was to determine whether a substance exhibited so much as a low degree of sensibility. In this method the sun's light was reflected horizontally through a large lens, and then transmitted through a small lens placed in the condensed beam. The small lens was covered by a small vessel with parallel sides of glass, containing a blue ammoniacal solution of copper, or else by a deep blue glass combined with a weak solution of nitrate or sulphate of copper. The object of the latter solution was to absorb the extreme red which is transmitted by a blue glass. The light coming through the lens was then analysed by a prism, being received directly into the eye, or else allowed to fall on a white object which had been previously ascertained not to change the refrangibility of the light incident upon it. I found clean white earthenware to serve very well for such an object, but each observer ought to test for himself the substance he employs. When a test object, such as white earthenware, is used, it is placed at the focus of the lens, and the spot of blue light formed upon it is analysed by a prism to see if the absorption is sufficient. When the visible rays are considered to have been sufficiently absorbed, the object to be observed is placed at the focus of the lens, and the spot of light formed upon it is viewed through a prism. The spectrum then seen is compared with that given by the test object. This method of observation is rather easier than that of a linear spectrum, and is at least as delicate if the object be merely to determine whether a substance is sensitive or not, but on the whole it is not near so useful. It may sometimes be used with advantage in the case of translucent bodies.

111. An extremely pale solution of nitrate or sulphate of copper is sufficient to absorb the extreme red transmitted by a deep blue glass. This is not the case with the ammoniacal solution, which does not absorb the extreme red till it is of a pretty deep blue. Its absorbing power is greatest, not at the extreme red, but about the orange, as may be seen by using candle-light, which is richer in red rays than daylight.

112. Another method of observation which is sometimes useful, consists in employing a large lens and absorbing medium, as described in Art. 110, but leaving out the additional small lens. The substance to be examined is placed in the condensed beam, and viewed through an absorbing medium which is approximately complementary to the former. This method is chiefly useful in examining a confused mass of various substances. The most minute fragments of sensitive substances show themselves in this manner.

Results obtained with a Linear Spectrum.

113. When this method is applied to the examination of common objects, it is found that the property of producing a change of refrangibility in the incident light is extremely common. Thus, wood of various kinds, cork, horn, bone, ivory, white shells, leather, quills, white feathers, white bristles, the skin of the hand, the nails, are all more or less sensitive. To make a list of sensitive substances would be endless work; for it is very rare to meet with a white or light-coloured organic substance which is not more or less sensitive. I am not now speaking of organic substances obtained in a state of chemical isolation, of which some are sensitive and others insensible. That substances of a dark colour should frequently prove insensible is only what might have been expected, because the dispersed light is not reflected from the surface, but emanates from all points of a stratum of finite thickness; and in order that dispersed light should be forthcoming, it is necessary that the active light entering, and the dispersed light of a different refrangibility returning, should both escape absorption on the part of the colouring matter. Such substances usually consist of a mixture of various chemical ingredients, of which one or more may very likely be sensitive, in which case the substance may be compared to a solution of sulphate of quinine mixed with ink. Frequently however the colouring matter is itself sensitive.

114. Among sensitive substances I have mentioned the skin of the hand, which stands rather low in the scale. I have found the back of the hand a convenient test object. When the sunlight is not strong enough to show with ease the derived spectrum in the case of the hand, there is little use in attempting to observe.

115. It is needless to say that papers washed with tincture of turmeric, or with a solution of sulphate of quinine, display their sensibility in a remarkable manner when examined in a linear spectrum. The sensibility of turmeric paper is rather impaired by exposing the paper to the light, but on the other hand is materially increased by washing it with a solution of tartaric acid.

116. Paper washed with an ethereal solution from dried archil exhibited very well the sensibility of that substance. The derived spectrum consisted chiefly of two distinct portions, one containing orange and a little red, the other consisting chiefly of green, just as in the beam of dispersed light, produced by white light taken as a whole, which the solution itself exhibited. Indeed, I have found that the prismatic composition of dispersed light could be determined even more conveniently by means of a linear spectrum than by means of the beam dispersed by a solution.

117. The inside of the capsules of the *Datura stramonium* is nearly white, and apparently uniform. But when the capsules are examined in a linear spectrum, certain patches shine out like bright clouds in the invisible rays. The whole of the inside is sensitive, as such substances almost always are, but these patches, which are probably spots against which the seeds have pressed, are remarkably so. The capsules were examined after they had begun to burst.

118. By means of a linear spectrum the sensibility of chlorophyll may be detected in a green leaf. It is exhibited by the appearance in the derived spectrum of a narrow pure red band of remarkably low refrangibility. The refrangibility is so low that I have always found this band separated from the derived spectrum due to other sensitive substances with which chlorophyll or one of its modifications might have been mixed.

119. The petals of flowers, so far as I have examined, are as a class rather remarkable for their insensibility, some appearing quite insensible, and others only slightly sensitive. The bright yellow chaffy involucre of a species of everlasting, proved, however, highly sensitive, and its sensibility was also displayed in an alcoholic solution. This medium was sensitive enough to exhibit a pretty copious dispersive reflexion of a pale greenish yellow light. Its sensibility was more confined than usual to the rays of very high refrangibility.

120. Among petals, the most remarkable which I have observed are those of the purple groundsel (*Senecio elegans*). These petals disperse a red light, more copious than is usual among petals. If a petal be placed behind a slit, and the transmitted light be analysed, it is found to exhibit three remarkable bands of absorption, much resembling those of blue glass, but closer together, and beginning later in the spectrum, the first appearing about the place of the orange. These bands are still better seen in a solution of the colouring matter in weak alcohol. On examining this medium by the third method, with a lens of shorter focus than usual, and looking down from above, the places of the absorption bands were indicated by tooth-shaped interruptions in the beam of light reflected from motes. The points of these teeth were occupied by red dispersed light, which did not appear in the intervening beams of light reflected from motes, from whence it appears that there is the same sort of connexion between the absorption and dispersion of this medium as was noticed in Art. 59, in the case of solutions of chlorophyll and its modifications.

121. A collection of sea-weeds appeared all more or less sensitive, most of them highly so. All, or almost all, except the white ones, exhibited in the derived spectrum the peculiar red band indicative of chlorophyll and its modifications. The transmitted light also exhibited more or less the absorption bands due to this substance, which was likewise, in the specimens tried, extracted by alcohol. But the most remarkable example of sensibility found in sea-weeds occurs in the case of the red colouring matter contained in orangy red, red, pink, and purple sea-weeds. To judge by its optical properties, this colouring matter appears to be the same in all cases, but to be mixed in different proportions with chlorophyll, or some modification of it, and probably other colouring matters, thus giving rise to the various tints seen in such sea-weeds. The derived spectrum exhibited by sea-weeds of this kind consists mainly of a band of unusual brightness, containing some red, followed by orange and yellow. This band fades away gradually at its less refrangible limit, where it is separated by a dark interval from the narrow well-defined red band of still lower refrangibility due to chlorophyll. At its more refrangible limit, however, it breaks off with unusual abruptness.

122. When the light transmitted through such a sea-weed is subjected to prismatic analysis, in addition to one at least of the absorption bands due to chlorophyll, there is seen a band obliterating the yellow, another dividing the green from the blue, and a third, far less conspicuous, dividing the green into two. The whole of the green is absorbed more rapidly than the blue beyond, and not merely than the red, which last is the final tint.

123. The red colouring matter is easily extracted by cold water from certain kinds of red sea-weed, if fresh gathered; but when once the plant has been dried, the colouring matter cannot be extracted in any way that I know of. It is apparently insoluble in alcohol and ether, and is decomposed by boiling. Cold water extracts only a trace of it after a long time.

124. A piece of recently gathered red sea-weed, on being mashed with cold water, readily gave out its red colouring matter. When the residue was treated with alcohol, the fluid was almost immediately coloured green by chlorophyll, whereas this substance is only very slowly and sparingly extracted by alcohol from dried sea-weeds. A dried sea-weed may apparently be assimilated to an intimate mixture of gum and resin, which it would be very difficult to dissolve, whether it were attacked by water or alcohol.

125. The solution of the red colouring matter was highly sensitive, exhibiting a copious dispersive reflexion of a yellowish orange light. The transmitted light was pink or red, according to the thickness through which the light passed. When this light was analysed, the same three absorption bands which have been already mentioned were perceived. The analysis of the light transmitted by the fronds of various red sea-weeds had rendered it extremely probable that the faint division in the

green did belong to the red colouring matter; but till I had obtained this matter in solution I did not feel certain that it might not have been due to chlorophyll, the spectrum of which exhibits a division in the green.

126. When this fluid was examined in Sir DAVID BREWSTER's manner, and the dispersed beam was analysed, the spectrum was found to consist of a broad band like that which has been already described as seen in the derived spectrum given by a frond of red sea-weed. When the solution, which happened to be very weak, was examined by the third method, the dispersion was found to be produced chiefly by a portion of the incident spectrum, having a breadth about equal to that of the interval between the two principal bands of absorption. To each of these bands corresponded a maximum of activity. The tint of the dispersed light was nearly uniform; but by the fourth method of observation some faint dispersed red could be made out, which appeared before the main part of the dispersion had come on. This medium affords a very good example of an intimate connexion between absorption and internal dispersion.

127. The colouring matters of birds' feathers appeared to be insensible, white feathers being most sensitive, pale ones next, and dark ones not at all: however, I have not examined a large collection.

128. Of coloured fruits, such as currants, &c., the colouring matter appeared, in the very few cases which I have examined, to be quite insensible.

129. A set of water colours were by no means remarkable for sensibility, but rather the contrary. The inorganic colours appeared quite insensible, except white lead, the sensibility of which was perhaps due to size, and offered nothing striking, either as to its character or as to its amount. Some lakes and other organic colours proved moderately sensitive. But I found one water colour, called Indian yellow, which stands pretty high among sensitive substances. In its mode of dispersion it much resembles turmeric, but it does not come up to that substance in the amount of sensibility. It is said to be composed of urate of lime, but I do not know how far it may be regarded as chemically pure.

130. Many of the substances used in dyeing, and dyed articles in common use, furnish very remarkable examples of sensibility. Archil, litmus and turmeric have been already mentioned; and I have been recently informed by a friend that the *Mercurialis perennis*, in which a striking instance of sensibility was observed, was formerly employed in dyeing. A piece of scarlet cloth, examined in a linear spectrum, gave a copious derived spectrum which was very narrow, consisting chiefly of the more refrangible red. With a vertical slit the bands H and fixed lines beyond were seen on a red ground. Paper washed with a solution of cochineal and afterwards with a solution of alum, when examined in a linear spectrum, displayed a pretty high degree of sensibility, the derived spectrum consisting in this case of a red band. If tartaric acid be used instead of alum, the dispersion is a good deal more copious.

Common red tape is another example in which the derived spectrum is very copious,

consisting mainly of a red band. Some red wool, dyed I suppose with madder, proved extremely sensitive. The derived spectrum in this case was pretty broad, but red was the predominant colour. Green wool, dyed I do not know with what, was also very sensitive, giving a pretty broad derived spectrum, in which green was the predominant colour. These examples may suffice, but the reader must not suppose that they form the only instances in which dispersion was observed among dyed substances. On the contrary, it is extremely common in this class.

131. Brazil wood, safflower, red sandal wood, fustic and madder, all gave rise to solutions having a pretty high degree of sensibility. The solutions here referred to were such as were obtained directly by water, &c., in which the colours which these substances are capable of producing were not brought out. The beautiful red colouring matters of logwood and camwood appear to be insensible; for a fresh-made solution of logwood in water exhibited no perceptible sensibility, and the slight sensibility exhibited by a similar solution of camwood seemed to have no relation to the red colouring matter.

132. Paper washed with a solution of madder in alcohol was sensitive in a pretty high degree, but the sensibility was greatly increased by afterwards washing with a solution of alum. Accordingly I found that a decoction of madder in a solution of alum exhibited a very high degree of sensibility, displaying a copious dispersive reflexion of a yellow light. In this medium the dispersion commenced about the fixed line D, and continued from thence onwards far beyond the extreme violet, so that the group of fixed lines *n* was seen with great ease.

133. Safflower red, examined in the shape in which it is sold on what is called a *pink saucer*, proved highly sensitive, giving a bright and narrow derived spectrum, which consisted chiefly of the more refrangible red. This substance possesses some other remarkable optical properties, which however do not belong to the immediate subject of this paper.

134. Metals proved totally insensible. I have examined gold, platinum, silver, mercury, copper, iron, lead, zinc and tin. Brass is like simple metals in this respect; but if the surface be lackered the lacker displays its own sensibility.

135. The non-metallic elements, carbon, sulphur, iodine and bromine, are insensible.

136. Among common stones I have found dark flint, limestone, chalk and some others which were sensitive, though only in a low degree compared with organic substances. To guard against any impurity of the surface, the stones were broken across, and the fresh surface examined. In the cases mentioned, the sensibility observed is not to be attributed to the chief ingredient of the stone, for quartz, chalcedony, Iceland spar and Carrara marble were insensible.

Compounds of Uranium.

137. Towards the end of last autumn, when the latenees of the season afforded but few opportunities for observation, I learned from different sources that the kind of yellow glass which has been already mentioned as possessing in so high a degree the property of internal dispersion was coloured with oxide of uranium. This rendered it interesting to examine other compounds of uranium; and I accordingly procured some crystallized nitrate of the peroxide, which, with a few other compounds formed from it, and some of the natural minerals which contain uranium, were examined by methods which have been already explained.

138. The crystals of the nitrate were not sufficiently large and perfect to admit of observation by the methods applicable to fluids and clear solids, but they could be readily observed by means of a linear spectrum. They proved to be sensitive in a very high degree, dispersing a green light which had the same very remarkable composition that has been already described in the case of the yellow glass. On placing a crystal in the continuation of the same linear spectrum with the glass, and viewing the whole through a prism, the five bright bands of which the derived spectrum given by each of the two media usually consisted, appeared to correspond to one another as regards their position in the spectrum. With great concentration of light I have seen an additional band of greater refrangibility in the spectrum of the crystals.

139. Some crystals of nitrate of uranium were gently heated so as to expel a good part at least of the water of crystallization. The residue after some time became opake and nearly white. In this state it was still more sensitive than the crystals. The dispersed light was not exactly of the same tint, but more nearly white; and the derived spectrum was found on being analysed to contain, in addition to the bright bands usually seen in the derived spectrum of the crystals, another blue band still more refrangible. The fused mass gradually attracted moisture from the air, its colour changed to that of the crystals, and the most refrangible of the bright bands disappeared from the derived spectrum. Although when the incident light was very much concentrated I have seen this band even in the crystals, it was faint compared with the preceding bands, whereas in the case of the whitish mass its intensity was not very different from that of the others. It appears therefore that the quality as well as the quantity of the dispersed light was altered by depriving the crystals of a part of their water.

140. A solution of nitrate of uranium in water is decidedly sensitive, though not sufficiently so to exhibit much dispersive reflexion. When the dispersed beam is analysed it is resolved into bright bands. When the solution is examined in a pure spectrum, the mode of dispersion is found to agree with that of canary glass. The dispersion commences abruptly at the same part of the spectrum as in the case of the glass, and after a rather narrow band in which light is copiously dispersed, there follows a remarkable minimum of sensibility, just as in the glass (see Art. 76.), where the dispersed light is almost imperceptible. After this the dispersion is resumed,

and offers nothing remarkable. The minimum of sensibility occurs at the very same place in the spectrum, whether the sensitive medium be a solution of nitrate of uranium or glass coloured yellow by uranium.

141. *Yellow Uranite.*—This mineral, when examined in a linear spectrum, proved to be sensitive in an extremely high degree. The derived spectrum consisted, as in the case of the glass, of bright bands arranged at regular intervals, but in this case six were seen, a band being visible in the faint red at the extremity of the spectrum which could not be made out in the case of the glass.

142. *Green Uranite, or Chalcolite.*—According to M. Peligot the formula of the yellow uranite of Autun is PhO^5, CaO, $2(U^2O^2O)$, 8HO, and the green uranite differs from the yellow only in having the lime replaced by oxide of copper*. Yet a specimen of green uranite on being examined in a linear spectrum proved totally insensible. The primitive spectrum showed however a very remarkable system of dark bands depending on the absorption of light by the mineral. In examining these bands, the previous prismatic decomposition of the light, so far from being necessary, is decidedly inconvenient. It is better to dispense with the prisms altogether, using only the lens, and placing the mineral so that the image of the slit is formed upon it. The bright line thus formed is viewed from a convenient distance through a prism, the eye being held out of the direction of regular reflexion. The position of any bands which may appear in the spectrum can then be determined by means of the fixed lines, which are seen at the same time; or, if it be desired to see the latter more distinctly, it will be sufficient to attach a fragment of paper to the mineral or other substance, placing it so that the image of the slit is formed partly on the paper and partly on the substance to be examined. I have frequently found this mode of observation convenient in examining the absorption of light by opake substances. The manner in which the absorption of the medium comes into play in this case will be considered in greater detail further on (see Art. 176.).

143. When green uranite was examined in this manner, it showed a very remarkable system of dark bands of absorption. These bands were seven in number, or at any rate six, and were arranged with all the regularity of bands of interference. The first was situated at about $b\frac{1}{2}F$, the second at F; the middle of the sixth fell a very little short of G; the third, fourth and fifth were arranged at regular intervals between the second and sixth; the seventh was situated about as far beyond the sixth as the sixth beyond the fifth. The spectrum was so faint in the region of the seventh band as to leave some slight doubts respecting its existence. There would not have been light enough to see bands further on.

144. Uranite is highly lamellar in its structure, from whence it is otherwise called uran-mica. The reader may perhaps suppose that the dark bands described in the last paragraph were bands of interference, which I had mistaken for bands of absorption, and that they were really of the nature of Newton's rings, or more exactly of

* Annales de Chimie, tom. v. (1842) p. 46.

the bands seen in an experiment due to the Baron von WREDE. There may, it will perhaps be said, have been a fissure parallel to the first surface, so as to separate a thin plate; and the interference of the two streams of light reflected respectively on the upper and under surface of this plate may have produced the bands observed. But various phenomena attending these bands are irreconcilable with such a supposition. Towards the edges of the crystal, where flaws did in fact exist, bands of the same nature as Von WREDE's were actually observed. But these had an appearance totally different from that of the others. The dark bands of the interference system were more intensely black and better defined than those of the other system, and were very variable, depending as they did upon the thickness of the plate by which they were formed, whereas the bands belonging to the first system were always the same. Besides, were these bands due to interference, there is no reason why they should be confined to one region of the spectrum, and that by no means the brightest. However, to take away all possible doubts respecting the nature of the bands, I detached a small scale from the crystal, and having placed it behind a slit in a beam of sunlight condensed by a lens, I analysed the transmitted light by a prism. Were the bands really due to absorption, they ought to be more distinct in the transmitted light, whereas, were they of the nature of Von WREDE's bands, they ought to be faint, and almost imperceptible. The spectrum of the transmitted light contained however four dark bands, which were well defined and intensely black. The whole of the spectrum beyond the place of the next band was absorbed, which is the reason why four bands only were visible.

145. The absorption bands of green uranite, though they showed great regularity with respect to their positions, did not appear very regular with regard to their intensities. The second, fifth and sixth seemed to me to be more conspicuous than the first, third and fourth. I cannot say for certain whether this ought to be attributed to fluctuations in the absorbing power of the medium, or fluctuations in the original intensity of the solar spectrum, but I am strongly inclined to prefer the former view.

146. The intervals between the absorption bands of green uranite were nearly equal to the intervals between the bright bands of which the derived spectrum consisted in the case of yellow uranite. After having seen both systems, I could not fail to be impressed with the conviction of a most intimate connexion between the causes of the two phenomena, unconnected as at first sight they might appear. The more I examined the compounds of uranium, the more this conviction was strengthened in my mind.

147. Yellow uranite exhibits a system of absorption bands similar to those of green uranite. Nitrate of uranium also shows a similar system. In a solution I have observed seven of these bands arranged at regular intervals. The first absorption band coincided with F, the fifth with G nearly. The absorption bands may also be seen by analysing the light transmitted through the crystals. The following arrangement exhibited at one view the absorption bands and those due to the light which had changed its refrangibility.

148. The sun's light was reflected horizontally by a mirror, and condensed by passing through a large lens. It was then transmitted through a vessel with parallel sides containing a moderately strong ammoniacal solution of a salt of copper. The strength of the solution, and the length of the path of the light within it, were such as to allow of the transmission of a little green besides the blue and violet. A crystal of nitrate of uranium was then attached to a narrow slit, and placed in the blue beam which had been transmitted through the solution, the crystal being turned towards the incident light. The light coming from the crystal through the slit was then viewed from behind, and analysed by a prism. A most remarkable spectrum was thus exhibited, consisting from end to end of nothing but bands arranged at regular intervals. The interval between consecutive bands appeared to increase gradually from the red to the violet, just as is the case with bands of interference. Although this interval appeared to alter continuously from one end of the spectrum to the other, the entire system of bands was made up of two distinct systems, different in appearance, and very different in nature. The less refrangible part of the spectrum, where only for the crystal there would have been nothing but darkness, was filled with narrow bright bands, due to the light which had changed its refrangibility. These bands were much narrower than the dark intervals between them, but they were not mere lines containing light of definite refrangibility. The more refrangible part of the spectrum was occupied by the system of bands of absorption. The interval between the most refrangible bright band and the least refrangible dark band of absorption appeared to be a very little greater than one band-interval, so that had there been one band more of either kind the least refrangible absorption band would have been situated immediately above the most refrangible bright band. With strong light I think I have seen an additional band of this nature.

149. *Pitchblende.*—This mineral proved to be quite insensible, and exhibited nothing remarkable.

150. *Hydrate of Peroxide of Uranium.*—Some crystallized nitrate of uranium was exposed to a heat a good deal short of redness, whereby most of the acid was expelled. The residue was of a deep brick-red colour, and consisted no doubt chiefly of anhydrous peroxide. It was quite insensible. In order to remove any undecomposed nitrate, it was boiled with water, whereby the undecomposed nitrate was dissolved, and the peroxide converted into a hydrate. This hydrate, after having been washed and dried at the temperature of the air, was of an extremely beautiful yellow colour, and was I suppose the hydrate U^2O^3+2HO described in chemical treatises. It was tolerably sensitive, in fact for an inorganic substance extremely so, though the sensibility was much less than that of nitrate of uranium, yellow uranite, or canary glass. The derived spectrum consisted as before of separate bright bands. A small portion of the powder was attached by water to blotting-paper, and dried before a fire. The powder thus obtained on paper was duller than before, and inclined a little more to orange, though the colour was not much deeper than that of the former hydrate.

From its colour and the circumstances of its formation, it was probably the other hydrate U^2O^3+HO. It proved on examination to be totally insensible.

151. *Acetate of Peroxide of Uranium*, prepared by dissolving the yellow hydrate of the peroxide in acetic acid, and evaporating to crystallize.—This salt is extremely sensitive, about as much so as the nitrate. The derived spectrum consisted of six bright bands arranged at regular intervals. It seemed to me that the last five of these were respectively a little more refrangible than the five bands given by the nitrate, and then a sixth band was visible in the faint red in the case of the acetate which was not ordinarily seen in the nitrate. However, this observation has need to be repeated under more favourable circumstances.

152. Nitrate and acetate of peroxide of uranium, yellow uranite, and canary glass, are all so highly sensitive as to allow the primary spectrum to be examined with a prism at some distance. In the first three media the bright bands are narrow, much narrower than the dark intervals between; in the glass they appear much broader than in the other media.

153. *Oxalate of Peroxide of Uranium*, prepared in the manner mentioned by M. Peligot, namely, by adding a saturated solution of oxalic acid to a solution of nitrate of uranium, washing and drying the precipitate.—This salt was sensitive, but only in a low degree. However, the derived spectrum bore prismatic examination sufficiently to show three or four bright bands. The absorption of the medium was examined by spreading some of the powder on glass along with water and allowing it to dry. The layer was then examined by different methods. The salt exhibits three very intense absorption bands in the highly refrangible part of the spectrum. The positions of these bands, by measurement, were F 0·31 G, F 0·58 G, F 0·85 G.

154. *Phosphate of Peroxide of Uranium*, prepared by precipitation from a solution of nitrate of uranium by adding a solution of common phosphate of soda.—This salt was sensitive, though not in a high degree. It was a good deal more sensitive than the oxalate, but I think not so much so as the hydrate of the peroxide. The derived spectrum consisted of bright bands as usual *.

155. *Uranate of Potassa*, prepared by dropping a solution of nitrate of uranium into a solution of caustic potash, stopping long before the alkali was neutralized.—This salt was found to be insensible, both in its original state as a gelatinous hydrate, and in various stages of drying.

156. *Uranate of Lime*, prepared in a similar manner with lime-water.—This salt, which after drying is of a fine orange colour, was like the preceding found to be insensible. It seemed interesting to examine these two salts, because the former contains two elements (not counting oxygen) in common with canary glass, and the latter two elements in common with yellow uranite. Yet the salts are insensible while the two other media are so remarkably sensitive.

157. *Solutions by means of alkaline carbonates*.—It is known to chemists that alka-

* See note C.

line carbonates, added in solution to a solution of nitrate of uranium, give yellow precipitates which are redissolved in an excess of the precipitant. The solutions thus obtained with the carbonates of potassa and soda, which were of a greenish yellow colour, were found to be totally insensible. They exhibited however four of those singular absorption bands so characteristic of salts of peroxide of uranium. Of these the third fell a little short of G, its more refrangible edge nearly coinciding with that fixed line; the first and second were situated between F and G, the distance of the first beyond F being somewhat greater than the interval between two consecutive bands. The fourth, which was situated beyond G, was fainter than the others. The second and third were the most conspicuous of the set.

158. The absorption bands due to peroxide of uranium afford an easy mode of detecting that substance in solution. For this purpose the solutions mentioned in the preceding paragraph are much preferable to the nitrate, for they produce much stronger bands when only a small quantity of uranium is present. The absorption bands of nitrate of uranium are visible, as might have been expected, in presence of a large quantity of nitrate of copper*.

Optical Tests of Uranium in Blowpipe Experiments.

159. When a bead of microcosmic salt is fused with oxide of uranium, and brought to its highest state of oxidation, it is yellow by transmitted light. Such a bead is sensitive in a very high degree, quite as much so as canary glass. When the light falls sideways on it, and it is held against black cloth or a dark object, it exhibits plainly the green colour due to internal dispersion. When properly examined by means of sunlight its sensibility is evident at once, and when the dispersed light is viewed through a prism it is resolved into bright bands. One of the most convenient modes of examining such minute objects consists in reflecting the sun's light horizontally through a large lens, intercepting by means of absorbing media all the rays except those of very high refrangibility, placing the object to be examined in the condensed beam, and viewing it through a prism. So delicate is this test when applied to uranium, that on one occasion, when engaged in examining a bead coloured green by chromium, which had been fused in the exterior flame, I observed the appearance given by uranium. This turned out to be actually due to uranium, of which a mere trace was accidentally present without my knowledge.

160. The green communicated to microcosmic salt by uranium after exposure to the reducing flame has a very peculiar composition, by means of which the presence of uranium may be instantly detected. For this purpose it is sufficient to view through a prism the inverted image of the flame of a candle formed by the bead, the latter being so held as to be seen projected on a dark object. The observation is perfectly simple, and occupies only a few seconds. The spectrum exhibits an isolated band at the red extremity, followed by a very intense dark band of absorption. A

* See note D.

similar dark band, but not quite so intense, occurs in the green: beyond the green there is usually but little light seen. As the absorption progresses the first dark band invades all the space from the red to the green, and the spectrum consists of an isolated red band and a green band divided into two. In its mode of absorption, the medium has a strong general resemblance to chlorophyll. The green due to copper or to chromium shows nothing remarkable when viewed through a prism, and could not possibly be confounded with the green due to protoxide of uranium. The absorption bands due to this oxide are not completely brought out till the bead is cold.

161. Uranium produces the same effects with borax as with microcosmic salt, but they are less distinct, or at least less easily produced.

162. When the uranium contained in a bead of microcosmic salt is thoroughly oxidized, and the bead is gently heated, so as just to be self-luminous, the light which it gives out is not red, like that of most substances at a low heat, but green, or rather greenish white.

163. Solutions of protoxide of uranium have a very remarkable effect on the spectrum, resembling more or less that of a bead of microcosmic salt coloured green by uranium. Of course the absorption can be observed much better by means of a solution than by a mere bead. I have observed several bands of absorption in such solutions, but the cases which I have hitherto examined are too few to justify me in entering into detail. Besides, the absorption bands due to protoxide of uranium do not belong properly to my subject, the compounds of this oxide, so far as I have examined, being insensible.

Appearance of highly Sensitive Media in a Beam from which the Visible Rays are nearly excluded.

164. When a large beam of sunlight is reflected horizontally into a darkened room, and transmitted through an absorbing medium, placed in the window, of such a nature as to let pass only the feebly illuminating rays of high refrangibility and the invisible rays beyond, various sensitive media have a very strange and unnatural appearance when placed in the beam, on account of the peculiar softness of the dispersed light with which the media appear as it were self-luminous, and the almost entire absence of strong light reflected from convexities. Among substances eminently proper for this experiment, may be mentioned a solution of the bark of the horse-chestnut, or of sulphate of quinine, or of stramonium seeds, a decoction of madder in a solution of alum, and above all, ornamental articles of canary glass. The appearance of a specimen of yellow uranite was curiously altered by this mode of examination. By daylight the mineral appeared much of the same colour as the stone in which it was imbedded, but when placed in a beam such as that above mentioned the uranite was strongly luminous, while the stone remained dark.

Natural Crystals.

165. Of natural crystals I have hitherto examined only a small number. For a long time I was occupied almost exclusively with vegetable products, the mineral kingdom not appearing promising. However, I have found internal dispersion in certain specimens of apatite, arragonite, chrysoberyl, cyanite, and topaz. In all these cases the dispersion appeared due, as in the case of fluor-spar, to some substance accidentally present in small quantity; so that yellow uranite is at present the only natural crystal to the essential constituents of which the property of internal dispersion has been found to belong.

166. Among the minerals just mentioned apatite was the most sensitive, though it fell very far short of yellow uranite. That the sensibility was not due to phosphate of lime, was plain from the circumstances that a colourless specimen was insensible, and that the amount of sensibility was found to be different in different parts of the same sensitive specimen. With the exception of the colourless crystal already mentioned, all the specimens of apatite examined were of a greenish colour, and all were sensitive. The dispersed light was something of an orange colour, but was not homogeneous orange. In one specimen it consisted of three distinct bright bands at regular intervals. The mode in which the sensibility of this crystal was connected with the refrangibility of the incident rays was very peculiar. In arragonite dispersion was found in the transparent specimens examined; the translucent specimens were found to be insensible. The dispersed light was of a brownish white colour. In the same crystal some parts were insensible and others more or less sensitive. The portions of equal sensibility were arranged in plane strata, just as in the case of fluor-spar, as has been noticed by Sir David Brewster. In a specimen which had been cut for showing conical refraction, the strata were in some places perpendicular to the plane of the optic axes, and in other parts parallel to the line bisecting the axes, and inclined to their plane at such an angle that the two directions of the strata must have been parallel to two of the commonest lateral faces. Another specimen showed strata parallel to an oblique terminal face. The strata are plainly due, as Sir David Brewster has remarked with reference to fluor-spar, to some substance taken up during crystallization. Accordingly, they preserve a sort of history of the growth of the crystal. In a twin crystal of fluor-spar, the direction of the strata in that part of the mass which was common to the geometrical forms of both crystals, showed to which crystal it really belonged. In fluor-spar the strata are parallel to the faces of the cube, at least in the specimens which I have examined, and the same has been observed by Sir David Brewster.

In chrysoberyl, cyanite and topaz, the dispersed light was red or reddish, and was too variable to allow of its being attributed to the essential constituents of the crystals. In these cases the sensibility was but slight; indeed in cyanite there was only a trace of dispersion when the crystal was examined under great concentration of light.

Coloured Glasses.

167. Besides canary glass, I have examined the common coloured glasses, including that coloured by gold, but with one exception have not met with any example in which the sensibility observed appeared to have any connexion with the colouring matter. The paler glasses exhibited a little internal dispersion, because the colour was not sufficiently intense to mask the dispersion which a common colourless glass would exhibit.

168. The exception occurred in the case of the pale brown glass, which has been already mentioned in connexion with my first experiment. This glass dispersed a red light under the influence of the highly refrangible rays. The colour of the light was not pure prismatic red, but red was predominant. A similar dispersion, due apparently to the same cause, was observed in the case of one of the common reddish brown German wine bottles. The sensibility of these glasses appears to be due to an alkaline sulphuret. A bead purposely coloured in this manner was in fact found to disperse a red light like the glasses. Moreover, in the confused masses obtained by fusing sulphate of soda and sulphate of potash on charcoal before the blowpipe, certain portions were found which dispersed a red light, and that pretty copiously for an inorganic substance. A similar dispersion was observed among the products obtained by fusing together sulphur and carbonate of potash, while other parts of the confused mass exhibited dispersion of a different kind. It seems plain that among the combinations of sulphur with the alkalies sensitive compounds exist, but what they are I have not examined.

Cautions with respect to the discrimination between true and false internal dispersion.

169. In the early part of this paper certain tests were given for distinguishing between true and false internal dispersion in a fluid. But it requires some experience in observations of this kind to be able readily to decide, and a too rigid adherence to one of the tests to the exclusion of the others might lead to error.

The first test relates to the continuous appearance of a truly dispersed beam. But sometimes solid particles exist in mechanical suspension, which are so fine and so numerous, that this test alone might lead the observer to mistake a falsely for a truly dispersed beam. On the other hand, if a fluid which itself alone exhibits no internal dispersion, true or false, hold solid particles in what is obviously mere mechanical suspension, we must not immediately conclude that the medium, taken as a whole, is incapable of changing the refrangibility of any portion of the light incident upon it. For we have seen that the fluid state is not in the least degree essential to the exhibition of sensibility, and of course a fluid will serve as well as anything else for the mere mechanical support of a sensitive substance.

170. Thus lycopodium is very sensitive, as appears by examining the powder in a linear spectrum. Accordingly, I found that when a little lycopodium was mixed with water, and the whole medium was examined by the fourth method, it displayed

its sensibility, although the beam of light which had changed its refrangibility was plainly discontinuous. When Indian yellow was used instead of lycopodium, the whole medium exhibited its sensibility when it was examined by the fourth method. In this case the suspended particles were so fine that the beam of light which had changed its refrangibility appeared to be continuous, though of course it was not really so. In observing with muddy fluids like these, it is almost necessary to employ absorbing media, since otherwise the effect of the light scattered at the surfaces of the prisms and large lens might lead the observer to conclusions altogether erroneous.

171. The next test relates to the polarization of a falsely dispersed beam. Being engaged on one occasion in examining the effects of acids and alkalies on a weak solution of a sensitive substance, employing sunlight which had been merely reflected through a small lens, I met with a beam which had every appearance of having been only falsely dispersed, but on viewing it from above through a doubly refracting prism I was surprised at first by finding it unpolarized. It soon occurred to me that the beam must have been due, not to solid motes, but to excessively small bubbles of carbonic acid gas, the existence of which was thus revealed, though they were too small to be seen directly. The light being incident on these bubbles at an angle of about 45°, which is very little less than the angle of total reflexion, the reflected light would be almost perfectly unpolarized*.

172. Water which had been merely boiled in a test tube gave a similar result. The unpolarized beam of falsely dispersed light was of course due in this case to the air which had been held in solution. This shows why long-continued boiling should be necessary, in order to free water from air. It is not that the affinity of water for air is so great as to be only gradually overcome, but that the air, immediately expelled from solution when the temperature rises sufficiently, is still retained in a state of mechanical mixture, forming excessively minute bubbles, the terminal velocity of which is insensible. Accordingly it is not till larger bubbles are formed, by the casual meeting of a number of these small bubbles, that the air rises to the surface and escapes.

173. With respect to the test of true dispersion depending on the change of refrangibility, it has been already remarked that in some cases the change is so slight, that if this test alone were applied, the observer might mistake true dispersion for false. However, it is only in rare cases that there is any danger of being deceived in this manner in the application of the test; but on the other hand, in observing a muddy fluid or a translucent solid by the fourth method, the observer, if not on his guard, might easily be deceived by the effect of scattered light, and be led to mistake false dispersion for true. Thus suppose the medium to be water holding in suspension particles of an insensible water colour, and the small lens to be placed a little beyond the commencement of the violet. Two beams of light would enter the lens, namely, a regularly refracted beam of violet, and a scattered beam of white light.

* See note E.

Of these the latter would be insignificant compared with the former, were it not that the illuminating power of the colours belonging to the middle of the spectrum is so very much greater than that of the violet. When the dispersed beam was analysed by a prism, it would be decomposed into a violet beam of definite refrangibility, followed by a dark interval, and then a broad band containing the colours of the brighter part of the spectrum in their natural order. This is what is constantly seen in cases of true dispersion; but the polarization of the beam, and its behaviour under the action of absorbing media, would reveal the counterfeit character of the dispersion.

On the Colours of Natural Bodies.

174. By this expression I mean to include only the colours to which it is usually applied, namely, those of leaves, flowers, paints, dyed articles, &c., which form the great mass of the colours that fall under our observation. I do not refer to colours due to refraction, such as those of the rainbow, or to diffraction, such as those of the coronæ seen about the sun and moon, or to interference, such as those seen in the clear wings of small flies, or to the colours which accompany specular reflexion, which last are usually but slight, though sometimes pretty intense.

In some few instances, as for example in the case of fluor-spar, various salts of peroxide of uranium, acid solutions of disulphate of quinine, &c., colours are observed, sufficiently strong to arrest attention, which have a remarkable and hitherto unsuspected origin. But I am not now speaking of colours arising from a change of refrangibility in the incident light. In the vast majority of cases these colours are far too feeble to form any sensible portion of the whole colour observed. The colours which dyed articles give out under the influence of the highly refrangible rays usually agree more or less nearly with those of which such substances commonly appear, and it is possible that the colour arising from a change of refrangibility may contribute in some slight degree to the brilliancy of the tint observed. If, however, the effect be sensible I am persuaded that it is but slight; and very brilliant colours may be produced without a change of refrangibility, as for example in the case of biniodide of mercury. For the present I shall neglect the light which may have changed its refrangibility.

175. Few, I suppose, now attach much importance to the bold speculations in which NEWTON attributed the colours of natural bodies to the reflexion of light from thin plates. Sir DAVID BREWSTER has shown how extremely different the prismatic composition of the green of the vegetable world is, from what it ought to be, according to NEWTON's theory, and what NEWTON supposed that it was. It is now admitted that the various colours of natural bodies are merely particular instances of one general phenomenon, namely, that of absorption. Absorption is most conveniently studied in a clear fluid or solid, but it does not the less exist in a body of irregular structure, such as a dyed cloth or a coloured powder.

The green colouring matter of leaves affords an excellent example of the identity of the effect produced on light by natural bodies and of ordinary absorption; for the same very peculiar system of absorption bands which are displayed by a clear solution of the colouring matter may be observed directly in the leaf itself. However, it is needless to bring forward arguments to support a theory now I suppose universally admitted; my present object is merely to point out the mode in which the colours which bodies reflect, or more properly scatter externally, depends upon the absorbing power of the colouring matter, so as to justify the conclusions deduced in Art. 142, from observations made in the manner there described.

176. Let white light be incident on a body having an irregular internal structure, such as a coloured powder. A portion will be reflected at the first irregular surface, but the larger portion will partly enter the particles, partly pass between them, and so proceed. In its progress the light is continually reflected in an irregular manner at the surfaces of the particles, and a portion of it is continually absorbed in its passage through them. For simplicity's sake, suppose the light incident in a direction perpendicular to the general surface, and neglect all light which is more than once reflected. Let t be the thickness of a stratum which the light has penetrated, I the intensity of the light at that depth, or rather the intensity of a given kind of light, so that the whole intensity may be represented by $\int \mathrm{I}d\mu$, μ being the refractive index in some standard substance. In passing across the stratum whose thickness is dt, suppose the fraction qdt of the light to be absorbed, and the fraction rdt to be reflected and scattered in all directions, then

$$d\mathrm{I}=-(q+r)\mathrm{I}dt.$$

Integrating this equation, and supposing I_0 to be the initial value of I, when $t=0$, we have

$$\mathrm{I}=\mathrm{I}_0e^{-(q+r)t}. \quad \ldots \ldots \ldots \ldots \quad \text{(a.)}$$

For the sake of simplicity, suppose the body viewed in a direction nearly perpendicular to the general surface; and of the light reflected and scattered in passing across the stratum whose thickness is dt, suppose that the fraction n would enter the eye if none were lost by absorption, &c. Then the intensity of the light coming from that stratum would be $nr\mathrm{I}dt$. But in getting back across the stratum whose thickness is t, the intensity is diminished in the ratio of I_0 to I. Hence if I′ be the intensity of the light actually entering the eye,

$$d\mathrm{I}'=nr\mathrm{I}_0^{-1}\mathrm{I}^2dt=nr\mathrm{I}_0e^{-2(q+r)t}dt.$$

If we suppose the thickness of the body sufficient to develope all the colour which the body is capable of giving, the superior limit of t will be ∞, and we shall have

$$\mathrm{I}'=\frac{nr}{2(q+r)}\mathrm{I}_0. \quad \ldots \ldots \ldots \ldots \quad \text{(b.)}$$

177. The colour which accompanies ordinary reflexion being usually but slight, I shall neglect the chromatic variations of r. It is q which is subject to extensive and apparently capricious variations, depending upon the refrangibility of the light.

Imagine two curves drawn whose abscissæ are proportional to μ, and ordinates proportional to the ratio of I to I_0 for the first, and the ratio of I' to I_0 for the second. These curves will serve to represent to the mind the composition of the light transmitted through a stratum of the body having a thickness t, and of that reflected from the body when seen in mass. It is plain that the maximum and minimum ordinates in the two curves will correspond to the same abscissæ; but unless t be very small, so small as to be insufficient to bring out the colour of the medium seen by transmission, the maxima and minima will be much more developed in the first curve, whose ordinates vary as e^{-qt}, than in the second, whose ordinates vary as $(q+r)^{-1}$. If, then, the absorbing power be subject to fluctuations depending on the refrangibility of the light, the bands of absorption may be observed either in the reflected or in the transmitted light, but they admit of being better brought out in the latter.

178. If the nature of the substance be given, q will be given. If now the body be of a loose nature, as for example blue glass reduced to a fine powder, r will be considerable. Hence, in accordance with the expression (b.), the quantity of light scattered externally will be considerable, but the tint will be but slight. If the powder be now wetted, the reflexions at the surfaces of the particles will be diminished, r will be diminished, and, as appears from (b.), the quantity of light scattered externally will be diminished, but at the same time the tint will be deepened, since the chromatic variations of I' are increased. If the body be compact and nearly homogeneous, r will be small, and therefore very little light will be returned, except what is regularly reflected at the first surface. The tint of the small quantity of light which is reflected otherwise than regularly, will be somewhat purer than before, inasmuch as the chromatic variations of I' tend to become the same as those of q^{-1}.

On the nature of False Dispersion, and on some applications of it.

179. It has been already stated that a beam of falsely dispersed light seen in a fluid has generally more or less of a sparkling appearance, indicating that it owes its origin merely to motes held in mechanical suspension. Sometimes, however, no defect of continuity is apparent. This is especially the case when two fluids are mixed together, of which one contains in solution a very small quantity of a substance which we might expect to be precipitated by the addition of the other, or when a slightly viscous fluid has remained quiet for a long time. If some part at least of a falsely dispersed beam be plainly due to motes, that does not of course prove for certain that there is no part which may have a different origin, and may be essentially connected with true dispersion; nor do the theoretical views which I entertain of the cause of the latter lead me to regard it as at all impossible that a beam polarized in the plane of reflexion, and having the same refrangibility as the incident light, may be a necessary accompaniment of true dispersion. However, observation, I think, points in a contrary direction; for although more or less of

false dispersion is almost always exhibited along with true dispersion, the quantity of the former seems to have no relation to the quantity of the latter, but does seem to have relation to the greater or less degree of clearness which we should be disposed to attribute to the fluid.

180. The phenomenon of false internal dispersion seems to admit of being applied as a chemical test to determine whether or not precipitation takes place. Thus, if a little tincture of turmeric be greatly diluted with alcohol, and then water be added, a yellow fluid is obtained which appears to be perfectly clear, exhibiting no sensible opalescence; but the occurrence of a copious false dispersion when the fluid is examined by sunlight, reveals at once the existence of suspended particles, though they are too minute to be seen individually, or even to give a discontinuous appearance to the falsely dispersed beam. Although such a precipitation could not, I suppose, be used as a means of mechanical separation, it might still be useful as pointing out the possibility of an actual separation under different circumstances as to strength of solution, &c.

181. One of the best instances of false dispersion that I have met with, best, that is, in forming a most excellent imitation of true dispersion, occurred in the case of a specimen of plate-glass which was made, as I was informed, with a quantity of alkali barely sufficient. This glass, which was very slightly yellowish brown, when viewed edgeways by transmitted light, had a bluish appearance when viewed properly, strongly resembling that of a decoction of the bark of the horse-chestnut, diluted with water till the dispersed light is no longer concentrated in the neighbourhood of the surface. But when the glass was examined by sunlight, the polarization of the dispersed beam, and the identity of its refrangibility with that of the incident light, showed that this was merely an instance of false dispersion. Another very good example of false dispersion is afforded by chloride of tin dissolved in a very large quantity of common water.

182. When a horizontal beam of falsely dispersed light is viewed from above, in a vertical direction, and analysed, it is found to consist chiefly of light polarized in the plane of reflexion. It has often struck me, while engaged in these observations, that when the beam had a continuous appearance, the polarization was more nearly perfect than when it was sparkling, so as to force on the mind the conviction that it arose merely from motes. Indeed, in the former case, the polarization has often appeared perfect, or all but perfect. It is possible that this may in some measure have been due to the circumstance, that when a given quantity of light is diminished in a given ratio, the illumination is perceived with more difficulty when the light is uniformly diffused than when it is spread over the same space, but collected into specks. Be this as it may, there was at least no tendency observed towards polarization in a plane perpendicular to the plane of reflexion, when the suspended particles became finer, and therefore the beam more nearly continuous.

183. Now this result appears to me to have no remote bearing on the question of

the direction of the vibrations in polarized light. So long as the suspended particles are large compared with the waves of light, reflexion takes place as it would from a portion of the surface of a large solid immersed in the fluid, and no conclusion can be drawn either way. But if the diameters of the particles be small compared with the length of a wave of light, it seems plain that the vibrations in a reflected ray cannot be perpendicular to the vibrations in the incident ray. Let us suppose for the present, that in the case of the beams actually observed, the suspended particles were small compared with the length of a wave of light. Observation showed that the reflected ray was polarized. Now all the appearances presented by a plane-polarized ray are symmetrical with respect to the plane of polarization. Hence we have two directions to choose between for the direction of the vibrations in the reflected ray, namely, that of the incident ray, and a direction perpendicular to both the incident and the reflected rays. The former would be necessarily perpendicular to the directions of vibration in the incident ray, and therefore we are obliged to choose the latter, and consequently to suppose that the vibrations of plane-polarized light are perpendicular to the plane of polarization, since experiment shows that the plane of polarization of the reflected ray is the plane of reflexion. According to this theory, if we resolve the vibrations in the incident ray horizontally and vertically, the resolved parts will correspond to the two rays, polarized respectively in and perpendicularly to the plane of reflexion, into which the incident ray may be conceived to be divided, and of these the former alone is capable of furnishing a reflected ray, that is of course a ray reflected vertically upwards. And in fact observation shows, that, in order to quench the dispersed beam, it is sufficient, instead of analysing the reflected light, to polarize the incident light in a plane perpendicular to the plane of reflexion.

Now in the case of several of the beams actually observed, it is probable that many of the particles were really small compared with the length of a wave of light. At any rate they can hardly fail to have been small enough to produce a tendency in the polarization towards what it would become in the limit. But no tendency whatsoever was observed towards polarization in a plane perpendicular to the plane of reflexion. On the contrary, there did appear to be a tendency towards a more complete polarization in the plane of reflexion.

M. Babinet has been led by the same reasoning to an opposite conclusion respecting the direction of the vibrations in polarized light, resting on an experiment of M. Arago's, in which it appeared that when light was incident perpendicularly on the surface of white paper, and the reflected or rather scattered light was viewed in a direction almost grazing the surface, it was found to be partially polarized in the plane of the sheet of paper*. But the actions which take place when light is incident on a broad irregular surface, like that of paper, bounding too a body which is so translucent that a great part of the light must enter it and come out again, appear

* Comptes Rendus, tom. xxix. p. 514.

to me to be too complex to allow us to deduce any conclusion from the result respecting the direction of vibration. Besides, the result itself admits of easy explanation, by attributing it to the light which has entered the substance of the paper and come out again, which might be expected to be polarized by refraction.

Effect of Heat on the Sensibility of Glass, &c.

184. The sensibility of glass is temporarily destroyed by heat. The glass may be heated by holding it in the flame of a spirit-lamp, as a heat much short of redness is sufficient. This takes place even with glass coloured by oxide of uranium, which is in general so highly sensitive. The sensibility returns again as the glass cools. A bead of microcosmic salt, containing uranium in its highest state of oxidation, is very sensitive when cold, but insensible when hot. The sensibility gradually comes on as the bead cools. A solution of nitrate of uranium in water on being heated has its sensibility impaired, very much so by the time the temperature reaches the boiling-point. The sensitive compounds, whatever may have been their precise nature, obtained by fusing the sulphates of soda and potassa on charcoal before the blowpipe, were insensible while hot. The few vegetable solutions which I have examined with this object did not seem to have their sensibility affected by being heated.

Effect of Concentration and Dilution.

185. In investigating the change of refrangibility produced by a sensitive substance in solution, it is almost always convenient to have the solution weak. This however is by no means merely a matter of convenience, for the quantity of light which the medium is capable of giving back with a changed refrangibility is often materially diminished by increasing the concentration of the solution. Thus a solution which, when in a concentrated state, exhibits no sensible dispersive reflexion, will often exhibit when much diluted a very copious appearance of that nature. On the other hand, the dilution may of course be carried too far, so as to render imperceptible the peculiar properties of the substance dissolved. Yet it is wonderful what a degree of dilution a highly sensitive solution will bear before its sensibility ceases to be perceptible.

That the sensibility will be diminished, and will at last become imperceptible, if only the dilution be carried far enough, is nothing more than might have been predicted with the utmost confidence. In such a case the light passes completely through the fluid long before it has produced all the effect which it is capable of producing. But that concentration should be an obstacle to the exhibition of the phenomenon is not perhaps what we should have expected, and deserves an attentive consideration.

186. Imagine a given sensitive substance to be held in solution, in a vessel of which the face towards the eye is plane, and the breadth in the direction of vision as great as we please; and suppose the solvent, or at least the fluid used for diluting the solu-

tion, to be itself colourless and insensible. Suppose the fluid to be illuminated by light of given intensity and given refrangibility entering at the face next the eye, and let the eye E from a given position look in the direction of a given point P in the nearer surface of the vessel. In short, let everything be given except the strength of the solution. For the sake of simplicity regard the eye as a point, and make E the vertex of an indefinitely thin conical surface surrounding the line EP. Call this conical surface C, and let *c* be the surface within the fluid generated by right lines coinciding with the refracted rays which would be produced by incident rays coinciding with the generating lines of the surface C. This latter surface we may if we please regard as cylindrical, since we shall only be concerned with so much of the fluid contained within it as lies at a distance from P less than that at which the light entering the eye in consequence of internal dispersion ceases to be sensible; and in the cases to which the present investigation is meant to apply this distance is but small compared with PE. Let the fluid within *c* be divided into elementary portions by planes parallel to the surface of the fluid at P, and at distances from P proportional to the strength of the solution. It is evident that an element of a given rank, reckoned from P, will contain a constant number of sensitive molecules, and the incident light in reaching this element has to pass through a thickness of the medium such that a plate of the same thickness, and having a given area, contains a given number of sensitive or absorbing molecules. The same is true of the dispersed light which proceeds from the element and enters the eye. Now it seems natural to suppose that if the strength of a solution be doubled, trebled, &c., or reduced to one-half, one-third, &c., the quantity of light absorbed will be the same provided the length of the path of the light be reduced to one-half, one-third, &c., or doubled, trebled, &c. This comes to the same thing as supposing that each absorbing molecule stops the same fractional part of the light passing it, whether the solution be more or less dilute. We should similarly be inclined to suppose that each sensitive molecule would give out the same quantity of light, when influenced by light of given intensity, whether it belonged to a stronger or a weaker solution. If we admit these suppositions, it is plain that the quantity of dispersed light which reaches the eye from the element under consideration will be independent of the strength of the solution. This being true for each element in particular will be true for the aggregate effect of them all, and therefore the quantity of light exhibited by dispersive reflexion will be independent of the strength of the solution. It may be readily seen that the result will be the same if we take into account the finite size of the pupil.

187. Now this is by no means true in experiment. On examining in a pure spectrum a highly concentrated solution of sulphate of quinine, a copious dispersion was observed to commence a little below the fixed line G. It remained very strong as far as H, and beyond. In the weak solution first mentioned in this paper, it will be remembered that the dispersion seemed to come on about G$\frac{1}{2}$H. The reason of this, or at least one reason, is evident, and was very prettily shown by the form of the

space to which the dispersed light was confined. On looking down from above, so that this space was seen in projection, it appeared in the case of the weak solution to have approximately the form of the space contained between one branch of a rectangular hyperbola, one asymptote, and a line parallel to the other, the first asymptote being the projection of the anterior surface, and the line parallel to the other being the course of the least refrangible of the active rays which were capable of producing a sensible quantity of dispersed light. The breadth of the illuminated space, which among the most highly refrangible rays was almost insensible, continually increased, until the space ended in a blue beam which went quite across the vessel. But in the case of the strong solution the illuminated space had throughout an almost insensible breadth, except just close to its lower limit, that is, the limit corresponding to the least refrangible of the active rays, where it ended in a sort of tail or plano-concave wedge, which penetrated to a moderate distance into the fluid. Hence one reason, though perhaps not the only reason, why the strong solution showed a copious dispersion from G to G$\frac{1}{2}$H, where the weak solution showed hardly any, is plain enough. But in the region of the invisible rays beyond the violet, the dispersion was plainly more copious with the weak than with the strong solution. It appears then that in such a case the sensitive molecules do not act independently of each other, but the quantity of light emitted by a given number of molecules is less, in proportion to the light (visible or invisible) consumed, than when a solution is more dilute. We should expect *à priori* that when a solution is tolerably dilute further dilution would make no more difference in this respect. This seems to agree very well with experiment. For when a pretty dilute solution and one much more dilute are compared with respect to the quantity of dispersed light given out in a given portion of the incident spectrum, they appear to be alike. I suppose the comparison to be made with respect to such a portion of the incident spectrum, or in the case of solutions of such strength, that the dispersed light is confined to a space extending to no great distance into the fluid in either solution. Under these circumstances the comparison may be made easily enough.

188. In the actual experiment, the elementary portions of light coming from the elementary strata of fluid situated at different distances from the anterior surface enter the eye together. Let us however trace the consequences of the very natural supposition, that in passing across a given stratum of fluid the quantity of light absorbed, as well as the quantity given out by dispersion, is proportional, *cæteris paribus*, to the intensity of the incident light. The incident light is here supposed to be homogeneous, and to belong indifferently to the visible or invisible part of the spectrum. In crossing the elementary stratum having a thickness dt, let the fraction qdt of the incident light be absorbed, and the fraction rdt dispersed in such a direction as to reach the eye; and of the latter portion let the fraction sdt be absorbed in crossing a stratum having a thickness dt, s being different from q on account of the change of refrangibility. Then by a very simple calculation similar to

that of Art. 176, we find for the intensity I′ of the dispersed light which enters the eye

$$I' = \frac{r}{q+s} I_0,$$

I_0 being the intensity of the incident light. Since a sensitive fluid is in general coloured, and the dispersed light is in general heterogeneous, s will in general be different for the different portions into which the dispersed light would be decomposed by a prism. However, if the fluid be colourless, or all but colourless, as is the case with a solution of sulphate of quinine, s will be insensible, so that I′ will be proportional simply to rq^{-1}. Hence from the observed variations in I′, arising from variations in the strength of the solution, we may infer the corresponding variations in rq^{-1}.

If, then, we represent by the ordinate of a curve the ratio of the quantity of light given out to the quantity of light absorbed by a given number of active molecules, the abscissa being the ratio of the quantity of diluting fluid to the quantity of the sensitive substance in solution, it appears that the curve will be concave towards the axis of the abscissæ, and will have an asymptote parallel to that axis.

On the Choice of a Screen.

189. We have seen that white paper, the substance commonly employed as a screen on which to receive the spectrum, gives back with a changed refrangibility a portion of the light incident upon it. This might in some cases lead an observer not aware of the circumstance to erroneous conclusions. Since the colour of dispersed light depends upon its refrangibility, which is different from that of the active light, the colours of a spectrum received on white paper must be somewhat modified. In truth the intensity of the light dispersed is so small compared with the intensity of the light scattered, that the modification is quite insensible except in the extreme violet. But beyond the extreme violet the spectrum seems to be prolonged with a sort of greenish gray tint, which belongs neither to that nor to any other part of the true spectrum. In experiments on absorption, if instead of receiving the light directly into the eye it be found convenient to form a pure spectrum on a screen of white paper, then, if the absorbing medium be placed in the path of the incident light, the scattered light forming any part of the spectrum cannot be cut off or weakened without at the same time cutting off or weakening the dispersed light coming from the same part of the screen. But if the absorbing medium be held in front of the eye, its effect on the spectrum will sometimes be very sensibly different from what it would be were the screen to send back none but scattered light.

It is true that the quantity of light dispersed by white paper is so small that this substance may very well continue to be used as a screen, without any danger of the observer's being deceived, if only he be aware of the fact of dispersion, so as to be on his guard. Still, it is not unreasonable to seek for a substitute for paper, which may be free from the same objection.

190. A porcelain tablet appeared to be unexceptionable in this respect, for it exhi-

bited no perceptible sensibility, even when examined by a linear spectrum. However, the translucency of the substance gave the spectrum a blurred appearance, and the fixed lines were not shown so well as on paper.

Chalk scraped smooth is well adapted, from its fineness, its whiteness and its opacity, for showing the most delicate objects. The finest fixed lines are beautifully seen on it, decidedly better than on paper. Its sensibility too, though not absolutely null, is much less than that of most kinds of white paper. Indeed, it would be an unnecessary refinement to seek for anything better, were it not that a piece of sufficient size might not always be at hand. From what I have seen, I believe that the best kind of screen will be obtained by the use of some white inorganic chemical precipitate, but my experiments in this department have not yet been sufficiently extended to authorize me in recommending any particular process.

191. The object of the observer may however be altogether different, and he may wish to extend the spectrum as far as possible, for the purpose of viewing the fixed lines belonging to the invisible part beyond the extreme violet, or making experiments on the invisible rays. For this purpose it would be próper to employ a clear and highly sensitive solid or fluid. A weak solution of sulphate or phosphate of quinine would do very well, or a weak decoction of the bark of the horse-chestnut (no doubt a solution of pure esculine would be better), or an alcoholic solution of the seeds of the *Datura stramonium*. But perhaps the most convenient thing of all would be a slab of glass coloured yellow by oxide of uranium. This would be always ready, and in point of sensibility the glass does not seem to yield to any of the solutions above mentioned, at least so far as relates to those rays which are capable of passing through glass*.

192. In making experiments on the invisible rays, it is well to get rid, as far as possible, of the glare arising from the bright part of the spectrum, and therefore a clear solid or solution is preferable to an opake screen. If it be desired to show the fixed lines in the visible and invisible parts of the spectrum at the same time, a screen may be employed consisting of paper washed with a moderately strong solution of sulphate of quinine, or an alcoholic solution of stramonium seeds. Turmeric paper is not, I think, quite so good for showing the fixed lines of very high refrangibility, but is at least equally good for the extreme violet and for the rays a good distance further on, especially if it has been washed with a solution of tartaric acid. It is likely that many other acids would do as well. Very excellent screens might probably be prepared by washing paper with a solution of esculine, or even of the bark of the horse-chestnut†, or by covering pasteboard with yellow uranite reduced to fine powder, and made to adhere by a weak solution of pure gum Arabic; but these I have not tried.

* See note F.

† See note G.

Application of internal dispersion to demonstrating the course of rays.

193. Solutions of quinine have already been employed for this purpose, and a weak decoction of the bark of the horse-chestnut appears to be decidedly better. But the effect is immensely improved by using absorbing media to cut off all the rays belonging to the bright part of the visible spectrum. A deep blue glass will answer very well for this purpose if its faces be even, so as not to disturb the regularity of the refraction. The appearance of the general pencil refracted through a rather large lens, with its caustic surface, its geometrical focus, &c., is singularly beautiful when exhibited in this way, on account of the perfect continuity of the light, and the delicacy with which the different degrees of illumination belonging to different parts of the pencil are represented by the different degrees of brightness of the dispersed light. The solution should be contained in a vessel with plane sides of glass, and ought to be very weak, or else only the part of the pencil which lies near the surface by which the light enters will be properly represented.

Application of internal dispersion to the determination of the absorbing power of media with respect to the invisible rays beyond the violet, and the reflecting power of surfaces with respect to those rays.

194. Hitherto no method has been known by which the absorbing power of a medium with respect to these rays could be determined for each degree of refrangibility in particular, except that which consists in taking a photographic impression of a pure spectrum, the light forming the spectrum having been transmitted through the substance to be examined. It is needless to remark how troublesome such a process is when contrasted with the mode of determining the absorption which media exercise on the visible rays. But the phenomenon of internal dispersion furnishes the philosopher, so to speak, with *eyes to see the invisible rays,* so that the absorbing power of the medium with respect to these rays may be instantly observed. For this purpose it is sufficient to form a pure spectrum, using instead of a screen a highly sensitive fluid or solid, such as one of those mentioned in Art. 191, and to hold before it the medium to be examined, or else to place the medium over the whole or a part of the slit.

195. In this way the transparency of glass coloured yellow by oxide of silver with respect to the violet rays and some of those still more refrangible, which has been remarked by Sir John Herschel*, may be at once observed. A set of green glasses were found to be very variable in the mode in which they absorbed the invisible rays, some absorbing the more refrangible of the rays capable of affecting a dilute solution of sulphate of quinine and transmitting the less refrangible, others absorbing the less and transmitting the more refrangible, and others again absorbing them all. These rays were absorbed by solutions of chromate and bichromate of potash so weak as to be almost colourless. A thickness of about a quarter of an inch of sulphuret of

* Philosophical Transactions for 1840, p. 39.

carbon was sufficient to absorb all the rays beyond $Hk1$, so that a hollow prism filled with this fluid would be useless in experiments on these rays. It should be remarked that the sulphuret of carbon employed was not yellow from dissolved sulphur, but apparently as colourless as water.

196. To determine qualitatively the reflecting power of a polished surface with respect to the invisible rays of each particular degree of refrangibility, it would be sufficient to form a pure spectrum as usual, reflect the rays sideways before they come to the focus of the larger lens, place a sensitive medium to receive them, and compare the effect with that produced on the same medium when the rays are allowed to fall directly upon it.

Effect of different Flames.

197. Want of sunlight proved to be such an impediment to the pursuit of these researches that I was induced to try some bright flames, with the view of obtaining some convenient substitute. Candle-light is very ill adapted to these experiments. The flame of a camphene-lamp proved no better, perhaps rather worse, for it abounds so much in rays belonging to the bright part of the spectrum that the glare of the light prevents all observation of faint objects; and the flame does not appear to be rich in invisible rays in anything like the proportion in which it is rich in visible ones. The flame of nitre burning on wood or charcoal produced a very good effect, exhibiting, when the combustion was most vivid, a copious dispersive reflexion in a weak solution of sulphate of quinine contained in a bottle held near it. The tint of the dispersed light appeared to be not quite the same as that given by daylight, but to verge a little towards violet. However, I do not place very strong reliance on the judgment of the eye under such circumstances. A still stronger dispersive reflexion was produced by a flash of gunpowder. The tint in this case appeared to be the same as that seen by daylight.

198. While engaged in some of these experiments on bright flames, I was surprised by discovering the strong effect produced by the flame of a spirit-lamp, the illuminating power of which is so feeble. When this flame was held close to a bottle containing sulphate of quinine, a very distinct dispersive reflexion was exhibited. The same was the case with several other sensitive solutions. However, the full effect of the flame is not thus exhibited, because a considerable portion of the rays which it emits is stopped by glass. It is best observed by pouring the solution into an open vessel, such as a wine glass or tumbler, holding the flame immediately over it, and placing the eye in or very little below the plane of the surface. In this way nothing is interposed between the flame and the fluid, except an inch or two of air, the absorption produced by which, it is presumed, is insensible; and the plane strata, parallel to the surface, into which the illuminated portion of the fluid may be conceived to be divided, are all projected into lines, whereby the intensity of the blue light is materially increased. It is to be observed further, that if the eye be held a

little below the plane of the surface, there enters it, not only the light coming directly from the blue stratum itself, but also that coming from its image formed by total internal reflexion. This mode of observation has already been employed by Sir JOHN HERSCHEL in the case of sunlight. As it is frequently useful in these researches it will be convenient to have a name for it, and I shall accordingly speak of it as the method of observing by *superficial projection.*

199. The opacity of a solution of sulphate of quinine appears to increase regularly and rapidly with the refrangibility of the light. Hence we may form an estimate of the refrangibility of any light by which the solution may be affected, by observing the degree in which the illumination is concentrated in the neighbourhood of the surface. For this purpose it is essential to employ a weak solution, since otherwise streams of invisible light of various degrees of refrangibility produce each their full effect in strata so very narrow, that they cannot be distinguished by the breadth of the stratum. Now to judge by the great concentration of the illumination produced by a spirit-lamp, even in the case of an extremely weak solution, as well as by the considerable degree in which the active rays were intercepted by glass, these rays, taken as a whole, must have been of very high refrangibility, such as to place them among the most refrangible of the fixed lines represented in the map, or perhaps even altogether beyond them. In making observations on the solar spectrum, it was plain that the prisms were by no means transparent with respect to the rays belonging to the group p of fixed lines. Yet these rays, before they produced their effect, had to pass twice through the plate-glass belonging to the mirror (except so far as regards the rays reflected at the first surface), then through three prisms, though to be sure as close as possible to the edges, then through a lens by no means very thin, and lastly, through the side of the vessel containing the fluid. Such a train of glass would be sufficient materially to weaken, if not even wholly to cut off the active rays coming from the flame of a spirit-lamp.

200. The flame of naphtha produces nearly the same effect as that of alcohol. The flame of ether is not so good; but whether this arises solely from its richness in visible rays, which only produce a glare, or likewise from a comparative poverty in highly refrangible invisible rays, it is not easy to say. The flame of hydrogen produces a very strong effect. The invisible rays in which it so much abounds, taken as a whole, appear to be even more refrangible than those which come from the flame of a spirit-lamp. In making some observations with the flame of hydrogen, when the gas was nearly exhausted, so that the flame was reduced to a roundish knob no larger than a sweet pea, and giving hardly any light, it was found still to produce a very marked effect when held over the surface of a solution of sulphate of quinine. The flame of sulphuret of carbon produces on most objects a much stronger effect than that of alcohol. It exhibits distinctly the blue light dispersed close to the surface of a solution of guaiacum in alcohol, which the flame of alcohol does not .It appears then that the flame of sulphuret of carbon is rich in invisible rays of such a

refrangibility as to place them among the groups of fixed lines *m*, *n*, or a little beyond, since when a solution of guaiacum is examined in the solar spectrum, it is found that that is the region in which the blue dispersed light is produced. The blue light dispersed by a solution of guaiacum may also be seen by using the blue flame of sulphur burning feebly. The poverty of the flame of a spirit-lamp, not only with respect to visible rays, but also with respect to invisible rays, except those of very high refrangibility, accounts for the circumstance that it does not exhibit, or at least hardly at all exhibits, the blue light dispersed by fluor-spar.

Mode of determining, by means of the light of a spirit-lamp, the transparency of bodies with respect to the invisible rays of high refrangibility.

201. If the body be a solid, and be bounded by parallel surfaces, its transparency with regard to these rays is easily tested. For this purpose it is sufficient to hold the flame of a spirit-lamp a little way above the surface of a weak solution of sulphate of quinine contained in an open vessel in a dark room, and then, placing the eye so as to see the dispersed light in projection, alternately to interpose and remove the plate to be examined.

202. On examining in this way various specimens of glass, I found none which did not show evident defects of transparency. The purest specimens of plate-glass appeared, I think, to be the least defective. I cannot say whether the observed defects of transparency were due to the essential ingredients of the glass, or to accidental impurities. It is possible that glass made with chemically pure materials might be transparent*. I believe that a mere trace of peroxide of iron, or of sulphuret of soda or potassa, would be sufficient to impair materially the transparency of glass with respect to these rays, and such impurities are very likely to be present. Quartz, however, appeared to be perfectly transparent, the active rays passing through the thickness of one or two inches, whether parallel or perpendicular to the axis, without any perceptible loss. The contrast between quartz and mica was very striking, for a plate of mica no thicker than paper produced a very sensible diminution in the illumination.

203. For the purpose of observing fluids, I procured two vessels consisting of sections of a wide glass tube, about an inch long, closed at one end with a disc of quartz. I shall call these for brevity quartz vessels, though of course the bottom is the only part in which there is any occasion to use quartz. When a fluid is to be examined it is poured into a quartz vessel, and then the vessel with its fluid contents is examined in the manner of a solid plate, as described in Art. 201. On account of the perfect transparency of quartz, the fluid is as good as suspended in air. When a

* Some specimens of glass belonging to Dr. FARADAY's experiments, which from the absence of colour and of internal dispersion seemed hopeful, could not be examined for transparency, on account of their irregular figure; and as they were only lent to me by a friend, I did not feel myself at liberty to get them cut and polished.

quartz vessel was partly filled with water, the addition of a very small quantity of nitrate of iron was sufficient to cause the absorption of the active rays. The solution was so weak as to be almost colourless when viewed through the thickness through which the rays would have to pass. A solution of perchloride of iron had a similar effect. These fluids I had specially examined by sunlight, and had not found in them the least trace of internal dispersion. When a fluid exhibits internal dispersion, it is almost always very opake with regard to rays of high refrangibility, as is shown, without any special experiment, in the course of the observations by which the internal dispersion is exhibited; but it by no means follows conversely, that when a fluid is very opake with regard to these rays, though nearly transparent with regard to the visible rays, it exhibits the phenomenon of internal dispersion.

204. I have little doubt that the solar spectrum would be prolonged, though to what extent I am unable to say, by using a complete optical train in every member of which glass was replaced by quartz. Such a train would be rather expensive, but would not involve any particular difficulty of execution. If solid prisms of quartz were used, half of the incident light would be lost, on account of the double refraction of the substance, unless the prisms were cut in a particular manner, which however would seem likely to involve some difficulties, both in the execution and in the observations. But hollow prisms holding fluids might be employed, having the two faces across which the light has to pass made of quartz plates. For a reason already mentioned, sulphuret of carbon cannot be employed for filling the prisms, and the dispersive power of water is very low, but there appears to be no objection to the use of a solution of some colourless metallic salt. At least saturated solutions of sulphate of zinc and of acetate of lead, the only salts I have tried with this view, showed no defects of transparency when examined in quartz vessels by means of the flame of a spirit-lamp and a solution of sulphate of quinine*.

Effect of Hydrochloric Acid, &c. on Solutions of Quinine. Optical evidences of combination in other instances.

205. Sir JOHN HERSCHEL, in his interesting paper already so often referred to, observes that it is only acid solutions of quinine which exhibit the peculiar blue colour, and that among different acids the muriatic seems least efficacious (page 145).

For my own part I have tried solutions of quinine (not disulphate) in dilute sulphuric, phosphoric, nitric, acetic, citric, tartaric, oxalic, and hydrocyanic acids, and also in a solution of alum. In all these cases the blue colour of the dispersed light was plainly seen by ordinary daylight, especially when the fluid was examined by superficial projection. It was not easy to say which solution answered best, but I am inclined to think that in which phosphoric acid was used.

206. But when quinine was dissolved in dilute hydrochloric acid the blue colour was not exhibited, not even when the fluid was held in the sunlight, and examined by superficial projection. Certain theoretical views led me to regard this as an evi-

* See note H.

dence of a more intimate union between quinine and hydrochloric acid than between quinine and the acids first mentioned, and to try whether the addition of hydrochloric acid to the solutions mentioned in the preceding paragraph would not destroy the blue colour. On trial this proved to be actually the case, so that even sulphuric acid is incapable of developing the blue colour in a solution of quinine in hydrochloric acid.

207. That the quinine was not decomposed when the blue colour due to sulphate of quinine was destroyed by hydrochloric acid, but only differently combined, was shown by adding a solution of carbonate of soda, which produced a white precipitate; and when this was collected on a filter, washed, and redissolved in dilute sulphuric acid, it exhibited the blue colour as usual.

208. The addition of a solution of common salt, instead of hydrochloric acid, to the solutions mentioned in Art. 205, likewise destroyed the blue colour. In the case of sulphuric acid this is only what might have been confidently anticipated; but we should not perhaps have expected that quinine in combination with a weak acid, such as citric, would decompose hydrochlorate of soda, giving rise to citrate of soda and hydrochlorate of quinine; yet this appears to be the nature of the reaction.

209. It might perhaps be supposed that the sulphuric acid was only partially expelled from sulphate of quinine by hydrochloric acid, and that the salt in solution was really a sort of double salt, in which the same base, quinine, was combined with sulphuric and hydrochloric acids in atomic proportion. But if so, it is probable, though not certain, that the same salt would be formed on adding hydrochloric acid to a solution of disulphate of quinine, even though the quantity were not sufficient to combine with the whole of the disulphate. On this supposition, if hydrochloric acid were added by small quantities at a time to a solution of disulphate of quinine, the blue colour ought not to be developed; and when acid enough had been added it ought to be incapable of being developed by the addition of sulphuric acid; whereas, if the whole of the sulphuric acid be expelled by hydrochloric acid, the blue colour ought to be first developed, by the conversion of a portion of the disulphate of quinine into a sulphate, and then destroyed, on the addition of more acid, by the conversion of the sulphate into a hydrochlorate. On trying the experiment with a solution of disulphate of quinine in warm water, it was found that the blue colour was actually first developed and then destroyed.

210. A practical conclusion which seems to follow from these results is, that in the employment of quinine in medicine it is of little consequence whether the sulphate, phosphate, acetate, or hydrochlorate be used, since the first three salts would be immediately converted by the common salt in the body into the hydrochlorate, and the small quantity of a neutral salt of soda resulting from the double decomposition could hardly, one would suppose, be worth considering. However, the common quinine is associated with cinchonine, the reactions of which may be different. According to Sir John Herschel, the latter alkaloid does not exhibit the blue colour, and therefore the optical tests do not apply to it. If it be desired to obtain a soluble

salt of quinine which shall not be converted by common salt, by double decomposition, into a hydrochlorate, it must apparently be sought for among the combinations of quinine with very weak acids, the affinity of which for soda does not much help that of hydrochloric acid for quinine. It seems likely enough that such salts may exist; for though acetate or citrate of quinine decomposes hydrochlorate of soda, hydrochlorate of quinine is decomposed by carbonate of soda; and it is probable that many vegetable acids behave like the carbonic in this respect.

211. The blue dispersion of a solution of sulphate of quinine is destroyed by hydrobromic and hydriodic acids just as by hydrochloric. In the experiment, solutions of bromide and iodide of potassium were used; but as a considerable excess of sulphuric acid was purposely added to the solution of quinine, the potassa introduced would merely remain inert in the solution as a sulphate, without impeding the observation. The same experiment was tried with phosphate of quinine with the same result.

212. It is stated in TURNER's Chemistry, that the play of colours observed in solutions of polychrome (*i. e.* esculine) is destroyed by acids, and heightened by alkalies. The destruction, or at least almost complete destruction, of the blue colour due to dispersed light in a decoction of the bark of the horse-chestnut, which is produced by acids, is readily observed; but I could not perceive that the addition of alkalies in the first instance to a fresh solution made any difference one way or other. If the blue colour had previously been destroyed by an acid, it was restored by the alkali. If the horse-chestnut had never been examined chemically, these observations alone would indicate that in all probability the principle to which the blue colour was due was capable of entering into firm combination with acids, but did not combine with alkalies. It is, in fact, as we know, a vegetable base.

213. A solution of nitrate of uranium in ether is insensible, as if some of the elements of the ether entered into firm combination with the oxide of uranium. In connexion with this circumstance, it is rather remarkable, that although the ether passes off by evaporation when the solution is left to itself in an open vessel, if heat be applied chemical action sets in, and the residue consists chiefly of a salt which has all the appearance of oxalate of uranium. This salt, when washed and examined in the moist state, without very great concentration of light, was found to be insensible*.

214. It is rare to meet with solutions so highly sensitive as those of quinine and esculine, but similar observations may be made on a great number of solutions, by employing suitable methods. The most searching method consists in forming a bright and tolerably pure spectrum, by transmitting the sun's light through a very broad slit, or even leaving out the slit altogether. It is desirable to use a lens of only moderate focal length in connexion with the prisms. The solution having been placed in the spectrum, the acid, or other agent whose reactions it is desired to study,

* See note I.

is to be added, and the effect, if any, observed. It is usually advantageous to cover the slit with a blue glass, or similar absorbing medium; but sometimes effects take place in the bright part of the spectrum, which is intercepted by such a medium. When false dispersion abounds, it is well to look down on the fluid through a NICOL'S prism, so as to stop all light which is polarized in the plane of reflexion.

Negative results with reference to a mutual action of the rays incident on sensitive solutions.

215. The antagonistic effects of the more and less refrangible rays, which have been observed in certain phenomena, induced me to try whether anything of the kind could be perceived in the case of internal dispersion. The following arrangement was adopted for putting this question to the test of experiment.

A tumbler was filled with a very dilute solution of sulphate of quinine, and placed in a pure spectrum. As usual, the illuminated portion of the fluid consisted of two distinct parts, one the blue beam of truly dispersed light, corresponding to the highly refrangible rays, the other the beam reflected from motes, exhibiting the usual prismatic colours, and corresponding to the brighter of the visible rays. The fluid was nearly free from motes, so that the first beam was by far the brighter of the two; and the second beam, without being bright enough at all to interfere with the observation, was useful as serving to point out where the red, yellow, &c. rays lay. A flat prism, having an angle of about 130°, was then held in front of the vessel, with its edge vertical, and situated in the more refrangible part of the visible rays. The rays forming the two beams were thus bent in opposite directions, and the beams made to cross each other within the fluid; and by turning the prism a little in both directions in azimuth, that is, round an axis parallel to the incident rays, it was easy to make sure that the beams did actually cross. But not the slightest perceptible difference in the blue beam was made by the passage of the red and other lowly refrangible rays across it.

216. Certain theoretical views having led me to regard it as doubtful whether the intensity of light internally dispersed was proportional to the intensity of the incident rays, other circumstances being the same, I was induced to try the following experiment.

The sun's light was reflected horizontally through a large lens, which was covered by a screen containing two moderately large round holes, situated in the same horizontal plane, and a good distance apart. The beams coming through the two holes converged of course towards the focus of the lens, and at the same time contracted in width, and became brighter from the concentration of the light. For our present purpose, they may be regarded as cylindrical beams converging towards the focus of the lens. When they had approached each other sufficiently, they were transmitted through a blue ammoniacal solution of copper, contained in a vessel with parallel sides. The object of this was of course to absorb all the bright visible rays, which

would not only be useless for exciting the solution which it was meant to try, but would materially hinder the observation by the glare which they would produce. The beams were then admitted into a vessel containing a decoction of the bark of the horse-chestnut, greatly diluted with water. In passing through the fluid they produced two blue beams of truly dispersed light, which converged towards a point a little way outside the vessel. A flat prism, with an angle of about 150°, was then held in front of the vessel, with its edge vertical, and situated between the incident beams. The blue beams of dispersed light were thus made to cross within the fluid; and by moving the prism in azimuth, it was easy to make one beam either fall above the other, cross it, or fall below it. Now on looking down from above with one eye only, and moving the prism backwards and forwards in azimuth, I could not perceive the slightest difference of illumination, according as the blue beams actually crossed each other, or were merely seen projected one on the other. In this experiment, then, it appeared that one beam of incident rays produced as much additional dispersed light in a portion of fluid already excited by the other beam, as it was capable of producing in a similar portion of fluid not otherwise excited.

Effect of an electric spark. Nature of its phosphorogenic rays.

217. For the use of the apparatus with which the following experiments were made, I am indebted to the kindness of Professor Cumming.

An electric spark produces an internal dispersion of light in a very striking manner in the case of an extremely dilute solution of sulphate of quinine. Having prepared a solution so weak, that when it was examined by superficial projection by the light of a spirit-lamp, nothing was seen but a pale gleam of light extending a good way into the fluid, and not only not confined to the surface, but not even showing any particular concentration in the neighbourhood of the surface, I placed it so as to be illuminated by the sparks from the prime conductor of an electrifying machine, which passed at no great distance over the surface. A very marked internal dispersion was produced, but the nature of the effect depended in a good measure on the character of the spark. A feeble branched spark, giving but little light, and making little noise, produced an illumination extending to a considerable depth, and very much stronger than that occasioned in the same solution by the flame of a spirit-lamp. The rays by which this was produced passed in a great measure through a plate of glass interposed between the spark and the surface of the fluid. But a bright linear spark, making a sharp crack, produced an illumination almost confined to an excessively thin stratum adjacent to the surface of the fluid; and the rays by which this was produced were cut off by glass, though transmitted through quartz. The same was the case with the discharge from a Leyden jar, which produced a bright light almost confined to the surface*.

218. The opacity of a solution of sulphate of quinine appears to increase regularly

* See note J.

and rapidly with the refrangibility of the rays incident upon it. Hence we are led to the conclusion that a strong electric spark is excessively rich in invisible rays of extremely high refrangibility. Glass is opake with respect to these rays, but quartz transparent.

219. It is known that the phosphorogenic rays of an electric spark, at least those which affect CANTON's phosphorus, pass very freely through quartz, but are stopped by a very moderate thickness of glass. This alone, after what has been already mentioned, would lead us to suppose that the phosphorogenic rays coming from such a spark are merely rays of very high refrangibility. If so, they ought to be intercepted by a very small quantity of a substance known to absorb such rays with energy.

After having made some experiments on the production of phosphorescence in CANTON's phosphorus by means of an electric discharge, and observed how the influence of the discharge was transmitted through quartz and stopped, or almost entirely stopped, by glass, I felt confident that my own observations were comparable with those of others. A small portion of the phosphorus was then placed on card, covered by an empty quartz vessel, and had the discharge of a Leyden jar passed over it. The phosphorescence was powerfully excited, being visible in a room which was by no means quite dark; and when the card was carried into a dark place, the phosphorescent light remained plainly visible for a good while. The experiment was then repeated with a fresh portion of the same phosphorus, the vessel this time containing water. The phosphorescence was produced as before, though not I think so copiously. But on taking a fresh portion of the phosphorus, and substituting for water a very dilute solution of sulphate of quinine, the influence of the spark was arrested, and the phosphorus was not rendered luminous. It was found that a solution containing only about one part of quinine in 10,000, with a depth of half an inch, was sufficient to prevent the generation of phosphorescence.

220. This result, it seems to me, would be sufficient, were proof wanting, to show that no part of the effect is attributable *directly* to the electrical disturbance. The effect produced when the phosphorus is at the distance of an inch or so from the points of the discharger seems exactly the same as when it is nearer, being merely somewhat weaker, as would naturally be expected, whatever view were taken of the nature of the influence. But at the distance of an inch, the influence of the spark, though it passes freely through quartz and water, is cut off by adding to the water an excessively small quantity of sulphate of quinine. It cannot be supposed that the electrical relations of the medium, or its permeability to electrical attractions and repulsions, are utterly changed by such an addition; while, on the other hand, the result is in perfect conformity with what we know respecting the stoppage of radiations by absorbing media. However, the principal object of the experiment was not to confirm the view which makes the influence of the spark to consist in the rays which emanate from it, a view which I suppose is pretty generally adopted, but to

investigate more fully the nature of these rays. Enough has, I think, been adduced to show that they are merely rays which there is no reason to suppose are physically different from those of light, but quite the contrary, and which are of very high refrangibility, and are therefore invisible, since they fall far beyond the limits of refrangibility within which the retina is affected. Indeed, it seems very likely that the highly refrangible rays never reach the retina, but are absorbed by the coats of the eye*. Hence the phenomena relating to the phosphorescence produced by an electric discharge afford no countenance to the supposition that it is possible to divide rays of a given refrangibility into phosphorogenic, chemical, luminous, &c. Of course the most unexceptionable mode of determining the refrangibility of the phosphorogenic rays would be by actual prismatic decomposition, but this would require the employment of a quartz train.

Points of resemblance and contrast between internal dispersion and phosphorescence.

221. As the term *phosphorescence* has been applied to several different phenomena, I must here explain that I mean the spontaneous exhibition of a soft light, independently of chemical changes, which some substances exhibit for a time after having been exposed to the sun's rays, or to an electric discharge, or to light from some other sources.

In many respects the two phenomena have a strong resemblance. Thus, the general features of internal dispersion cannot be better conceived than by regarding the sensitive medium as self-luminous while under the excitement of the active rays. Again, it is well known that the rays of the solar spectrum by which the phosphorescence of Canton's phosphorus, sulphuret of barium, and other phosphori, is produced, are those of high refrangibility, as well as the invisible rays beyond; and these are precisely the rays which in the great majority of cases are most efficient in producing internal dispersion. I do not however know how far it may be true that when phosphorescence is excited by homogeneous light the refrangibility of the incident light is a superior limit to the refrangibilities of the component parts of the light emitted. Indeed, according to Professor Draper, when the phosphorescence of Canton's phosphorus is excited by the rays from incandescent lime, the active rays belong to the red extremity of the spectrum†. If this result be confirmed, it follows that the most striking law relating to internal dispersion is not obeyed in the case of phosphorescence.

In the same paper Professor Draper remarks, "Some time ago I determined the refrangibility of the rays of an electric spark which excite phosphorescence in sulphuret of lime; they are found at the violet extremity of the spectrum." In what way Professor Draper determined the refrangibility of rays with respect to which glass is so opake, he does not give the least hint. Being perfectly in the dark as to the evidence on which the conclusion is based, I cannot accept it in contradiction to

* See note K.

† Philosophical Magazine, vol. xxvii. (Dec. 1845) p. 436.

my own experiments. Perhaps, however, "at the violet extremity" may mean nothing more than somewhere in the highly refracted region beyond the visible rays. If so, Professor Draper's statement is in accordance with my own conclusions.

222. When one part of a phosphorus has been excited, the phosphorescence is found gradually to extend itself to the neighbouring parts. In this respect a substance which exhibits internal dispersion presents a striking contrast. The finest fixed lines of the spectrum are seen sharply defined, whether in a solution, or in a clear solid, or on a washed paper.

223. Of course, theoretically, there ought, to a certain extent, to be a communication of illumination from one part of a sensitive fluid to another, on account of the light which is twice, three times, &c. dispersed. This however must be excessively small; for the mean refrangibility of the dispersed light is usually much lower than the refrangibility of the active light, perhaps lower than that of any light capable of exciting the solution. However, generally some few of the dispersed rays would have a refrangibility sufficiently high to be dispersed again. But practically the intensity of the light twice dispersed in this manner would be so very small that it may safely be altogether disregarded.

224. But by far the most striking point of contrast between the two phenomena, consists in the apparently instantaneous commencement and cessation of the illumination, in the case of internal dispersion, when the active light is admitted and cut off. There is nothing to create the least suspicion of any appreciable duration in the effect. When internal dispersion is exhibited by means of an electric spark, it appears no less momentary than the illumination of a landscape by a flash of lightning. I have not attempted to determine whether any appreciable duration could be made out by means of a revolving mirror.

225. There appears to be no relation between the substances which exhibit a change of refrangibility and those which phosphoresce, either spontaneously, or on the application of heat. Thus the sulphurets of calcium and barium, on being examined for internal dispersion, were found to be insensible, as was also Iceland spar. The last substance phosphoresced strongly on the application of heat. So far as was examined, the minerals which did exhibit a change of refrangibility showed no special disposition to phosphoresce. Sir David Brewster has remarked, that a specimen of fluor-spar which exhibited a blue light by internal dispersion, exhibited when heated a blue phosphorescent light; but this appears to have been merely a casual coincidence*.

On the Cause of True Internal Dispersion, and of Absorption.

226. In considering the cause of internal dispersion, we may I think at once discard all supposition of reflexions and refractions of the vibrations of the luminiferous ether among the ultimate molecules of bodies. It seems to be quite contrary

* Report of the Meeting of the British Association at Newcastle in 1839, p. 11.

to dynamical principles to suppose that any such causes should be adequate to account for the production of vibrations of one period from vibrations of another.

All believers, I suppose, in the undulatory theory of light are agreed in regarding the production of light in the first instance as due to vibratory movements among the ultimate molecules of the self-luminous body. Now in the phenomenon of internal dispersion, the sensitive body, so long as it is under the influence of the active light, behaves as if it were self-luminous. Nothing then seems more natural than to suppose that the incident vibrations of the luminiferous ether produce vibratory movements among the ultimate molecules of sensitive substances, and that the molecules in turn, swinging on their own account, produce vibrations in the luminiferous ether, and thus cause the sensation of light. The periodic times of these vibrations depend upon the periods in which the molecules are disposed to swing, not upon the periodic time of the incident vibrations.

227. But in the very outset of this theory an objection will probably be urged, that it is quite as much contrary to dynamical principles to suppose the periodic time of the ethereal vibrations capable of being changed through the intervention of ponderable molecules as without any such machinery. The answer to this objection is, that such a notion depends altogether on the applicability of a certain dynamical principle relating to indefinitely small motions, and that we have no right to regard the molecular vibrations as indefinitely small. The excursions of the atoms may be, and doubtless are, excessively small compared with the length of a wave of light; but it by no means follows that they are excessively small compared with the linear dimensions of a complex molecule. It is well known that chemical changes take place under the influence of light, especially the more refrangible rays, which would not otherwise happen. In such cases it is plain that the molecular disturbances must not be regarded as indefinitely small. But vibrations may very well take place which do not go to the length of complete disruption, and yet which ought by no means to be regarded as indefinitely small. Furthermore, it is to be observed that if in the cases of indefinitely small molecular displacements the forces of restitution be not proportional to the displacements, the principle above alluded to will not be applicable however small the disturbance may be; and if in the expressions for the forces of restitution the terms depending on first powers of the displacements (supposed finite), though not absolutely null, be very small, the principle will not apply unless the molecular excursions be extremely small indeed. In consequence of the necessity of introducing forces not proportional to the displacements, it would be very difficult to calculate the motion, even were we acquainted with all the circumstances of the case, whereas we are quite in the dark respecting the actual data of the problem. But certainly we cannot affirm that in the disturbance communicated back again to the luminiferous ether none but periodic vibrations would be produced, having the same period as the incident vibrations. Rather, it seems evident that a sort of irregular motion must be produced in the molecules, periodic only in the

sense that the molecules retain the same mean state; and that the disturbance which the molecules in turn communicate to the ether must be such as cannot be expressed by circular functions of a given period, namely, that of the incident vibrations.

228. It is very remarkable with what pertinacity a particular mode of internal dispersion attaches itself to a particular chemical substance. Thus the singular dispersion of a red light exhibited by the green colouring matter of leaves is found in a green leaf, or in a solution of the green colouring matter in alcohol, ether, sulphuret of carbon, or muriatic acid. The dispersion exhibited by nitrate of uranium is found in a solution of the salt in water, as well as in the crystals themselves, which are doubly refracting. In all probability therefore the molecular vibrations by which the dispersed light is produced are not vibrations in which the molecules move among one another, but vibrations among the constituent parts of the molecules themselves, performed by virtue of the internal forces which hold the parts of the molecules together. It is worthy of remark that it is chiefly among organic compounds, the ultimate molecules of which we are taught by chemistry to regard as having a complicated structure, that internal dispersion is found. It is true that peroxide of uranium furnishes many examples of internal dispersion; but then the anhydrous peroxide is itself insensible, it is only some of the compounds into which it enters that are so remarkably sensitive; and the chemical formulæ of these compounds, so far as they are known, are not by any means extremely simple, although it is true that they may not be more complicated than formulæ relating to other oxides. Why this particular oxide should be disposed to enter into tottering combinations I do not pretend even to conjecture; but it seems not a little remarkable that peroxide of uranium, which is so peculiar with respect to its optical properties, should also present some singularities in its mode of chemical combination, which led M. Peligot to regard it as the protoxide of a compound radical.

229. We are, I conceive, at present far from an explanation of the phenomena of internal dispersion in all their details. They appear to be associated with the inmost structure of chemical molecules, to such a degree as to throw even the phenomena of polarization into the shade. In this respect, indeed, absorption seems superior to polarization, since most of the phenomena of polarization refer rather to the state of crystalline aggregation of the molecules than to their constitution; but the phenomena of internal dispersion appear to be much more searching than those of absorption. There is one law however relating to internal dispersion so striking and so simple, that it seems not unreasonable to look for an explanation of it; I allude to that according to which the refrangibility of light is always lowered in the process of dispersion. I have not hitherto been able altogether to satisfy myself respecting a dynamical explanation of this law, but the following conjectures will not perhaps be deemed altogether unworthy of being mentioned.

230. Reasons have already been brought forward for regarding the molecular vibrations as performed under the influence of forces not proportional to the dis-

placements. For simplicity's sake, let us suppose for the present the parts of the forces of restitution depending upon first powers of the displacements to be absolutely null. Then, when a molecule is disturbed, its atoms will be acted on by forces depending upon the second and higher powers of the displacements. These forces must tend to restore the atoms to their mean positions; otherwise the equilibrium would be unstable, and the atoms would enter into new combinations, either with one another, or with the atoms of the surrounding medium; so that, in fact, such compounds could never be formed. The condition of stability would require the parts of the forces depending upon squares of the displacements to vanish, but this is a point which need not be attended to, all that is essential to bear in mind being, that we have forces of restitution varying in a higher ratio than the displacements. If the parts of the forces of restitution which depend upon first powers of the displacements, though not absolutely null, be very small, the remaining parts must still be such as to tend to restore the atoms to their positions of equilibrium; otherwise the stability of the molecule, though not mathematically null, would be so very slight, that such compounds would probably never form themselves, but others of more stability would be formed instead. Or, even were such unstable compounds formed, they would probably be decomposed on attempting to excite them in the manner in which sensitive substances are excited in observing the phenomena of internal dispersion; so that whether they exist or not, they may be set aside in considering these phenomena.

231. Now when vibrations are performed under the action of forces which vary in a higher ratio than the displacements, the periodic times are not constant, but depend upon the amplitudes of vibration, being greater or less according as the amplitudes are less or greater. Suppose the molecular and ethereal vibrations already going on, and imagine the amplitudes of the former kept constant by the application of external forces. According to the value of the epoch of the vibrations of a particular molecule, the ethereal vibrations will tend, in the mean of several successive undulations, to augment or to check the vibrations of the molecule. For some time there will be a tendency one way, then for some time a tendency the other way, and so on, the opposite tendencies balancing each other in the long run. The lengths of the times during which the tendency lies in one direction, will depend upon the periodic times of the molecular and ethereal vibrations, being on the whole greater or less according as the two periodic times are more or less nearly equal. But since no external forces actually act to keep the amplitudes constant, when the ethereal vibrations are favourable to disturbance the molecule is further disturbed, and therefore its periodic time is diminished; and when they are favourable to quiescence the disturbance of the molecule is checked, and therefore its periodic time is increased. If, then, the ether be vibrating more rapidly than the molecule, when the action is favourable to disturbance the periodic time of the molecular vibrations is rendered more nearly equal to that of the ethereal vibrations, and therefore the time

during which the action is favourable to disturbance is prolonged; but when the action is favourable to quiescence, the effect is just the reverse. Hence, on the whole, there is a balance outstanding in favour of disturbance. But if the ether be vibrating more slowly than the molecule, it appears from similar reasoning that there will be a balance the other way. Hence it is only when the periodic time of the ethereal vibrations is less than that of the molecular, that the latter vibrations can be kept going by the former.

232. But it will probably be objected to this explanation, that when a periodic disturbing force affects the mean motion of a planet, the mean motion is a maximum, not when the force tending to augment it is a maximum, but at a time later by a quarter of the period of the force, namely, when the force vanishes in changing sign; and that in a similar manner the change in the periodic time of the vibrations of a disturbed molecule will affect equally the duration of the time during which the action is favourable to increased disturbance, and that during which it is favourable to quiescence, or more exactly will not alter either, since the effects in the first and second halves of those times will neutralize each other. The answer to this objection is, that we must not treat a molecule as if it were isolated, like a heavenly body, since it is continually losing its motion by communication, perhaps to neighbouring molecules, but at any rate to the luminiferous ether; for without a communication of the latter kind there would be no dispersed light. Hence we must consider the *immediate* tendency of the disturbing forces rather than their tendency in the long run.

233. When a molecule itself vibrates in an irregularly periodical manner, the vibrations which it imparts to the ether are of course of a similar character. The resolution of these into vibrations corresponding to different degrees of refrangibility, involves some very delicate mathematical considerations, into which I do not propose to enter. But without this it is evident that when the ether is agitated by the vibrations of an immense number of molecules, in all possible states as regards amplitude, and consequently periodic time of vibration, the disturbance of the ether must consist of a mixture of periodic vibrations, having their periods comprised between the greatest and least of those belonging to the molecular vibrations; and corresponding to these different periods there will be portions of light of different degrees of refrangibility found in the dispersed beam. These refrangibilities will range between two limits, an inferior limit equal to the refrangibility corresponding to the periodic time of indefinitely small vibrations, and a superior limit equal to the refrangibility of the active light.

234. This theory seems to accord very well with the general character of dispersed beams, as regards the prismatic composition of the light of which they consist. When analysed by a prism, these beams are sometimes found to break off abruptly at their more refrangible border, but I do not recollect ever to have met with an instance in which a beam broke off abruptly at the opposite border, except when the whole beam was almost homogeneous. This is just as it ought to be according to

the above theory, because the amplitude of vibration decreases indefinitely in approaching the less refrangible limit. In the case of a solution of chlorophyll, we may suppose that the part of the molecular forces of restitution depending on first powers of the displacements is considerable, on which supposition, the effect ought to approach to what would take place were there no other part. But were the forces of restitution strictly proportional to the displacements, the vibrations would be isochronous, and could only be excited by ethereal vibrations having almost exactly the same period, but would be powerfully excited by such. Accordingly, in a solution of chlorophyll the dispersion comes on very suddenly; a large part of it is produced by active light of nearly the same refrangibility as the dispersed light; and the latter, by whatever active light produced, has nearly the same refrangibility that it had at first. This supposition, combined with the preceding theory, accounts also for the transparency of the fluid with respect to rays of less refrangibility than the first absorption band, for the great intensity of that band, for the rapidity with which opacity comes on at its less refrangible border, and the comparatively slow resumption of transparency on the other side. A difference of the same nature on opposite sides of a maximum of opacity seems to be a very common phenomenon in absorption. On the other hand, in those numerous cases in which the dispersion comes on gradually, in the manner described in Art. 44, we may suppose the part of the forces of restitution depending on first powers of the displacements to be but small.

235. It may appear at first sight to be a formidable objection to the theory here brought forward, that in the experiment mentioned in Art. 216, the intensity of the dispersed light did not appear to be more than doubled when the intensity of the incident disturbance was doubled; and that in the experiment described in Art. 215, the rays of low refrangibility did not appear to exercise any protecting influence. But the difficulty may, I think, be got over by a very reasonable supposition. It seems very natural to suppose that a given molecule remains for the greater part of the time at rest, or nearly so, and only now and then gets involved in vibrations. On this supposition, it is only a very small per-centage of the molecules that at a given instant are vibrating to an extent worth considering. Conceive now a stream of light consisting of the highly refrangible rays to be incident on a sensitive medium, and to cause 1 per cent. of the sensitive molecules to vibrate considerably, the rest vibrating so little that they may be regarded as at rest. Now imagine a second stream, similar in all respects to the first, to influence the medium which is already under the influence of the first stream. Of the 1 per cent. of the molecules already vibrating, many are vibrating, we may suppose, nearly with their maximum amplitude, and consequently are not much affected. Besides, it is a great chance if the epoch of the ethereal vibrations belonging to the second stream is such as to produce any great tendency either towards quiescence or towards disturbance in a molecule just for the short time that it is vibrating strongly under the influence of the first stream. But of the 99 per cent. of quiescent molecules 1 per cent. are made to

vibrate. Hence the effect of the two streams together is very nearly the same in kind as that of one alone, but double in intensity.

236. The apparent absence of a protecting influence in the less refrangible rays seems at first more difficult to account for, but perhaps the following reasoning may be thought satisfactory. We ought not to attribute more influence in the direction of protection to a second beam of rays of low refrangibility, than in the contrary direction to a second beam of rays of high refrangibility. Now if the effect of a beam of rays of high refrangibility be to throw 1 per cent. of the molecules into a state of vibration, it would be a commensurate effect in a beam of rays of low refrangibility to stop the vibrations of 1 per cent. of the molecules, if they were all vibrating. But since only 1 per cent. are actually vibrating, the real protecting effect amounts to no more than stopping the vibrations of one molecule in every 10,000, an effect which may be regarded as insensible.

237. The simple consideration that work cannot be done without the expenditure of power, shows that when light incident on a medium gives rise to dispersed light, a portion at least of the absorption which the medium is observed to exercise must be due to the production of the dispersed light. If the dispersed light really arises from molecular disturbances, and for my own part I think it almost beyond a question that it does, it follows that in these cases light is absorbed in consequence of its being used up in producing molecular disturbances. But since we must not needlessly multiply the causes of natural phenomena, we are led to attribute the absorption of light in all cases to the production or augmentation of molecular disturbances, unless reason be shown to the contrary. It might seem at first sight that the production or non-production of dispersed light establishes at once a broad distinction between different kinds of absorption. I do not think that much stress can be laid on this distinction. In the first place it may be remarked, that we have no reason to suppose that vibrations which are of the same nature as those of light are confined to the range of refrangibility that the human eye can take in. If, therefore, no dispersed light be perceived, it does not follow that no invisible rays are dispersed. If the incident light belong to the visible part of the spectrum, the dispersed rays (if any), being of lower refrangibility than the incident light, can only be invisible by having a refrangibility less than that of red light, and would manifest themselves solely or mainly by their heating effect. However, though invisible rays of this nature are in all probability emitted by the body in consequence of the absorption of visible light, we are not bound to suppose that in their mode of emission they precisely resemble the visible rays observed in the phenomena of internal dispersion. In most cases, perhaps, they are more nearly analogous to the visible rays emitted by solar phosphori. It is possible to conceive, and it seems probable that there exist, various degrees of molecular connexion from mere casual juxtaposition to the closest chemical union. A compound molecule may vibrate as a whole, by virtue of its connexion with adjacent molecules, or it may vibrate by itself, in the

manner of an isolated vibrating plate or rod, and between these extreme limits we may conceive various intermediate modes of vibration. Hence, without departing from the general supposition that the absorption of light is due to the production of molecular disturbances, we may conceive that the modes in which the ether communicates its vibrations to the molecules, and the molecules in turn communicate their disturbances to the ether, are very various.

I do not bring forward the idea that the absorption of light is due to the production of molecular disturbances as new, though possibly the communication of the ethereal vibrations to the molecules may hitherto have been supposed necessarily to imply the existence of synchronous vibrations among the molecules. The change in the periodic time of vibrations which takes place in the process of internal dispersion would hardly have been suspected, had it not been for the singular phenomenon which pointed it out.

238. The only theory of absorption, so far as I am aware, in which an attempt is made to deduce its laws from a physical cause is that of the Baron Von Wrede, who attributes absorption to interference*. The Baron's paper is in many respects very beautiful, but it has always appeared to me to be a fatal objection to his theory that it supposes vibrations to be annihilated. It is true that two streams of light may interfere and produce darkness, but then to make up for it more light is produced in other quarters. Light is not lost by interference, but only the illumination differently distributed. Were the disappearance of light in the direction of a pencil admitted into a medium merely a phenomenon of interference, the full quantity of light admitted ought to be forthcoming in side directions. Were a series of vibrations incident on a medium, without producing any progressive change in its state, or any disturbance issuing from it, it would follow that work was continually being annihilated. But we have reason to think that the annihilation of work is no less a physical impossibility than its creation, that is, than perpetual motion.

List of highly sensitive substances.

239. For the sake of any one who may wish to make experiments in this subject, I subjoin a list of the more remarkable of the substances which have fallen under my notice. It will be seen that most of these substances were suggested by the papers of Sir David Brewster and Sir John Herschel.

Glass coloured by peroxide of uranium: yellow uranite: nitrate or acetate of the peroxide. Probably various other salts of the peroxide would do as well. The absorption bands of the salts, whether sensitive or not, of peroxide of uranium ought to be studied in connexion with the change of refrangibility.

A solution of the green colouring matter of leaves in alcohol. To obtain a solution which will keep, it is well previously to steep the leaves in boiling water. The alcohol should not be left permanently in contact with the leaves, unless it be wished

* Poggendorff's Annalen, B. xxxiii. S. 353; or Taylor's Scientific Memoirs, vol. i. p. 477.

to observe the changes which in that case take place, but poured off when the strength of the solution is thought sufficient. Also, the solution when out of use must be kept in the dark.

A weak solution of the bark of the horse-chestnut.

A weak solution of sulphate of quinine, *i. e.* a solution of the common disulphate in very weak sulphuric acid. Various other salts of quinine are nearly if not quite as good.

Fluor-spar (a certain green variety).

Red sea-weeds of various shades: a solution of the red colouring matter in cold water. If a solution be desired, a sea-weed must be used which has never been dried. Sometimes even a fresh sea-weed will not answer well.

A solution of the seeds of the *Datura stramonium* in not too strong alcohol.

Various solutions obtained from archil and litmus (see Arts. 65 to 72).

A decoction of madder in a solution of alum.

Paper washed with a pretty strong solution of sulphate of quinine, or with a solution of stramonium seeds, or with tincture of turmeric. The sensibility of the last paper is increased by washing it with a solution of tartaric acid. This paper ought to be kept in the dark.

A solution, not too strong, of guaiacum in alcohol.

Safflower-red, scarlet cloth, substances dyed red with madder, and various other dyed articles in common use.

Many of the solutions here mentioned are mixtures of various compounds. Of course if the sensitive substance can be obtained chemically pure it will be all the better.

Conclusion.

240. The following are the principal results arrived at in the course of the researches detailed in this paper:—

(1.) In the phenomenon of true internal dispersion the refrangibility of light is changed, incident light of definite refrangibility giving rise to dispersed light of various refrangibilities.

(2.) The refrangibility of the incident light is a superior limit to the refrangibility of the component parts of the dispersed light.

(3.) The colour of light is in general changed by internal dispersion, the new colour always corresponding to the new refrangibility. It is a matter of perfect indifference whether the incident rays belong to the visible or invisible part of the spectrum.

(4.) The nature and intensity of the light dispersed by a solution appear to be strictly independent of the state of polarization of the incident rays. Moreover, whether the incident rays be polarized or unpolarized, the dispersed light offers no traces of polarization. It seems to emanate equally in all directions, as if the fluid were self-luminous.

(5.) The phenomenon of a change of refrangibility proves to be extremely common, especially in the case of organic substances such as those ordinarily met with, in which it is almost always manifested to a greater or less degree.

(6.) It affords peculiar facilities for the study of the invisible rays of the spectrum more refrangible than the violet, and of the absorbing action of media with respect to them.

(7.) It furnishes a new chemical test, of a remarkably searching character, which seems likely to prove of great value in the separation of organic compounds. The test is specially remarkable for this, that it leads to the independent recognition of one or more sensitive substances in a mixture of various compounds, and shows to a great extent, before such substances have been isolated, in what menstrua they are soluble, and with what agents they enter into combination. Unfortunately, these observations for the most part require sunlight.

(8.) The phenomena of internal dispersion oppose fresh difficulties to the supposition of a difference of nature in luminous, chemical, and phosphorogenic rays, but are perfectly conformable to the supposition that the production of light, of chemical changes, and of phosphoric excitement, are merely different effects of the same cause. The phosphorogenic rays of an electric spark, which, as is already known, are intercepted by glass, appear to be nothing more than invisible rays of excessively high refrangibility, which there is no reason for supposing to be of a different nature from rays of light.

NOTES ADDED DURING PRINTING.

Note A. Art. 23.

Shortly after the preceding paper was forwarded to the Royal Society, I found M. Edmond Becquerel's map of the fixed lines of the chemical spectrum, which is published in the 40th volume of the 'Bibliothèque Universelle de Genève' (July and August 1842). I had seen in Moigno's 'Repertoire d'Optique Moderne,' that the map had been presented to the French Academy, and naturally felt anxious to obtain it; but not finding any further notice of it either in that work or in the 'Comptes Rendus,' I supposed that it had not yet been published. The principal lines in this map I recognized at a glance. M. Becquerel's broad band I is my *l*; his group of four lines M with the preceding band forms my group *m*; his group of four lines N forms the first four of my group *n*; his line O is my *n*. It is only in the last group that there can be any doubt as to the identification; but I feel almost certain that M. Becquerel's P is my *o*, and the next two lines, the last in his map, are the two between *o* and *p*. It is difficult at first to believe that the strong line *p* should have been left out, while the two faint lines between *o* and *p* are represented, but the difficulty is, I think, removed by considering the feeble photographic action in that part of

the spectrum. M. BECQUEREL expressly states that lines were seen beyond the last he has represented, though they were hardly distinct; and on comparing together his map, Mr. KINGSLEY's photographs, and my own map, I think hardly any doubt can remain as to the identification.

I take this opportunity of referring to another very interesting paper of M. BECQUEREL's, entitled 'Des effets produits sur les corps par les rayons solaires,' which is published in the Annales de Chimie, tom. ix. (1843) p. 257, with which I was not acquainted till lately, or I should have referred to it before. This paper contains, among other things, an investigation of the effects of transparent and coloured screens on the luminous, chemical, and phosphorogenic rays, in which it is shown, that, notwithstanding the great difference in the action of a given screen on the three classes of rays, when we study the effect of the incident rays as a whole, its action is the very same when we confine our attention to rays of any one refrangibility. Among the media employed by M. BECQUEREL, are some whose absorbing effect I have mentioned in the present paper, as having been determined by methods depending upon the change of refrangibility. In such cases my own results, as might have been anticipated, are in perfect harmony with those of M. BECQUEREL. With respect to the results at which I have arrived regarding the nature of the phosphorogenic rays of an electric spark, which are mentioned towards the end of the paper, I have been in a good measure anticipated by M. BECQUEREL. Yet I do not think that even he was aware that so much of the effect of the spark was due to rays of such high refrangibility.

Note B. Art. 105.

I have since succeeded, by a particular arrangement, in seeing so far into the "lavender" rays as to make out the groups of fixed lines *m*, *n*, *p* by means of light received directly into the eye, and even to perceive light beyond that.

As to the colour of these rays when they are well isolated, I think the corolla of the lavender gives as good an idea of it as could be expected from the circumstances. They seem to me to want the luminousness of the blue and the ruddiness of the violet. No doubt much error and uncertainty has hitherto existed both as to the colour and as to the illuminating power of these rays, because the gray prolongation of a spectrum formed on paper by projection has been mistaken for the lavender rays.

Note C. Art. 154.

On adding common phosphoric acid to a solution of nitrate of uranium no effect seemed to be produced, but on examining the vessel some days afterwards, a precipitate was found to have fallen. This precipitate proved to be sensitive in a very high degree.

Note D. Art. 158.

I have since observed in a mineral solution a system of absorption bands so remarkable, and so closely resembling in many respects those found in the salts of peroxide of uranium, though they occur in a totally different part of the spectrum, that I think no apology is needed for mentioning the circumstance. The medium referred to is a solution of permanganate of potassa, in fact, red solution of mineral chameleon. In order to see the bands, it is essential to employ a dilute solution,

or else to view it in small thickness, since otherwise the whole of the region in which the bands occur is absorbed. The bands are five in number, and are equidistant, or at least very nearly so. The first is situated at about three-fifths of a band-interval above D; the last coincides with F, or, if anything, falls a little short of it. The second and third are the most intense of the set. I have carefully examined the solution for change of refrangibility, and have not found the least trace. Ferrate of potassa shows nothing remarkable.

By means of the bands just mentioned, the colour of permanganate of potassa may be instantly and infallibly distinguished from that of certain other red solutions of manganese, the colour of which some chemists have been disposed to attribute to permanganic acid (see a paper by Mr. PEARSALL 'On red Solutions of Manganese,' Journal of the Royal Institution, New Series, No. IV. p. 49).

Note E. Art. 171.

If we suppose the angle of incidence *exactly* equal to 45°, assume $\frac{4}{3}$ for the refractive index of the fluid, and apply FRESNEL's formulæ to calculate the ratio of the intensity of light reflected at the exterior surface of a bubble, and polarized in a plane perpendicular to the plane of incidence, to that of light similarly reflected and polarized in that plane, we find 0·228 to 1, a ratio which certainly differs much from one of equality. But in order to render the two intensities equal, it is sufficient to increase the angle of incidence by only 3° 35′; and in fact, as a matter of convenience, the position of the observer was usually such that the deviation of the light was somewhat greater than 90°, and therefore the angle of incidence somewhat greater than 45°.

Note F. Art. 191.

I have since received a slab of glass of the kind here recommended, which has been executed for me by Mr. DARKER of Lambeth, and which answers its purpose admirably, the medium being eminently sensitive. Besides its general use as a screen, this slab, from its size and form, has enabled me to trace further than I had hitherto done (Arts. 75, 76) the connexion between certain fluctuations of transparency which the medium exhibits and corresponding fluctuations of sensibility.

Note G. Art. 192.

Paper washed with a mere infusion of the bark of the horse-chestnut is quickly discoloured; but a piece washed with a solution which had been purified by chemical means remained white, and proved exceedingly sensitive.

Note H. Art. 204.

I have since ordered a complete train of quartz, of which a considerable portion, comprising among other things two very fine prisms, has been already executed for me by Mr. DARKER. With these I have seen the fixed lines to a distance beyond H more than double that of p; so that the length of the spectrum, reckoned from H, was more than double the length of the part previously known from photographic impressions. The light was reflected by the metallic speculum of a SILBERMANN's heliostat, which I have received from M. DUBOSCQ-SOLEIL. With the glass train the group p was faint, but with the quartz train there was abundance of light to see not only the group p, but the fixed lines as far as $Hp1$, or thereabouts. From the group n to about the middle of the new region, the lines are less bold and striking than in the region of the groups H, l, m, n, but the latter

part of the new region contains many lines remarkable both for their strength and for their arrangement. I hope to make a careful drawing of these lines as shown by the complete train with a summer's sun.

I have some reasons for believing that the photographic action of these highly refrangible rays is feeble, perhaps almost absolutely null. In the second of the papers referred to in Note A. (p. 300), M. Becquerel describes an experiment in which a prism of quartz was employed to form a spectrum; and yet the impressed spectrum formed by rays which had traversed the quartz alone was hardly longer than that formed by rays which, in addition to the quartz, had traversed a screen of pure flint-glass a centimetre in thickness. It is possible, I am inclined to think probable, that glass made with *perfectly* pure materials would be transparent like quartz, but all the specimens I have examined were decidedly defective in transparency. Besides, M. Becquerel, who may be allowed to be the best judge of his own experiments, considered the result just mentioned as a proof that the impressed spectrum formed by rays which had traversed quartz only did not extend, except a very trifling distance, beyond that formed by his train of glass; and yet his map, formed by means of the latter, does not take in the line *p*.

However, among the multitude of preparations capable of being acted on by light, it is probable that there may be some which are acted on mainly by rays of unusually high refrangibility, and which, on that very account, would not be suitable for the ordinary purposes of photography. With these it is possible that the new region of the solar spectrum might be taken photographically.

Note I. Art. 213.

I have since examined the salt, or product, whatever it may be, in the dry state, and under more favourable circumstances, and have found it sensitive, though not by any means in a high degree. It exhibits also the absorption bands which seem to run through the salts of peroxide of uranium.

In connexion with the insensibility of a solution of nitrate of uranium in ether, it seems interesting to mention a fact which I have since observed, namely, that the sensibility of a solution of nitrate of uranium in water is destroyed by the addition of a little alcohol.

Note J. Art. 217.

On repeating this experiment on a subsequent occasion, I could not satisfactorily make out the difference of character of a strong and of a weak spark from the prime conductor, perhaps because the machine was in less vigorous action; but the difference between the effects of a mere spark and of the discharge from a Leyden jar was plainly evident. I would here warn the reader, that in order to perform the experiment in such a manner as to obtain a striking and perfectly decisive result, it is essential to employ an excessively weak solution. The reason of this is evident.

A severe thunder-storm which visited Cambridge on the evening of July 16, 1852, afforded me a good opportunity of observing the effect of lightning on a solution of quinine, and other sensitive media. From the copiousness of the dispersed light, it was evident that the proportion of the active, and therefore highly refrangible rays to the visible rays was very far greater in the radiation from lightning than in daylight. A difference of character was observed between the effects of a weak distant flash, and of a bright flash nearly overhead, similar to that which has been described with reference to the effects of a spark from a machine, and of the discharge from a Leyden jar. In

artificial discharges, the stronger the spark the more the rays of excessively high refrangibility seem to abound, in proportion to the whole radiation. Now a flash of lightning is a discharge incomparably stronger than that of a Leyden jar. It might have been expected, therefore, that the radiation from lightning would be found to abound in invisible rays of excessively high refrangibility. Yet I could not make out in a satisfactory manner the absorption of the rays by glass, even by common window-glass. I do not wish to speak positively regarding the result of this observation, for of course observations with lightning are more difficult than those made with a machine which is under the control of the observer. Yet it did seem as if the spark from a Leyden jar was richer than lightning in rays of so high a refrangibility as to be stopped by glass. If this be really true, it must be attributed to one of two things, either the non-production of the rays in the first instance, in the case of lightning, or their absorption by the air or clouds in their passage from the place of the discharge. If they were not produced, that may be attributed to the rarity of the air at the height of the discharge, that is, at the height of the thunder-cloud. No doubt the metallic points of the discharger belonging to the electrical apparatus may have had an influence on the nature of the spark; but I am inclined to think that this influence, so far as it went, would have acted in the wrong direction, that is, would have tended to produce rays of lower, at the expense of those of higher refrangibility.

Note K. Art. 220.

My attention has recently been called to a paper by M. Brücke (Poggendorff's Annalen, B. v. (1845) S. 593), in which he describes some experiments which show that the different parts of the eye, and especially the crystalline lens, are far from transparent with respect to the rays of high refrangibility. The eyes employed were those of oxen and some other animals; and the inquiry was carried on by means of the effect which light that had passed through the part of the eye to be examined produced on a film of tincture of guaiacum that had been dried in the dark. Of course the phenomena described in the present paper afford peculiar facilities for such an inquiry, and I had frequently thought of entering upon it, but have not yet made any observations. Independently of the facility of the observations, and the advantage of being able to examine readily light of each degree of refrangibility in particular, the results obtained by means of sensitive media seem to be more trustworthy on this account, that it would be possible to employ fresh eyes. The experiments of M. Brücke necessarily occupied a considerable time, and it may be doubted whether the eye, especially after dissection, might not have changed in the interval, and whether the results so obtained are applicable to the eye as it exists in the living animal.

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N.B. The figures refer to the articles, the letters to the notes.

XXII. *The Reproduction of the* Ascaris mystax. *By* Henry Nelson, *M.D.* *Communicated by* Allen Thomson, *M.D., F.R.S., Professor of Anatomy in the University of Glasgow.*

Received May 22,—Read June 5, 1851.

Having been led to investigate the mode of generation and development of the *Ascaris mystax*, I venture to lay before the Royal Society the results of my observations, in the hope that they may not be devoid of interest.

The worm in question has been long known to helminthologists (Plate XXV*. figs. 3 and 4); it varies from 1 to 3 inches in length, is fusiform, and covered with a striated horny cuticle, which is projected on either side of the head to form the alæ characteristic of the species (fig. 1 *a*).

The *Ascaris mystax* is found within the intestinal canal of the domestic Cat. So common is it, that out of about thirty examined for the purpose I have not failed to find them in more than three or four cases.

The part in which they are generally found in the greatest abundance is that portion of the duodenum between the pylorus and the opening of the bile-duct. When the cat has fasted for some hours, the Ascarides pass the pyloric valve into the stomach, apparently in search of food; but this is never the case during digestion; on the contrary, they appear to be then swept further down the intestinal canal than the part they normally occupy.

The male *Ascaris mystax* (fig. 3) is about an inch and a quarter in length; while the female (fig. 4) is from 2 to 3 inches, and occasionally even 4 inches.

The males are easily distinguishable by their size and the peculiar curve of their tails (fig. 3 *a*), which are coiled round on the ventral aspect, while those of the females increase slightly in thickness to within a short distance of their termination, and are perfectly straight without any curvature of the point (fig. 4 *a*).

This creature derives its specific name from two lateral projections on either side of the head resembling moustaches (fig. 1 *a*). These projections or alæ are flat, transparent and striated, being covered by the horny cuticle which envelopes the whole body (fig. 1 *d*). In front is the mouth with its three lobes (fig. 1 *b*) opening directly into the intestinal canal (fig. 1 *c*). The entire body is covered with cartilaginous rings placed side by side and exactly of the same breadth, the whole forming one continuous cuticular envelope, appearing to be regularly striated when viewed externally (fig. 1 *d*).

The posterior extremity of the female becomes suddenly narrowed to a blunt point (fig. 4 *a*), a short distance from which is the anal aperture.

The tail of the male, as I have said, is abruptly curved on itself (fig. 3 *a*), so that its dorsum is convex (fig. 5 *a*) and its ventral aspect concave (fig. 5 *b*). A short distance from the conical apex and on the ventral surface (fig. 2 *b*) is the anus (figs. 2 and 5 *c*), and a little in front of this is the genital orifice (figs. 2 and 5 *d*).

On either side of the concave surface (fig. 2 *b*) in the male is a projecting ridge (fig. 2 *f*), consisting of a number of conical tubercles placed in a row and supporting a horny membrane that stretches between them. This membrane is finely serrated; the teeth look towards the tail, and no doubt serve an important part in giving it a secure hold while embracing the body of the female.

The intestinal canal is a straight tube (figs. 1 *c* and 5 *i*) passing from the mouth to the anus, situated in the axis of the body and surrounded by loose cellular tissue. It usually contains undigested particles and portions of intestinal villi of the Cat.

After this general description, I pass to the consideration of the reproductive organs, and commence with the generative apparatus of the male as being the more simple.

By squeezing the tail carefully between plates of glass two spicula (fig. 2 *g*) are forced out of the genital orifice (fig. 2 *d*), already mentioned. These spicula are slightly curved, the curvature looking towards the body, and in the ordinary state of the parts are entirely retracted within the trunk. They are placed one before the other, they are about $\frac{1}{20}$th of an inch in length and $\frac{1}{1800}$dth in breadth; their consistence is horny, their structure tubular with a joint at the base (fig. 6 *f*), which allows of the spiculum being withdrawn (fig. 5 *g*). That part beyond the joint (fig. 6 *c*, *f*) which always remains within the body even when the spiculum is protruded, is cartilaginous in appearance, and is furnished with tooth-like projections for the attachment of the muscles engaged in its protrusion and retraction, fig. 6 *d*. Near the apex is an aperture (fig. 7 *b*) by which the seminal fluid escapes, when forced along the tubular spiculum (figs. 6 and 7 *a*).

Of the way in which copulation takes place I can speak with certainty, having in my possession a specimen in which the tail of the male is wound round so as to embrace that portion of the trunk of the female where the orifice of the vagina (Plate XXX. fig. 91 *a*) is situated, by which means the spicula are directed into its cavity.

The internal organs of generation in the male consist of a single tube variously contracted and dilated, but without any branching or division (Plate XXV*. fig. 5 *h*, *k*, *m*, *n*, *o*). This tube is placed between the integument and the intestinal canal (fig. 5 *i*) and originates in a very fine cæcal extremity (fig. 5 *o*). As it gradually enlarges it becomes much contorted, doubled backwards and forwards, surrounding the intestinal canal and occupying the posterior half of the body.

Commencing at the narrow extremity (fig. 5 *o*) is a very long tubular portion (fig. 5 *o*, *m*) that answers to the testicle; joined to this is the seminal vesicle (fig. 5 *k*),

which is dilated, and several times the diameter of the testicular tube. Lastly, the seminal vesicle contracts slightly and forms the sheath of the spicula (fig. 5 *h*). The membrane forming the cæcal extremity (fig. 8 *a*) is very thick but soon becomes thin, so that the upper portion of the testicle (fig. 5 *o*) is perfectly transparent, and at the same time homogeneous (fig. 8 *b*). On examining the membranous walls of the generative tube lower down, they first present a granular appearance (fig. 9 *b*), and then longitudinal striæ along with the finely granular structure (fig. 10 *b*). A little above the seminal vesicle (fig. 5 *k*) the tube becomes muscular, presenting transverse rugæ (fig. 11), intended no doubt to force forward the contents. This muscular portion, which may be called the vas deferens (fig. 5 *m*), connects the testicle (fig. 5 *o*, *n*) with the seminal vesicle (fig. 5 *k*), and from its contractility remains constantly empty and therefore transparent. The vesicle itself (fig. 5 *k*) is covered by reticulations of long muscular fibrillæ, giving it the appearance of being enclosed in a net. When these fibres contract, the contained semen must be expelled with considerable power. The spicula also are provided with special muscles for protrusion and retraction, but into the particulars of these I shall not enter, passing to the far more important investigation of the mode in which the seminal particles are developed.

I have already described the apex of the cæcal extremity (fig. 8 *a*) as composed of a very thick membrane, but this membrane, although perfectly well defined on the exterior is not so within. Externally homogeneous, it becomes internally more and more granular till its inner surface appears almost entirely composed of very minute granules (fig. 8 *c*).

This is the true secreting organ, for the granules when thrown off begin immediately to swell and form nucleated cells (figs. 8 and 9 *d*). The homogeneous portion of the testicle is filled with little else than these cells of various sizes, floating in a transparent fluid (fig. 8), but as it becomes gradually striated, the cells are obscured by an immense number of minute opake granules (fig. 9 *a*).

The nucleated cells (fig. 9 *d*) and granules (fig. 9 *a*) are at first intermixed without any order, but further down the granules group themselves round the cells (fig. 9 *f*), forming envelopes for each individually (fig. 10 *a*). On rupturing the testicular tube at its commencement, the nucleated cells (Plate XXVI. fig. 18 *a*) are protruded, floating in a granular fluid (fig. 18 *b*); they have a very transparent cell-wall and a nucleus attached to one side (fig. 18 *d*). About the middle we find the granular masses (fig. 19 *a*) irregular in form, but within which the nucleated spermatic cells (fig. 19 *b*) may be distinctly seen. The masses are however so delicate that the slightest pressure destroys them altogether (fig. 19 *c*). Passing as far down as the muscular portion of the testicular tube, the vas deferens (Plate XXV*. fig. 5 *m*), we find the masses much smaller in size as well as more regular in shape (Plate XXVI. fig. 20 *a*). The granular envelope is globular with a well-defined margin and perfectly opake, so as to render invisible the included cell except when ruptured by

pressure (fig. 20 *b*). This (fig. 20 *a*) is the utmost development the semen undergoes, as long as it remains in the male organs. These granular envelopes (fig. 20 *a*) appear to perform the important function of preserving and preventing the enlargement of the spermatic cells contained within them (fig. 20 *c*), for sometimes, but very rarely, a spermatic cell may be seen, which having escaped from its granular covering, has swollen up to three times its former size (fig. 20 *d*), an occurrence which, if it happened more frequently, would prevent their passage through the spiculæ, whose calibre is only capable of admitting a single granular mass at a time.

Although the further changes which the spermatic cells undergo take place within the female, and consequently totally unconnected with the generative apparatus of the male, for the sake of continuity and to prevent confusion, I shall treat of them here.

On examining the uterine contents of a recently impregnated *Ascaris mystax*, a granular fluid is observed in which a number of nucleated cells are floating, but I have never been able to observe the granular masses already described as seen in the male. The disappearance then of the granular envelope (fig. 20 *a*, *b*) is the first visible change in the constitution of the semen, and can be accounted for in many ways.

The loose granules are the debris of the cell cases; while the nucleated cells (Plate XXVI. figs. 21–36) are simply the spermatic cells (fig. 20 *c*) much enlarged, apparently by the spontaneous imbibition of the surrounding fluid. By this enlargement, a most beautifully transparent spherical cell (figs. 21, 22 *a*) is produced $\frac{1}{900}$th of an inch in diameter, enclosing, or rather having attached to its inner surface a round, discoidal nucleus (fig. 21 *b*), and within this a nucleolus (fig. 21 c), sometimes even two nucleoli.

Before I describe, however, the transformation of these nuclei into spermatic particles, it will be useful to examine cursorily the statements of others on the subject.

Wagner and Leuckardt, in their article on Semen, in Todd's Cyclopædia, speaking of the *Ascaris acuminata*, say, "the nucleus has at first a roundish shape (fig. 12 *a*), but gradually stretches itself more and more, and projects more or less outwards with its point (figs. 13, 14, 15 *a*); thus metamorphosing itself into the peduncle-like appendix of the spermatozoon, the body of which is formed from the persisting membrane of the seminal cell"(figs. 12 to 15 *b*).

Kolliker states that these cells (fig. 16 *a*) are formed four at a time within other larger cells (fig. 16 *b*); and that the elongated nuclei of Siebold and Wagner (figs. 14, 15 *a*) are mere bundles of undeveloped spermatozoa, whose form he supposes, but has never seen, to be capillary.

Reichert, in his researches on the development of the spermatozoa of the *Ascaris acuminata*, indicates a spermatic cell containing a nucleus and nucleolus (fig. 17). The cell increases in size (fig. 17 *a*), as does the nucleolus (fig. 17 *c*); but the nucleus becomes less definite, and retains its former size (fig. 17 *b*).

This, then, according to him, is the fully-formed spermatozoon, consisting of a spherical cell (fig. 17 *a*), a nucleolus (fig. 17 *c*), and indistinct nucleus (fig. 17 *b*).

This is the substance of previous investigations respecting the development of the spermatic particles. I now proceed to state my observations as to their formation in the *Ascaris mystax*, the phenomena of which will be found to differ materially from those just described.

I have previously stated, that in the more developed condition of the spermatic cells they appear as transparent vesicles, without any granular envelope (figs. 21, 22 *a*), but containing a nucleus (figs. 21, 22 *b*) and nucleolus (figs. 21, 22 *c*). The nucleus appears discoidal when seen from above (fig. 21 *b*), but lenticular when viewed in profile (fig. 22 *b*), apparently enclosed between two portions of the membranous cell-wall (figs. 22 *a* and *d*). The internal margin of the nucleus (fig. 22 *d*) soon loses its clear and defined outline (fig. 23 *b*): the granular mass constituting the nucleus undergoes a marked increase in volume, projecting in a more or less conical form towards the centre of the cell (figs. 23–28 *b*). A membrane is then formed over the whole of that part of the nucleus which is in contact with the wall of the spermatic cell (figs. 23, 24, *f*). This membrane is very distinctly seen to separate the granular matter of the nucleus from the external cell, with which it is in accurate contact. It is also an entirely new formation not to be found in any of the former stages, and is apparently produced at the expense of the most external granules of the nucleus. The margin, however, of this discoidal membrane (figs. 23, 28 *f*) is not in apposition with the cell-wall, but has a tendency to surround and enclose the nucleus; giving the membrane the form of a watch-glass, whose convexity is in contact with the spermatic cell, while in its concavity are contained the granules of the nucleus, of which it forms a part. The watch-glass form, however, is soon lost, the membrane acquiring a tendency to become more convex at its centre, assumes first the appearance of a cup, filled to overflowing with the granular substance (fig. 31 *f*), then that of a rounded cone whose margin is everted or bell-shaped (fig. 32 *f*); though this is sometimes not to be seen from the nuclear matter which surrounds and hides it from view (fig. 33 *f*).

When this convexity takes place exactly in the middle, the external cell-wall is projected in the form of a papilla (figs. 24 to 27 *a*), but this is probably only an accidental occurrence, and not the general rule.

Although at variance with the statements of WAGNER, I say this with the more confidence, as Dr. ALLEN THOMSON, with whom I had an opportunity of examining these changes, most fully coincides in the view I have here taken. A slight projection of the external cell-wall is indeed common at one period; but in no instance have I observed a greater amount of protrusion than that figured in (fig. 27 *a*). The elongation of the nucleus into a tail must be regarded as doubtful in the *Ascaris acuminata*, as it is certainly not the case in the *Ascaris mystax*.

How then is the spermatic particle formed? I have described the tendency of the nuclear membrane to become more and more convex; as this convexity increases the apex impinges more or less obliquely against the cell-wall (figs. 28, 32, 33 *a*), protruding it slightly at the same time that it is itself diverted from the straight course, becoming bent into a curved form (figs. 29, 30 *f*). By this time the granular portion of the nucleus has become much diminished in volume, still presenting, however, the nucleolus (fig. 34 *c*); while the nuclear membrane has passed from the conical to a cylindrical shape (fig. 34 *f*). During these changes in form, the nuclear membrane likewise increases in thickness, and presents a double outline (figs. 29 to 34 *f*), and refracts light strongly.

By contraction in its transverse diameter and elongation in the other, it gradually assumes the form of a test-tube (figs. 35, 36 *f*), bent, however, to adapt itself to the concavity of the spermatic cell (figs. 35, 36 *a*), often making a curve of a quarter of a circle.

From the period at which a double outline is first visible, the granules begin to disappear (figs. 29, 30 *b*), till at last the nucleus (fig. 22 *b*) becomes entirely transformed into an elongated cæcal tube (figs. 35, 36 *f*) with very thick sides, the cavity being occupied by a dark homogeneous substance, while at its mouth is found the nucleolus (figs. 35, 36 *c*) and a few granules that have not yet disappeared (figs. 35, 36 *b*); the whole, however, still enclosed within the spherical spermatic cell (figs. 35, 36 *a*).

The blind extremity next enlarges slightly, while the enveloping cell dissolves, and a flask-shaped body, the true spermatic particle, is thus set free (fig. 38).

Although the disappearance of the spermatic cell occurs normally at this period, it often happens much sooner, and thus we find the spermatic particles set free in all stages of their development, from the primitive nucleus (fig. 37 *b*) to the perfect condition (fig. 38).

In many of these spermatic particles the nucleolus (figs. 37, 38 *c*) still remains; but it also in course of time disappears, leaving the mouths of these hollow bodies apparently open (fig. 38 *g*).

Originally the nucleated cells (fig. 18 *a*) in the cæcal extremity of the testicle (fig. 8) are not more than $\frac{1}{10000}$th of an inch in diameter; as they descend they become enveloped by granules, at first forming irregular masses about $\frac{1}{1000}$th of an inch (fig. 19), but by gradual consolidation become the round, opake bodies which I have called granular masses (fig. 20 *a*), measuring $\frac{1}{1700}$th of an inch in diameter, and containing each a single spermatic cell, now, however, increased to $\frac{1}{2500}$th of an inch (fig. 20 *c*).

After the spermatic cells are introduced into the female uterus they enlarge rapidly, and are met with measuring from $\frac{1}{1000}$th to $\frac{1}{700}$th of an inch (figs. 21 to 36). The nucleus is about one third the diameter of the cell, or about $\frac{1}{2700}$th

of an inch (figs. 21 to 26 *b*); while the breadth of the spermatic particles varies from $\frac{1}{5000}$th to $\frac{1}{3000}$th of an inch, and their length from $\frac{1}{1000}$th to $\frac{1}{700}$th of an inch (figs. 37, 38 *f*, *g*).

The spermatic particles (figs 37, 38) have long been known, but their nature has not hitherto been fully determined.

Cloquet, in his elaborate work on the *Ascaris lumbricoides*, mistook them for undeveloped ova. More lately, Kolliker imagined them to be bundles of capillary spermatic filaments.

Wagner believes them to form the tail only of the spermatozoon (figs. 14, 15 *a*); while Reichert evidently did not recognise their function, as he makes a nucleated cell (fig. 17) his "Reifes Saamen Körperchen."

Lastly, Siebold conjectures these corpuscles to be spermatozoa, having seen them in contact with the ova, but appears to have gone no further into the investigation.

That these flask-shaped bodies (fig. 38) are the mature spermatic particles cannot now be doubted, as we meet with them the highest in the oviduct of the female, while near the external orifice of the vagina none but nucleated cells, or cup-shaped nuclei only are seen.

As already stated, I have never observed a spermatic particle forcing out the cell-wall, except at a very early period of its development (figs. 24 to 29); but when perfectly formed, it is set free by the total disappearance of the spermatic cell. Nothing then remains except a few granules surrounding the nucleolus (fig. 38 *b*, *c*): and, although the cell-wall is very frequently lost much earlier, the development of the nuclear membrane, which constitutes the spermatic particle, seems to go on as long as any granules remain to be transformed. Wagner, therefore, is not correct in supposing these bodies to be mere tails, because they never project as such from the spermatic cells; also, when they become free, it is by the total disappearance of the cell; and lastly, because they alone are found highest in the oviduct.

Again, speaking relatively, what immense spermatozoa these cells (figs. 21 to 36) would make, if the opinions of Wagner and Reichert were correct, from $\frac{1}{1000}$th to $\frac{1}{700}$th of an inch in diameter, or one third that of the ovum in the same animal! a circumstance altogether unparalleled.

Neither do I see any reason to believe Kolliker's statement, that the spermatic cells are formed four at a time, within a mother-cell (fig. 16), never having observed such an occurrence.

The whole of the changes I have described may easily be traced by examining the uteri of the larger individuals, as in these impregnation has in all probability not occurred till very lately; their size apparently depending on the non-evolution of fertile ova.

However, from the delicacy and transparent nature of the spermatic cell, when at its greatest development, the very best microscopes must be employed in order to detect it.

Having now fully traced the formation of the spermatic particles, from the nucleated granule, thrown off by the cæcal extremity of the testicular tube (Plate XXV*. fig. 8 *d*) to the flask-shaped body met with in the oviduct (Plate XXVI. fig. 38), I now pass to the description of the female organism, previous to entering on the development of the ovum.

The female *Ascaris*, as I have already said (Plate XXV*. fig. 4), is larger than the male (fig. 3); its tail also is straight (fig. 4 *a*) and not curled up (fig. 3 *a*).

The orifice of the vagina (Plate XXX. fig. 91 *a*) is placed about one third the length of the animal from the head; it is a simple circular opening, but so small as to be seen with the greatest difficulty. The ovarian tubes, on the other hand, may be seen filling the body with convolutions, reaching from the tail to within a short distance of the head.

The reproductive apparatus is not firmly fixed in one position, but moves backwards and forwards with considerable facility in the space between the intestinal tube and integument.

The following is the method I employed for the extraction of these generative tubes. Cut off the head and neck a little above the convolutions, and seizing the tail with forceps, by very gradually increased pressure, commencing at the tail and passed along the body towards the cut extremity, squeeze out the whole visceral contents, leaving nothing but skin behind.

To prevent entanglement of the different parts it is best to effect this expression under water. When it has been accomplished, with needles unravel the convolutions and remove the intestinal canal, which is easily recognised by its straightness and great relative diameter. If carefully performed, we obtain two very long, almost capillary tubes (fig. 92 *a*, *b*, *c*, *d*), which after enlarging (fig. 92 *f*) become united into one canal (fig. 92 *g*), whose termination has already been described as the generative orifice (fig. 91 *a*).

On examining the reproductive apparatus (fig. 92) we observe it to be composed of several portions, differing in appearance as well as structure.

These tubes commence near the tail in two extremely delicate cæcal ends (fig. 92 *a*), gradually increase in size as well as in opacity, and after performing various convolutions backwards and forwards throughout the greater part of the body (fig. 92 *b*) they suddenly become constricted (fig. 92 *c*). This opake portion is the ovary, or rather the ovarian tube, and is from 4 to 6 inches in length; it is followed by an almost transparent piece from a quarter to half an inch in length, with a constriction at either end, separating it from the ovary on the one side and the part that immediately succeeds on the other (fig. 92 *c*, *d*). Each tube now becomes greatly dilated, so as to be several times the diameter of the former portions, forming what are termed the uteri (fig. 92 *f*).

The uteri are also half an inch long, lie parallel to each other, and unite together to form a vagina of about the same length (fig. 92 *g*). The opacity of the ovaries, uteri,

and vagina (fig. 92 *a*, *b*, *f*, *g*) is dependent on the eggs with which they are distended, while the transparent portion (fig. 92 *c*, *d*) contains hardly any (fig. 90 *c*, *d*).

This part has not received any definite name, nor have its limits been pointed out; but as it is here that the important process of fecundation takes place a name becomes indispensably necessary. To this, therefore, I shall apply the name of Oviduct (figs. 90 and 92 *c*, *d*).

The upper extremity of the ovary is formed of a membranous and perfectly transparent tube (Plate XXVII. fig. 39 *h*), much thicker at the very end (fig. 39 *a*). It is from this thick portion (fig. 39 *a*), which presents a finely granular structure, that the germinal vesicles are thrown off; it constitutes therefore the true ovary.

A short distance from the extremity, one or more apparent invaginations occur (fig. 39 *b*, *b*), as if other cæcal tubes were contained within the first; but these appearances are probably caused by casts of the internal surface of the upper end of the ovary (fig. 39 *a*), which are probably thrown off at intervals.

On examining a portion of the ovarian tube where it begins to be opake (fig. 40 *h*), we observe it marked with very faint lines and minute granules; but as we descend they gradually increase in distinctness. The sides also become thicker, so that about midway the tube is formed of a homogeneous membrane externally (fig. 41 *h*), continuous with that forming the cæcal extremity (fig. 39 *h*), and a number of longitudinal ridges or striæ internally (fig. 41 *l*). Each of these longitudinal striæ (fig. 41 *l*) contains a number of granules imbedded in them, causing them to project into the interior of the tube, and giving it somewhat the appearance of a rifle barrel (fig. 41 *k*).

This is the structure presented by the ovarian tube for the greater part of its length (fig. 41), commencing as soon as granules are seen to surround the germinal vesicles (fig. 40 *k*), and becoming gradually more distinct to within a short distance of the oviduct, where the striæ disappear and the external membrane alone remains (fig. 42 *h*).

The spot where the ovary terminates and the oviduct (fig. 42 *m*) begins is marked, as already stated, by a constriction (Plate XXX. fig. 90 *c*), causing such an amount of narrowing in the calibre of the generative tube (Plate XXVII. fig. 42 *m*) as only to admit the passage of one ovule at a time (fig. 42 *o*). The whole length of the oviduct (Plate XXX. fig. 90 *c*, *d*) is characterized by transverse markings, evidently of a muscular or contractile nature, and most developed at the ends where the constrictions (Plate XXVII. fig. 42 *m*) occur. While the exterior of the oviduct is transversely ribbed (fig. 42 *n*), its interior is lined with large cells, distended with a dark granular fluid, projecting into the cavity of the tube, and having all the appearance of secreting cells.

That these cells do secrete some sort of fluid, is proved by the fact that they become turgid when the *Ascaris* has been feeding, and are, on the contrary, almost invisible from flaccidity when it has had no food.

While the ovary is filled with an almost solid mass of ovules (Plate XXX. fig. 90 *b*),

the oviduct (fig. 90 *c*, *d*) contains a transparent, finely-granular fluid with only an ovule here and there. It is therefore probable that a peristaltic action being set up by the transverse muscular fibres of the oviduct, the ovules are detached and forced forwards singly into the uterus (fig. 90 *f*). Another constriction (fig. 90 *d*), but not so well marked as the first (fig. 90 *c*), indicates the termination of the oviduct, beyond which the tube becomes suddenly dilated into the uterus (fig. 90 *f*), which is several times the diameter of the preceding portions.

The uterus commences by a rounded extremity or fundus (fig. 90 *f*), into which the oviduct (fig. 90 *c*, *d*) opens, and gradually becomes narrower until it unites with its fellow (fig. 92 *f*) to form one tapering vagina (fig. 92 *g*). The uteri are formed of an external membrane (Plate XXVII. fig. 43 *h*) lined with broad, flat, quadrangular cells (fig. 43 *r*), each of which presents an oval or round nucleus (fig. 43 *s*) and a central nucleolus (fig. 43 *t*), of great beauty and distinctness.

Lastly, the vagina presents some transverse muscular rugæ externally, with nucleated flat cells internally, completing thus the complex structure of the female reproductive apparatus.

The cæcal extremity of the ovary (fig. 39 *a*) throws off, as I have said, small rounded granules (fig. 39 *c*), which enlarge rapidly and form the germinal vesicles (fig. 39 *d*).

These granules are formed by the thickened apex, and give its substance a semi-transparent structure (fig. 39 *a*). A fluid likewise fills the upper part of the ovarian tube, of an albuminous nature, and apparently the secretion of its walls. The germinal particles or granules when first thrown off, by the internal surface of the cæcal apex, are only $\frac{1}{10,000}$th of an inch in size (fig. 39 *c*), but begin almost immediately to increase to several times their original bulk, become vesicular (fig. 39 *d*), and present a nucleus within each cell (fig. 39 *f*). These are the germinal vesicles, and their included nuclei are the germinal spots (fig. 39 *d*, *f*).

Kölliker describes the invaginated appearance of the upper end of the ovary (fig. 39 *b*) as large cells whose nuclei are the germinal spots, which are set free by the successive openings of these large cells. (See Müller's Archiv for 1843.)

He further states that the germinal spot is first formed, and around it the germinal vesicle is developed like a primitive cell round its nucleus. My observations, however, lead me to believe that a germinal particle is first formed (fig. 39 *c*), which appears semiopake and solid (fig. 44 +); and that by the imbibition of the surrounding fluid the external membrane or layer of this germinal particle is distended, and thus forms the germinal vesicle (fig. 44 *b*), leaving the solid contents to form a central nucleus or germinal spot (fig. 44 *a*).

The germinal vesicles (fig. 39 *d*) are now $\frac{1}{3500}$th of an inch in size; as they pass down the ovary they disappear (fig. 40 *k*), becoming enveloped by opake granules (fig. 39 *g*).

At first, that is to say in the upper part of the ovarian tubes, these granules (fig. 39 *g*) are perfectly free, floating loosely in the fluid that surrounds them and lying in con-

tact, though not adherent to the germinal vesicles (figs. 39, 40 *d*). A little further down we observe the whole contents of the tubes to become gelatinous, and to consist apparently of nothing but granules. On applying gentle pressure so as to rupture this portion of the tube, the granular contents are easily forced out and may be seen to break up into semitransparent masses (figs. 45, 46).

Each mass (figs. 45, 46 *d*) constitutes an ovule, small and imperfect, but containing all the parts essential to an ovule. In the centre of each is a germinal vesicle (figs. 45, 46 *b*) enclosing a germinal spot (figs. 45, 46 *a*), and surrounded by opake granules (figs. 45, 46 *c*) imbedded in a transparent jelly-like substance.

These granules (figs. 45, 46 *c*) are of a fatty nature, and constitute the yolk in its primitive condition.

The ovule in this stage of its development is very irregular in form (figs. 45, 46), sometimes caudate, sometimes triangular, and at others round. They are all flattened and transparent, so that the contained germinal vesicle and spot may be distinctly recognised (figs. 45, 46 *a*, *b*) when viewed separately.

The distinctness of the outline which the transparent substance of the ovule presents (figs. 45, 46 *d*), led me at first to suppose that it was caused by a delicate enveloping membrane; but from further research I am led to believe that no such membrane exists, but that the distinct and regular outline is owing to the solidity and high refractive power of the clear substance (figs. 45, 46 *d*).

When the ovule is of a caudate or triangular form (fig. 45 *d*), the clear substance does not present so distinct an outline at the apex as elsewhere. From its being invariably placed nearest the centre of the ovarian tube it is probably the last formed, and therefore less consolidated than the other parts. It is doubtful whence the vitelline granules are formed (figs. 45, 46 *c*); they may be either thrown off by the cæcal extremity along with the germinal particles, separate spontaneously from the surrounding fluid, or be formed by the striated walls of the ovary.

I believe that the large granules (fig. 39 *g*), first seen to surround the germinal vesicles, are thrown off by the blind extremity, and become attached to them by the coagulation, or rather the consolidation of the surrounding albuminous fluid (figs. 45, 46). But when the ovule arrives at the striated portion (fig. 41) the number of yolk-granules is greatly increased (Plate XXVIII. figs. 48, 51 *c*), leading to the supposition that the granules contained within the striæ (Plate XXVII. fig. 41 *k*) are thrown off, and become imbedded in the clear substance of the ovule (Plate XXVIII. figs. 48, 51 *d*).

The possibility of such an occurrence cannot be doubted, both from the absence of a distinct limiting membrane round the ovule, as well as from the gelatinous nature of the clear substance, having a tendency to entangle and detain anything that may come in contact with it.

The germinal vesicles maintain exactly the same appearance, but increase a little in size (fig. 48 *b*), become globular in shape, and contain a highly refractive fluid

which makes them easily distinguishable. But while the germinal vesicle enlarges so slightly, the ovule increases rapidly as it descends the ovarian tube, passing from $\frac{1}{1000}$th to $\frac{1}{250}$th of an inch in diameter. This increase is not only owing to the granules which appear to be accumulated from the walls of the ovary, but also to an increase in the amount of the clear substance.

Some have supposed the increase in the number of the granules to be owing to their spontaneous division, while others have thought that they are formed by the germinal vesicle. To the first theory it may be objected, that the granules remain very much of the same size throughout, while those contained in the same ovule correspond exactly, which would not be the case if each granule were dividing into two or more. That the granules are not produced by the germinal vesicle is evident from its remaining very much of the same size and perfectly entire.

On the other hand, the reasons for supposing them to be formed from the walls of the ovary are, that the granules become more and more numerous as the ovule passes down the tube. Secondly, that the granules (Plate XXVIII. figs. 48, 51 *c*) exactly resemble in size, colour and form, those produced by the internal surface of the ovary (Plate XXVII. fig. 41 *k*). Lastly, because the striæ (fig. 41 *l*) of the ovary cease just where the ovule ceases to become larger (fig. 42 *h*).

The fact that the ovula were packed edgewise in this part of the tube (Plate XXVIII. fig. 49 *c*) was pointed out to me by Dr. Allen Thomson; also that three or four occupied the same plane (fig. 50 *d*). Hence the necessity of the ovula assuming the triangular form (Plate XXVII. fig. 47); the only form met with as long as the ovula present the appearance of a dense opake mass (Plate XXVIII. fig. 49).

The ovary is about $\frac{1}{120}$th of an inch broad at its widest part; and here it is that we find the ovula packed generally four on a plane (fig. 50 *d*), with their edges presenting externally (fig. 49 *d*), appearing long and narrow when viewed in profile (fig. 49 *c*), but broad and triangular when seen in front (fig. 49 *b*), entirely filling the transverse section of the tube (fig. 50 *w*).

As the granules increase in number so the clear substance becomes more and more obscured, till at length the whole ovule appears to be composed entirely of vitelline granules. The germinal vesicle (fig. 48 *b*) can no longer be distinguished, the ovule being perfectly opake (fig. 50 *d*). In this state they are met with near the entrance of the oviduct (fig. 50 *d*); they now become separated, detached singly from the mass, lose their triangular form (fig. 51), and by passing through the first constriction (Plate XXX. fig. 90 *c*) enter the glandular portion of the reproductive apparatus (Plate XXVII. fig. 42 *n*; Plate XXX. fig. 90 *c*, *d*).

When impregnation has taken place, the ovule first meets with the spermatic particles (Plate XXVII. fig. 42 *p*) in this part of the oviduct (fig. 42 *n*), but I think it may be best to trace the further changes it undergoes in the unimpregnated state, as they enable us to explain, or what is of more consequence, to contrast the appearances presented by the fecundated ovum with those of the non-vivified egg.

As soon as the ovule enters the glandular portion (Plate XXX. fig. 90 *c*, *d*) it floats free in a clear fluid secreted by the cells lining the oviduct. The ovule becomes thicker and more rounded, losing the flattened form it assumes that of an oblong sphere (Plate XXVIII. fig. 52). A granular chorion (fig. 54 *f*) begins to form, surrounding the ovule and constituting an elastic shell (fig. 55 *f*). While this external chorion is forming, however, the internal contents are undergoing change, the vitelline granules become much smaller in size (fig. 52 *c*), the germinal vesicle and spot disappear, and in their place we find a number of large transparent globules (fig. 53 *e*), having much more the appearance of oil than of being formed by cells. These globules are apparently formed partly by the disappearance of the germinal vesicle, and partly by the separation of the oil from the granules of the vitellus. After a time these oily globules (fig. 54 *e*) approach the circumference and disappear (fig. 55 *e*), while the minute opake particles (figs. 54, 55 *c*) pass towards the centre. The ovule now begins to shrink, the chorion becomes more granular and thicker (fig. 56 *f*), while from its inner surface a delicate membrane separates (fig. 57 *o*), and contracting on the particles contained within it, forms one molecular mass, spherical and perfectly opake (fig. 56 *o*).

This false ovum (fig. 56), if I may be allowed to call it so, is surrounded externally by a peculiar granulated chorion or shell of an irregularly ovoid form (fig. 56 *f*), within which, but separated from it by a clear fluid, is the opake spherical mass already described, with its delicate membrane (fig. 56 *o*). On rupturing one of these false ova, the opake mass is found (fig. 57 *c*) to consist of granules and a few oil-globules. While the fertile egg does not remain stationary but advances, stage after stage, till the worm is produced, the false ovum undergoes no further development, but when expelled from the vagina of the *Ascaris* rapidly decays.

In all those instances in which the spermatic fluid was imperfectly developed, or had not ascended higher than the uterus, false ova alone existed (fig. 56), having all the characters of those just described, and incapable of further change.

From this we may infer, that after the formation of the chorion (figs. 54 to 56 *f*) impregnation of the ovule is impossible; and as the chorion is formed while the egg is yet within the oviduct (Plate XXX. fig. 90 *c*, *d*), fertilization must take place, if at all, while the ovule is still in the oviduct, and before the formation of that envelope.

Let us now return to the fertile condition of the female, and trace the development of the ovum as it threads the oviduct (fig. 90 *c*, *d*).

The ovula are here surrounded by the spermatic particles; here it is that fecundation takes place,—by which alone fertile ova can be produced.

This conversion of the ovule into the fertile ovum, I shall endeavour to describe according to my own repeated observations.

Immediately that the ovule passes the constriction which terminates the ovary and enters the oviduct (Plate XXVII. fig. 42 *n*), it comes in contact with the semen (fig. 42 *p*), and a marked change in its form and appearance takes place. It is at this

period of an oblong shape (Plate XXVIII. fig. 58), and appears to be entirely made up of distinct opake granules, which completely hide the germinal vesicle and spot (figs. 58 to 60 *c*).

The edge at first is uniform, that is to say, not ragged, but presents a distinct outline (fig. 58 *d*). Sometimes a faint trace of the germinal vesicle may be perceived (fig. 60 *b*), although the ovule is never so transparent as to allow objects on the other side to be seen through it.

In the upper portion of the oviduct (Plate XXVII. fig. 42 *n*) we find the spermatic particles (fig. 42 *p*) in their greatest perfection (Plate XXVI. fig. 38 *g*), presenting the flask-like appearance, and in such numbers as to fill all the interstices between the ovulum and the sides of the oviduct.

At first the margin of the ovule (Plate XXVIII. fig. 51 *d*) is entire, that is to say, the clear substance presents a distinct, even and continuous surface, but a little further we perceive the spermatic particles to be closely applied against the ovule, depressing its surface in some places (fig. 60 *g*). Another step in advance, and all the ovula present an irregular rupture in some part or other of their periphery (figs. 58 to 60), generally at one side (fig. 58), and frequently in several places at the same time (figs. 59, 60).

Dr. Allen Thomson, having frequently examined these appearances, permits me to say that he agrees entirely with the above description of the phenomena.

That these appearances do not arise from pressure applied during the examination I am perfectly satisfied, having repeated my observations above a hundred times, and varied them so as to remove all possibility of such an occurrence. The effect of pressure on the ovule is also very different from that just described, as I have repeatedly seen; for if an entire ovule be squeezed between glasses, the vitelline granules coalesce, the clear substance dilates, and at last the whole lapses into a yellowish fluid.

From the immense number of ovula in which I have seen this partial rupture of the vitellus, and the want of success in my endeavours to cause a similar protrusion (figs. 58 to 60 *c*) of the yolk-granules by compression, I am irresistibly led to the conclusion that this is a vital phenomenon consequent on some natural cause, and not the result of accidental violence. Further, to make all certain, I divided the oviduct at both ends, washed away all loose ovules by a gentle stream of water, and then superimposed, as carefully as possible, a piece of the thinnest glass. No other pressure than the weight of the thin piece of glass was applied, yet every one of the ovules, as they slowly found their way out of the oviduct, presented in some part or another this rupture of the surface.

The reason why all the granules do not escape, arises from their being imbedded, not in a fluid, but in a sort of gelatinous substance, easily broken up, it is true, but sufficiently coherent to retain the granules in their places.

To what then are these appearances owing, and how are they produced? I have

already said that the spermatic particles become first applied to the periphery of the ovule (Plate XXVIII. fig. 60 g); and that a little later they are seen to indent the surface.

Still lower we find the spermatic particles imbedded in the substance of the ovule to a greater or less degree, and surrounded by the vitelline granules that have been displaced (fig. 58 *g*). Sometimes only one is seen to be thus imbedded (fig. 58*g*), but more commonly several spermatic particles are applied at the same place, with their closed ends directed in general towards the centre (figs. 59, 60 *gg*). Penetration then takes place, the particles passing into the substance of the ovule, amongst the vitelline granules, and surrounded by the clear substance (figs. 59, 60 *h*).

I have seen the spermatic particles in all stages of penetration, from mere contact (fig. 60 *g*) to perfect involvement within the ovule (figs. 59, 60 *h*). In their course they appear to create very little displacement, passing readily in all directions amongst the granules; their transparency and high refractive power rendering them easily distinguishable when near the surface (figs. 59, 60 *h*).

Of the possibility of penetration, no one who has ever seen an ovule of this *Ascaris* can have the slightest doubt, composed as it is of a clear gelatinous substance, without an enveloping membrane; while the very granules it encloses (fig. 51 *c*), there is great reason to believe, have at some period entered from without.

With regard to the probability, there is the breaking up of the surface of the ovule in certain places (figs. 58 to 60 *c*), rendering it still easier for the entrance of the spermatic particles.

Secondly, the application and adhesion of the seminal particles to the broken edge (figs. 58 to 60 *g*).

Lastly, their having been seen (figs. 59, 60 *h*) within the ovule, imbedded in its substance, and surrounded on all sides by the vitelline granules.

As I have shown the possibility and even the probability, it remains only for me to prove the correctness of my observations.

That the spermatic particles seen by me (figs. 59, 60 *h*) might possibly have been external to the ovule, will naturally occur to every microscopic observer. But this doubt is manifestly unfounded, for many reasons.

First. The particles (figs. 59, 60 *h*) could not have been lying upon the ovule, because vitelline granules were visible above them; that is to say, by distancing the object-glass from the object, the spermatic particles became indistinct, and a layer of granules came into focus, entirely covering the space they occupied.

Secondly. They could not have been below, for the ovules are much too opake to be seen through.

Thirdly. The seminal particles observed were only in focus, when the margin of the ovule was in focus, and must therefore have been on the same plane. But, as the ovule is a more or less spherical body, the focus of its margin corresponds with

that of its centre; hence the particles already mentioned could only have existed in the substance of the ovule.

Another objection might be raised; admitting the particles to have been situated in the substance of the ovule, might they not have been oil-globules, like those seen in the unfecundated egg?

The reply to this is most satisfactory; for not only were the particles elongated and cylindrical (figs. 59, 60 *h*), but the one extremity was closed, and the other open; hence they could only have been the spermatic particles of the male.

I cannot enter here upon the consideration of the changes which occur in the mammiferous ovum, but confine myself to a few remarks.

Dr. Martin Barry says, "On one occasion, in an ovum of $5\frac{1}{4}$ hours, I saw in the orifice of the membrane" (the external membrane of the ovum) "an object very much resembling a spermatozoon which had increased in size . . . I am not prepared to say that this was certainly a spermatozoon, but it seems proper to record the observation."

Now, whether we believe Dr. Barry to have really seen the penetration of the spermatozoon into the mammiferous ovum, or whether we agree with Bischoff and most other distinguished authors, and deny the correctness of Dr. Barry's observation, as well as the possibility of any such occurrence, the present investigations appear to be the first in which the fact of the penetration of spermatozoa into the ovum has been distinctly seen and clearly established, in one of the most highly organized of the entozoa.

The accuracy of these observations is satisfactorily borne out, and will be more readily admitted, as we continue to trace the progress of the ovule, the changes of the spermatic particles, and the development of the ovum.

The accompanying drawings present these appearances as exactly as possible, being taken originally from actual specimens, by means of a camera lucida. In some we find the broken margin of small extent (figs. 58, 60 *c*); in others it embraces nearly one half of the circumference (fig. 59 *c*). Some give a faint indication of the contained germinal vesicle (fig. 60 *b*); but in most the granules are too opake to admit of its being seen at all (figs. 58, 59). Let us now pass to the examination of the changes which take place in the ovule subsequent to the penetration of the spermatic particles.

The ovule, it must be remembered, is still in the oviduct; the secretive portion of the system, and that in which the false ova receive an enveloping membrane (figs. 54 to 57 *f*). Almost immediately therefore that the ovule has entered the oviduct, traces of a chorion, as a very delicate membrane, begin to appear (fig. 61 *f*), covering at first those portions only of the surface that remain smooth and entire (figs. 61, 62 *f*); but as we descend, enclosing by degrees the ruptures themselves, and surrounding thus the whole ovule in one continuous envelope (fig. 63 *f*), with

its germinal vesicle, vitelline granules, and the spermatic particles that have penetrated its substance (fig. 63 *h*). Having entered the vitellus in different directions, and to various depths, the spermatic particles (figs. 61, 62 *g*) lose their characteristic form, and swell into irregular masses (figs. 61 to 63 *h*), transparent, presenting a distinct outline, and highly refractive.

These transformed spermatic particles (figs. 61 to 63 *h*), being situated in the midst of the opake granules (figs. 61 to 63 *c*), give the ovum a most peculiar mottled appearance (figs. 62, 63).

That they are spermatic particles cannot be doubted, as we meet with them in all stages of transformation (figs. 61, 62 *g*, *h*); while the regularity in shape presented by the large oil-globules of the false ovum (figs. 53 to 55 *e*) is absent, as well as the running together of the granules (figs. 54, 55 *c*) by which they are formed.

The ovum immediately after the entrance of the spermatic particles (figs. 58 to 60 *g*) begins to acquire a chorion (figs. 61 to 63 *f*). The formation of this chorion does not appear to be dependent at all on the penetration, but to be owing to the ovum having reached that part of the oviduct by which the membrane is secreted, because we find it occurring even in the unfecundated state.

This chorion, however (figs. 61 to 63 *f*), differs from the granular shell of the false ovum (figs. 54 to 57 *f*), in being perfectly smooth, membranous and transparent (figs. 61 to 63 *f*), appearing as a single dark line, imperfect indeed at first, where protrusion of the granules exists (figs. 61, 62 *f*), but gradually encircling the whole ovum (fig. 63 *f*).

When first formed, the chorion is flaccid (fig. 63 *f*), and the ovum appears of an irregular shape (fig. 63); but by the imbibition of fluid it swells up, becomes tense and spherical (Plate XXIX. fig. 64 *f*).

The spermatic particles (Plate XXVIII. figs. 61 to 63 *h*), after becoming enclosed and swollen, begin to disappear (Plate XXIX. fig. 64 *h*), probably by solution, leaving in their place a transparent fluid (fig. 65 *m*).

The vitelline granules that previous to the impregnation of the ovum formed one uniform, opake mass (Plate XXVIII. fig. 58 *c*), and partially broken up (figs. 59 to 63 *c*) by the penetration and swelling of the spermatic particles (figs. 59 to 63 *h*), are now still further excavated, separated, and detached into distinct masses by the solution of those particles (Plate XXIX. figs. 64, 65 *c*). This appears not only to be owing to the solution of the spermatic particles themselves, but to some direct influence that their solution has on the yolk, for many of the granules disappear entirely; while others are changed, both as regards colour and size (fig. 66 *n*), a transformation totally different to that which I have described as taking place in the false ovum (Plate XXVIII. fig. 56 *c*).

Sometimes the whole vitellus is thus broken up (Plate XXIX. fig 65 *c*), giving the ovum a beautifully mottled appearance; but more commonly it is only the surface

that is first affected (figs. 64, 66 *c*), the process of disintegration gradually passing towards the centre.

This mottled appearance (fig. 65) has been noticed by REICHERT in the egg of a *Strongylus*, and ascribed by him to the formation of cells within the yolk, which is certainly not the case in the fertile ova of the *Ascaris mystax*.

When there is much disintegration, the germinal vesicle may be seen with its germinal spot or nucleus (figs. 64 to 66 *b*, *a*), and occasionally within this (figs. 64, 65 *a*), again, another nucleolus (figs. 64, 65 *k*).

But in general the germinal vesicle (figs. 64, 65 *b*) cannot be seen; for, as the erosion of the yolk commences on the surface (fig. 66 *c*) and gradually passes towards the centre, the vesicle is always covered by a layer of opake granules.

As the solution of the yolk goes on, the opake granular mass in the centre becomes less and less, leaving a clear margin of fluid surrounding it on all sides (fig. 66 *c*). Some granules, however, escape, and are seen floating in the fluid; but they are larger in size, and more transparent than the original yolk-granules (fig. 66 *n*).

About this period the ovum acquires another chorion, consisting generally of two membranes, and becomes more elliptical in form.

The three membranes that surround the ovum at this period, as they are all formed in the same way, and are exactly alike, may be considered as three layers of the same chorion (fig. 66 *f*), secreted at different times by the oviduct.

When the granular mass has been much reduced in size (fig. 66 *c*) it suddenly loses its opacity; and thus the whole vitellus is transformed into a few large, nearly transparent granules (fig. 67 *n*), among which we look in vain for a germinal vesicle; and only now and then are we able to distinguish one granule to be a little larger than its fellows (fig. 67 *a*), and to contain within it a dark spot (fig. 67 *k*).

In short, the germinal vesicle (figs. 64 to 66 *b*) ruptures, when disintegration has gone on to a certain length, and its disappearance is immediately followed by the transformation of the remaining vitelline granules.

I propose to call these transformed or altered granules by the name of embryonic granules (figs. 67, 68 *n*), since they appear about the same time as the embryonic vesicle, and with it assist in forming the embryo.

After the rupture of the germinal vesicle (figs. 64, 65 *b*), the interior of the egg is filled with the embryonic granules (fig. 67 *n*), not, however, packed close like the vitelline, but floating loose. About the centre of these granules, one a little larger than the rest (fig. 67 *a*) may sometimes be seen, having within it an opake spot (fig. 67 *k*). On comparing these with the nucleus and nucleolus of the germinal vesicle (fig. 64 *a*, *k*) before its rupture (fig. 64 *b*), I found them to resemble each other completely, having the same size, shape, and appearance, the same degree of refraction, and the same position.

The germinal vesicle immediately before its rupture (figs. 64, 65 *b*) is $\frac{1}{1000}$th of an

inch in diameter; its nucleus, or as it is commonly called, the germinal spot (figs. 64, 65 *a*), is $\frac{1}{4000}$th of an inch, and the contained nucleolus $\frac{1}{8000}$th of an inch (figs. 64, 65 *k*).

After the rupture, the nucleus and nucleolus (fig. 67 *a*, *k*) are of exactly the same size, $\frac{1}{4000}$th and $\frac{1}{8000}$th of an inch respectively.

But soon the nucleus, which is at first solid (fig. 67 *a*), begins to enlarge, swells up, and constitutes a transparent cell (fig. 68 *a*); while the nucleolus remains of the same size (fig 68 *k*), forming, in short, an embryonic vesicle and spot.

As soon as the embryonic vesicle (fig. 68 *a*) begins to form, a membrane separates from the internal surface of the chorion (fig. 69 *f*), and gradually contracts (fig. 69 *o*) on the embryonic granules (fig. 69 *n*), till a perfect sphere is formed, whose breadth is nearly equal to the lesser internal diameter of the ovum (fig. 70 *o*).

How, or from what this membrane (fig. 70 *o*) is formed, I cannot speak with certainty, further than that it separates from the inner surface of the chorion (fig. 69 *f*), and contracts on the embryonic granules (fig. 69 *n*) to form a true or embryonic yolk, exactly in the same way that it does (Plate XXVIII. fig. 56 *o*) in the false ovum (fig. 56) to form the opake or false yolk (fig. 56 *c*).

It is probable, therefore, that the formation of the yolk-membrane (Plate XXIX. fig. 70 *o*) is a physical process, unconnected with the fertility, or the individual vitality of the egg, as it takes place exactly at the same time, and in the same manner in the unfecundated or sterile ovum, into which the spermatic particles have never entered.

When the membrane of the embryonic yolk first separates from the inner surface of the chorion, it encloses not only the embryonic granules (fig. 69 *n*), vesicle (fig. 68 *a*) and spot (fig. 68 *k*), but likewise the clear fluid in which they float; although, as contraction goes on, this fluid passes through the membrane (fig. 70 *o*) and occupies the space between it and the external envelopes (fig. 70 *f*). It therefore acts the part of a sieve, allowing the fluid to pass, but retaining the granules, and bringing them within the influence of the embryonic vesicle (fig. 70 *a*).

The embryonic yolk is at first large and irregular in shape (fig. 69 *o*), but it soon becomes perfectly spherical (fig. 70 *o*), $\frac{1}{600}$th of an inch in diameter, enclosing an embryonic vesicle and spot, whose sizes are $\frac{1}{2000}$th and $\frac{1}{8000}$th of an inch respectively (figs. 70 *a*, *k*).

At this period the egg (fig. 70) is oval, its longer diameter being $\frac{1}{310}$th of an inch, and its shorter $\frac{1}{350}$th of an inch; its membranes are firm and resisting (fig. 70 *f*); and with this amount of organization it is expelled from the body of the mother.

The perfect ovum, therefore, consists of two or three homogeneous membranes, united to form one oval shell (fig. 70 *f*), some limpid fluid (fig. 70 *m*), a spherical embryonic yolk membrane (fig. 70 *o*), embryonic granules (fig. 70 *n*), an embryonic vesicle (fig. 70 *a*), and its nucleus, the embryonic spot (fig. 70 *k*).

Compare the fecundated (fig. 70) with the unfecundated ovum (Plate XXVIII. fig. 56), and one is immediately struck with the immense difference that exists between them.

In the false ovum there is no embryonic vesicle, no embryonic spot; while the substance that does exist, is apparently the colouring matter of the vitelline granules, collected into a structureless yolk (fig. 56 *c*), surrounded by a membrane (fig. 56 *o*), and the whole enclosed in a granular chorion (fig. 56 *f*) instead of a laminated shell (Plate XXIX. fig. 70 *f*).

The formation of the embryonic yolk membrane is not the effect of fecundation (fig. 70 *o*), because we see one produced in the sterile ovum; but after the entrance, swelling up, and solution of the spermatic particles, certain other changes are produced within the ovule, which do not occur otherwise.

The spermatic particles, by penetrating into the ovule, exert over it an influence of three distinct and somewhat opposed kinds.

First. A preservative effect, preventing the decay, disappearance, and blending together of the vitelline granules, the germinal vesicle and spot.

Secondly. A destructive or solvent influence, by which the vitelline granules and germinal vesicle are, after a time, gradually dissolved.

Thirdly. A power of transformation, by which the vitelline are changed into embryonic granules.

The preservative, destructive and transformative influences commence, as we have seen, with the union of the spermatic particles and ovule; they are conferred by the spermatic particles on the ovule, which continues to exist, while the sperm is destroyed by the act; and lastly, they appear all three to be of a purely chemical nature.

These properties once acquired, continue not only throughout the whole life of the creature, but remain after the death of the individual.

To one or other of these influences may all the changes that take place in the living body be ascribed, with the exception of those that are referrible to life alone.

But before entering on the consideration where life commences, and in what part it resides, it is essentially necessary that we make ourselves acquainted with the changes it effects in the ovum, by which the egg is transformed into an embryo, in all respects similar to the parent *Ascaris*; like it, capable of voluntary motion, assimilation, and the power to produce other ova.

These are most beautifully seen in the egg of the *Ascaris mystax*; but as they have been already described by far abler authors, I shall confine myself to a very brief outline of the changes as they occurred under my own observation.

The first alteration that the ovum undergoes is the division of the embryonic spot (Plate XXIX. fig. 72 *k*), and elongation of the embryonic vesicle (fig. 72 *a*).

This division is sometimes seen even before the germinal vesicle has disappeared (fig. 65 *k*), but does not take place normally till after the formation of the true or embryonic yolk (fig. 70 *o*).

The division of the nucleus (fig. 72 *k*) is immediately followed by that of its cell, the embryonic vesicle (fig. 73 *a*); and thus two embryonic vesicles are formed, each containing a separate nucleus or spot (fig. 73 *k*).

As soon as this has occurred, the two cells (fig. 73 *a*) are seen to separate and approach the opposite sides of the yolk; a portion of the yolk membrane (fig. 73 *o*) is protruded outwards, by the application of one of the embryonic cells against it. At first this protrusion is very slight (fig. 73 *o*), but by the continued movements of the vesicle (fig. 74 *a*) it becomes more and more increased, till at last the yolk assumes an oblong shape, with a constriction about the middle (fig. 74 *o*). The constriction gradually deepens, till at last two yolks are formed (fig. 75 *o*) by the sudden division of the investing membrane.

I have repeatedly watched this process as it occurred under the microscope, and found that while the division of the embryonic vesicles takes from five to ten hours, the division of the yolk does not take more than thirty minutes.

The separation of the yolk into two parts is, I think, entirely a mechanical effect, and not produced by vitality inherent either in the yolk-granules (fig. 75 *n*) or membrane (fig. 75 *o*). For, besides the rapidity of its accomplishment, I have observed that the embryonic vesicles (figs. 73, 74 *a*) continue during the progress of the division to revolve round and round in circles; the one moulding the newly projected portion of the yolk membrane (figs. 73, 74 *o*) into a spherical form, while the other prevents the original part from collapsing.

Sometimes, when the formation of a yolk has been prevented by immersion in some preservative fluid, the division of the embryonic vesicle still takes place (fig. 71 *a*), and the two are generally seen occupying opposite ends of the egg, but without any membranous or granular investment.

As soon as the yolk has divided into two (fig. 75 *o*), a pause occurs. The two embryonic vesicles (fig. 75 *a*) remain stationary, their nuclei (fig. 75 *k*) subdivide, they themselves elongate, and ultimately separate into two each. Thus four embryonic vesicles are formed, two within each yolk mass, which, by the repetition of the same process, is redivided into four (fig. 76 *o*).

Occasionally, when one embryonic vesicle divides more rapidly than its fellow, three yolk masses are produced; but this is rare, and not usually the case.

As this process is repeated from time to time, the number of the yolk masses increases from 4 (fig. 76 *o*) to 8, 16, 32, 64, 128, 256, &c., till they (fig. 77 *o*) become so numerous and so minute as to appear like granules (fig. 78 *o*); yet each granule (fig. 78 *o*) is composed of a nucleus, an embryonic cell, yolk substance, and yolk-membrane.

From the immense amount of subdivision and the number of interspaces caused by the spherical form of the granules, the whole egg is filled with them, giving it a dark or opake appearance (fig. 78).

A membrane appears to form on the external surface of this opake mass, corre-

sponding to the internal surface of the chorion; the production of which is attended with a loss of some of the most superficial granules.

Next, a depression (fig. 79 *p*) of this membrane takes place on one of the sides.

The depression is at first slight (fig. 79 *p*), but it gradually increases, forcing some of the granules before it, while others disappear by solution into a limpid fluid, which, passing through the membrane, occupies the space between it and the shell (fig. 79 *r*).

A hemispherical mass is thus produced, but the central portion continues to advance, forming first a cup-like depression (fig. 80 *p*), and, ultimately touching the membrane of the opposite or convex side, unites with it, to produce a thick circular ring (fig. 81 *p*).

This fleshy ring (fig. 81 *p*) soon presents a constriction at one point (fig. 82 *s*). The constriction, by deepening, divides the ring, which is thus transformed into a cylindrical worm, bent round, so that the two ends are in apposition (fig. 83 *s*); and covered externally with the membrane (fig. 83 *p*), now become thick, while internally we still see nothing but granules (fig. 83 *o*).

As the body elongates, the two ends overlap, and are seen to be pointed. At first the overlapping is slight (fig. 83 *s*), but it gradually increases (Plate XXX. figs. 84, 85, 86 *s*), till at length the little worm forms nearly two turns of a spiral (figs. 87, 88 *p*), surrounded on all sides by fluid (figs. 87, 88 *r*) and the chorion, or shell (figs. 87, 88 *f*).

By rupturing the egg, the embryo worm is set free (fig. 89), and is seen to possess the three-lobed mouth (fig. 89 *s*), peculiar to the genus, and a very thick cuticle (fig. 89 *p*) enclosing a number of untransformed granules (fig. 89 *o*).

The development of the embryo is best observed by placing the females entire in spirits of turpentine for a fortnight or three weeks; at the end of which I have found the ovaries distended with ova, all of which contained young worms, not only fully developed (figs. 84 to 88), but alive, endeavouring their utmost to rupture the chorion, by rolling themselves up into a tight spiral (fig. 88 *p*), and then suddenly reversing the coil.

Let us now return to the consideration as to which of these changes are vital, and which physical.

The most remarkable, as well as most apparent change that takes place in the ovum subsequent to penetration, is the division of the yolk, a phenomenon, which, although peculiar, seems to be entirely mechanical.

For the yolk membrane when at rest, as seen before the division of the yolk, assumes the spherical form, by its own molecular attraction (Plate XXIX. fig. 70 *o*); but when drawn out by the embryonic vesicles, acquires first a cylindrical shape, then that of an hour-glass, because that part of the membrane occupying the centre of the cylinder, having nothing to keep it distended, collapses; while the two ends are prevented from doing the same by the constant movements of the embryonic vesicles (fig. 74 *a*).

When the hour-glass form (fig. 74 *o*) has been once attained, the molecular attrac-

tion of the membrane tends no longer to draw it into a single sphere, but into two globules; and thus the division of the yolk is completed.

This view is further confirmed by the fact, that while the first steps of this process take, comparatively speaking, a long time, as soon as the hour-glass form has been once acquired, complete division is effected so suddenly that it is invisible; while a violent oscillatory motion is from the same cause communicated to the yolk masses, taking some little time to subside.

The division and movements of the embryonic vesicle, on the contrary, can only be ascribed to vitality.

That this division cannot be produced by the action of the spermatic particles on the embryonic vesicle, is evident from the fact, that it does not take place from without inwards, but from within outwards. The embryonic spot divides first; and this I have even seen to take place before the germinal vesicle has been ruptured, while it was still entire (fig. 65 *k*), and consequently long before the seminal fluid could possibly exert any influence over its nucleoli, imbedded as they are in the substance of the yet solid nucleus (fig. 65 *a*), surrounded by fluid, and protected by the germinal vesicle (fig. 65 *b*).

The fissiparous growth of the embryonic spot proves beyond a doubt that its division is caused by vitality inherent in it.

The embryonic vesicle, although owing its division to the nuclei it encloses, is also alive, because it grows in size, and when divided we see it move, not as some suppose, by mere electric repulsion; for I have most distinctly seen it continue to revolve in different directions, and in circles of various diameter.

I am inclined to the belief that these movements of the embryonic vesicles are caused by vibratile cilia, from a certain amount of commotion among the yolk granules immediately surrounding the vesicle, observable only when the latter is in motion.

But the embryonic vesicle and spot are nothing else than the nucleus and nucleolus of the germinal vesicle. Is this, then, alive? Yes: because, when first thrown off by the ovary as a germinal particle (Plate XXVII. fig. 39 *c*), it is solid; the external layer of which by growth forms a vesicle (fig. 39 *d*), while the interior remains solid some time longer, and constitutes its nucleus (fig. 39 *f*).

This nucleus has been already shown to possess vitality, and as it exists in the germinal particle, it also must be alive.

The growth of the germinal vesicle, therefore, from the germinal particle, is as vital as the growth of the embryonic vesicle from its nucleus, the germinal spot.

We have seen that life does not originate at the period of fecundation; we have traced the vitality possessed by the ovum as far back as the very commencement of the ovule; we must therefore admit that it is derived from the mother.

For as the germinal particle is living when thrown off by the ovary, and as the

ovary, being part of the female, shares her life, the vitality possessed by the germinal particle can only be derived from that of the mother.

From this, it appears that the embryo or young *Ascaris mystax* obtains its vitality solely from the mother, but that certain conditions are necessary for the continuance, maintenance and development of that life; and that these conditions are alone furnished by the changes effected by the product of the male, on the matters immediately surrounding the living cell.

When the male secretion is not present, when the above conditions are not fulfilled, life ceases; the vital point dies; and although the surrounding substance does not immediately perish, yet it no longer encloses a germinal vesicle, or even a germinal spot.

Finally, I would desire to draw attention to the beautiful analogy that exists between the products of the ovarian and testicular tubes. The cæcal extremities, of both the male and female reproductive systems, throw off solid particles of the same size, shape and appearance; both kinds soon present spots in their centres, and both swell up into nucleated cells. Yet the one is a seminal, and the other a germinal vesicle.

Granules are now accumulated round both. Both might be called ovules with equal propriety; so analogous are they in structure, that size alone distinguishes them. But the one is an ovule, and the other a granular mass.

The granular matter of the ovule dissolves, the germinal vesicle enlarges and disappears, setting free its nucleus.

The seminal mass likewise loses its granular covering; the seminal vesicle enlarges, and by disappearing, its nucleus is also set free.

Thus far the analogy is complete, but here it ends; the transformed nucleus of the male cell enters the granular vitelline substance of the female ovule, perishing by solution; while the nucleus of the germinal vesicle enlarges, divides, subdivides and redivides, till a mass of granules are formed, each possessed of an individual existence, and together capable of producing a living whole; a worm in every respect like its parent, endowed, like it, with the powers of assimilation, locomotion, and reproduction.

A new life therefore is not generated during the development of a new being, by the happy combination of physical forces; but the same life bestowed by God at the creation, continues without intermission, transmitted from mother to offspring, pervading and redeveloping itself in each individual member of the species.

Explanation of the Plates.

PLATE XXV*.

Fig. 1. Head of *Ascaris mystax*. *a*. The lateral ala or moustache, finely striated. *b*. The mouth with its three lips. *c*. The intestine commencing narrow at the mouth, and passing in a straight line along the axis of the body. *d*. The striated or horny bands of cuticle covering the body. Magnified 30 diameters.

Fig. 2. Tail of male. *a*. The dorsal aspect. *b*. The ventral. *c*. The anus. *d*. The genital orifice. *f*. Membranous projections, serrated on the edge and supported by tubercles. *g*. Spiculæ protruded to the natural extent. Magnified 30 diameters.

Fig. 3. Male *Ascaris mystax*, natural size. *a*. The tail coiled up on the ventral aspect.

Fig. 4. Female, natural size. *a*. The tail, not coiled, and coming to a blunt point.

Fig. 5. Reproductive apparatus of the male. *a*. The convex dorsum of the tail. *b*. The concave ventral surface. *c*. The anus. *d*. The genital orifice. *f*. The part of the spiculum which is never protruded naturally, and is covered with tubercles for the attachment of the protractor and retractor muscles. *g*. The horny opake portion of the spiculum, which is protruded in coitus. *h*. The sheath of the spiculæ into which they are withdrawn when not employed. *i*. The inferior portion of the intestinal canal. *k*. The seminal vesicle, covered by a reticulation of muscular fibres. *m*. The vas deferens. *n*. The testicular tube, filled with semen. *o*. The cæcal extremity of the testicle. Magnified 20 diameters.

Fig. 6. Part of the spiculum. *a*. The protrusive portion, opake and tubular. *c*. The cartilaginous or non-protrusive portion, semi-transparent and of larger calibre than the horny. *d*. The tubercles to which muscles are attached. *f*. The joint between the two portions of the spiculum.

Fig. 7. Part of the spiculum. *a*. The horny portion. *b*. The opening through which the semen escapes.

Fig. 8. The cæcal extremity of the testicle. *a*. The thick granular membrane of the extremity from which the seminal particles are thrown off. *b*. The transparent and thin membrane of the testicular tube. *c*. The inner surface of the thickened extremity, showing its granular and irregular aspect. *d*. The seminal particles as they are first thrown off. Magnified 330 diameters.

Fig. 9. Upper portion of the testicle. *b*. The membranous wall beginning to be slightly granular. *a*. The granules thrown off by the walls of the testicle.

d. The seminal vesicles with an enclosed nucleus. *f.* The seminal vesicle surrounded by the granules in a very irregular manner. Magnified 330 diameters.

Fig. 10. Lower portion of the testicle. *a.* The granular or seminal masses more or less formed. *b.* The walls of the tube presenting striæ as well as granules. Magnified 330 diameters.

Fig. 11. Portion of the vas deferens, showing the muscular or contractile fibres that encircle it. Magnified 330 diameters.

PLATE XXVI.

Figs. 12—15. Drawings representing the formation of the spermatozoon in the *Ascaris acuminata.* Copied from Wagner and Leuckardt's article on 'Semen' in Todd's Cyclopædia. *a.* The nucleus of the seminal cell, which by elongation constitutes the tail. *b.* The seminal cell, which is persistent and forms the body of the spermatozoon.

Fig. 16. Formation of the seminal cells, copied from Kolliker. *a.* The four seminal cells formed within. *b.* The mother-cell.

Fig. 17. Fully-formed spermatozoon of the *Ascaris acuminata* after Reichert. *a.* The seminal cell. *b.* The indefinite nucleus. *c.* The enlarged nucleolus. Magnified 300 diameters.

Fig. 18. Contents of the upper part of the testicle. Magnified 500 diameters. *a.* The seminal or spermatic cells. *b.* The granular fluid in which they float. *d.* The nucleus.

Fig. 19. Granular seminal masses in an imperfect state of formation. *a.* The irregular mass of granules. *b.* The enclosed spermatic cell. *c.* A mass subjected to pressure, by which it is resolved into a granular fluid. Magnified 500 diameters.

Fig. 20. Granular masses fully formed. *a.* The perfect mass, presenting nothing but granules and quite opake. *b.* Masses subjected to pressure, showing the contained spermatic cell. *c.* Nucleated spermatic cells set free by pressure. *d.* A spermatic cell, which, having escaped from its granular envelope, has swollen up to several times its former size. Magnified 500 diameters.

Fig. 21. Spermatic cell seen in front. *a.* The transparent delicate cell-wall. *b.* The discoidal, granular and opake nucleus. *c.* The nucleolus.

Fig. 22. Spermatic cell seen in profile. *a.* The cell-wall. *b.* The granular nucleus now appearing lenticular. *c.* The nucleolus. *d.* The membrane covering the internal surface of the nucleus, apparently a portion of the cell-wall.

Fig. 23. Spermatic cell seen in profile. The nucleus *b* enlarged and having lost its

internal limiting membrane. *f.* The nuclear membrane formed at the expense of the outermost granules of the nucleus.

Figs. 24—27. Spermatic cells in various stages. *a.* The cell-wall more or less protruded. *b.* The nucleus. *f.* The nuclear membrane, which, by increasing most at the centre, has forced the cell-wall out.

Figs. 28—30. Spermatic cells. *a.* The cell-wall recovering its protrusion. *b.* The nucleus, becoming smaller and more surrounded by the nuclear membrane. *f.* The nuclear membrane become tubular and bent by the elasticity of the cell-wall.

Figs. 31—34. Spermatic cells. *a.* The cell-wall now become spherical. *c.* The nucleolus still persistent. *f.* The nuclear membrane, first cup-shaped, then conical, and lastly tubular.

Figs. 35, 36. Spermatic cells in the last stage. *a.* The cell-wall. *b.* The remains of the nuclear granules. *c.* The nucleolus. *f.* The nuclear membrane, now become tubular or flask-shaped, adapting itself to the curve of the cell-wall.

Fig. 37. Spermatic particles in a state of imperfect development. *b.* The nuclei of spermatic cells set free. *c.* The nucleoli of the same. *f.* The nuclear membrane in various degrees of perfection.

Fig. 38. Perfect spermatic particles. *b.* The remaining granules of the nucleus. *c.* The nucleoli. *f.* The fully-formed nuclear membranes or spermatic particles. *g.* Their open mouths.

Figs. 21—38. Magnified 500 diameters.

PLATE XXVII. Magnified 330 diameters.

Fig. 39. The commencement of the ovary. *a.* The cæcal extremity, composed of a thick, semiopake granular membrane, from the internal surface of which the germinal particles are thrown off. *b.* Apparent invaginations, probably formed by casts of the cæcal extremity thrown off at intervals. *c.* The solid germinal particles as first thrown off. *d.* The germinal vesicle formed by the swelling up of the external surface of the germinal particle. *f.* The germinal spots or the central portion of the solid germinal particle. *g.* Large opake yolk-granules floating free. *h.* The delicate homogeneous and transparent membrane forming the upper portion of the ovarian tube.

Fig. 40. Portion of the ovary. *d.* The germinal vesicles. *h.* The membranous tube, giving indications of striæ and granules. *k.* The granules thrown off by the ovary to form the vitellus.

Fig. 41. Middle portion of the ovary. *h.* The external homogeneous membrane. *k.* The granules contained within. *l.* The striæ.

Fig. 42. The upper constriction, or that between the ovary and oviduct. *h*. The external membrane of the lower portion of the ovary without striæ. *m*. The constriction where the ovary ends and the oviduct commences. *n*. The transverse striæ of the oviduct, by which the ovules are forced along singly. *o*. An ovule detached from the general mass filling the ovary, and about to enter the oviduct. *p*. The spermatic particles fully developed, occupying the whole length of the oviduct.

Fig. 43. A small portion of the uterus. *h*. The external transparent and structureless membrane. *r*. Large flat cells by which it is lined. *s*. The nuclei, and *t*. The nucleoli of these cells.

Fig. 44. Contents of the cæcal portion of the ovary. *a*. The germinal spot, which is solid. *b*. The germinal vesicle, transparent, containing a fluid and the germinal spot. +. The germinal particle as first thrown off, the swelling up of whose outer layer forms the germinal vesicle, while the central portion remains unaltered to form the germinal spot.

Figs. 45, 46. Ovula at a very early stage of development. *a*. The germinal spot. *b*. The germinal vesicle, whose margin however is not visible. *c*. The vitelline granules surrounding the vesicle, and apparently the produce of the walls of the ovary. *d*. The clear gelatinous substance in which the granules are imbedded, and which forms the margin of the ovule.

Fig. 47. An ovule further advanced, of a triangular form. *a*. The germinal spot. *b*. The germinal vesicle. *c*. The vitelline granules. *d*. The clear substance forming a distinct outline.

PLATE XXVIII. Magnified 330 diameters.

Fig. 48. An ovule still further developed, presenting a discoidal form. *a*. The germinal spot. *b*. The germinal vesicle. *c*. The vitelline granules. *d*. The clear substance.

Fig. 49. Mass of ovules squeezed out of the ovary. *b*. Ovula seen on their flattened surface, and showing the contained germinal vesicles. *c*. Ovula in the natural position they occupy in the ovary, presenting only their edges externally. *d*. The external portions of the flat ovules, which lie against the wall of the ovary.

Fig. 50. Portion of the ovary at its inferior extremity, showing the disposition of the contained ovules. *d*. Four ovula forming one plane, and exactly filling the transverse section of the ovarian tube. *w*. The wall of the ovary containing a few granules.

Fig. 51. A fully developed ovule, of an elliptical or ovoid form, and in the state in which it leaves the ovary to enter the oviduct. *c*. The vitelline granules,

too opake to allow the germinal vesicle to be seen. *d.* The clear substance forming a distinct and clear outline or margin to the ovule.

Fig. 52. An unfecundated ovule, which having entered the oviduct has begun to alter. *c.* The vitelline granules which have begun to coalesce, and have lost their granular character entirely. *d.* The margin of the clear substance.

Fig. 53. An unfecundated ovule, in which the vitelline granules have wholly disappeared, their oily portion having run together into large globules. *e.* The large oil-globules thus formed. *d.* The margin of the clear substance still distinct.

Figs. 54, 55. An unfecundated ovule further changed, by which the oil-globules have begun to disappear, and the colouring matter of the original vitelline granules to approach the centre; while the whole ovule has become enclosed in a chorion. *c.* The colouring matter of the vitelline granules. *e.* The oil-globules. *f.* The granular chorion of the unfecundated ovum, which is secreted by the oviduct.

Fig. 56. An unfecundated or false ovum at its maximum degree of organization. *c.* The colouring matter of the vitelline granules collected into an opake or false yolk, but containing no germinal vesicle or spot. *f.* The granular chorion or shell so characteristic of the false ovum. *o.* The yolk-membrane, that, separating from the interior of the chorion, contracts on the colouring matter to form the spherical opake yolk.

Fig. 57. A false ovum ruptured by pressure. *c.* The colouring matter, which, with a few oil-globules, formed the opake yolk. *f.* The granular chorion. *o.* The yolk-membrane.

Fig. 58. An ovule during fecundation. *c.* The vitelline granules displaced by the application of a spermatic particle. *d.* The clear substance presenting a distinct margin all round the ovule, except where the protrusion of the granules has taken place. *g.* A spermatic particle which is on the point of entering the vitellus, being partially imbedded in its substance.

Fig. 59. An ovule during fertilization, whose vitellus is much broken up by the spermatic particles. *c.* The vitelline granules displaced by the spermatic particles. *g.* The spermatic particles which are entering the substance of the ovule. *h.* The spermatic particles that have penetrated and become wholly imbedded in the substance of the vitellus.

Fig. 60. An ovule during fecundation, presenting more than one rupture of the surface and more transparent than usual. *b.* The germinal vesicle, generally invisible at this period. *c.* Protruded vitelline granules. *g.* Spermatic particles applied against the surface of the ovule. *h.* Spermatic particles that have entered the vitellus.

Figs. 61, 62. Fertile ova, in which the penetration of the spermatic particles has

nearly ceased and the formation of a chorion begun. *c.* The vitelline granules much broken up. *f.* The first layer of the chorion, perfectly transparent and structureless. *g.* The spermatic particles entering at those places in which the chorion is as yet imperfect. *h.* The spermatic particles, which having entered have begun to swell, giving the vitellus a mottled appearance.

Fig. 63. A fertile ovum, with the chorion entire. *c.* The vitelline granules much broken up round the margin. *f.* The chorion, which has wholly encircled the ovum. *h.* The swollen spermatic particles.

PLATE XXIX. Magnified 330 diameters.

Fig. 64. Ovum become spherical either from the external imbibition of fluid or the liquefaction of the contained granules. *a.* The germinal spot. *b.* The germinal vesicle slightly enlarged, and become visible by the breaking up of the vitellus. *c.* The vitelline granules much broken up, and partially dissolved. *f.* The chorion become tense and spherical. *k.* The embryonic spot seen within the germinal spot, forming the nucleolus of the germinal vesicle. *h.* The swollen and partially dissolved spermatic particles.

Fig. 65. Ovum beginning to assume the oval form. *a.* The germinal spot. *b.* The germinal vesicle. *c.* The vitelline granules broken up into detatched masses. *k.* Two embryonic spots seen within the germinal spot. *m.* Fluid resulting from the solution of the spermatic particles and vitelline granules.

Fig. 66. Ovum still containing a germinal vesicle. *a.* The germinal spot. *b.* The germinal vesicle. *c.* A few vitelline granules surrounding the vesicle. *f.* The three layers of the chorion. *n.* The transformed or embryonic granules.

Fig. 67. Ovum in which all traces of the vitellus and germinal vesicle have disappeared. *a.* The germinal spot set free by the disappearance of its vesicle, but still solid. *f.* The chorion. *k.* The embryonic spot enclosed within the germinal spot. *n.* The transformed or embryonic granules floating free in a clear fluid.

Fig. 68. Ovum with the first trace of an embryonic vesicle. *a.* The embryonic vesicle formed by the swelling up of the germinal spot. *k.* The embryonic spot. *n.* The embryonic granules still free.

Fig. 69. Ovum with the separation of a yolk-membrane. *f.* The three layers of chorion. *n.* The embryonic granules. *o.* The yolk-membrane separating from the interior of the chorion, and contracting irregularly on the embryonic granules.

Fig. 70. A perfect ovum, in which state it is expelled from the vagina of the female.

a. The embryonic vesicle. *f*. The chorion. *k*. The embryonic spot. *m*. Clear fluid filling the space between the yolk and the chorion. *n*. The embryonic granules. *o*. The yolk-membrane become spherical, and enclosing the granules.

Fig. 71. An imperfect ovum placed in creosote water, by which the separation of a yolk-membrane was prevented, and yet the embryonic vesicle divided. *a*. The two divisions of the embryonic vesicle occupying opposite extremities of the egg, but without any membranous or granular investment.

Fig. 72. A perfect ovum, in which the embryonic spot has divided. *a*. The embryonic vesicle elongated. *k*. The divisions of the embryonic spot.

Fig. 73. Ovum in which the embryonic vesicle has divided and the protrusion of the yolk-membrane commenced. *a*. The divisions of the embryonic vesicle. *k*. Their embryonic spots. *o*. The protruded portion of the yolk-membrane.

Fig. 74. Ovum whose yolk presents the hour-glass appearance. *a*. The embryonic vesicles, which by constant gyrations have forced the yolk-membrane into an hour-glass shape. *o*. The yolk-membrane constricted in the middle by its own elasticity.

Fig. 75. Ovum, with the yolk divided into two. *a*. The embryonic vesicles. *k*. Redivisions of the embryonic spots. *n*. The two yolks. *o*. The divisions of the yolk-membrane, caused by the joint operation of its own elasticity and the traction of the embryonic vesicles.

Fig. 76. Ovum whose yolk has divided into four. *o*. The four portions of the yolk.

Fig. 77. Ovum whose yolk has divided into thirty-two. *o*. The divisions of the yolk.

Fig. 78. Ovum the whole interior of which is filled with the granules formed by often-repeated division of the yolk. *o*. The granules thus formed.

Fig. 79. Ovum in which the embryonic membrane or cuticle of the worm is first seen. *p*. The depressed portion of the membrane. *r*. The fluid filling the interspace between the membrane and the chorion.

Fig. 80. Ovum in which the depression of the embryonic membrane has given a cup-like form to the mass of subdivided granules. *p*. The depressed portion of the embryonic or cuticular membrane.

Fig. 81. An ovum in which the two sides of the cuticular membrane having united, a fleshy ring has been formed. *p*. The united portion of the cuticular membrane.

Fig. 82. An ovum where the fleshy ring begins to present a constriction at one point. *s*. The constriction.

Fig. 83. An ovum in which the head and tail of the worm are visible. *o*. Untransformed granules. *p*. Cuticular membrane covering and giving shape to the worm. *s*. The head and tail of the embryo.

PLATE XXX.

Fig. 84. Ovum enclosing an embryo. *s.* The overlapping extremities. Magnified 330 diameters.

Figs. 85, 86. Ova containing young worms variously coiled. Magnified 330 diameters.

Figs. 87, 88. Ova in which the worms have arrived at the period of hatching. *f.* The three layers of the chorion. *p.* The worm enclosed in its cuticle, forming nearly two turns of a spiral, by reversing which, the chorion is ruptured. *r.* Clear fluid surrounding the worm. Magnified 330 diameters.

Fig. 89. The embryo worm just escaped from its shell. *o.* The untransformed granules which constitute its body. *p.* The thick enveloping cuticle. *s.* The three-lobed mouth. Magnified 330 diameters.

Fig. 90. The glandular portion of the female reproductive apparatus in which the ovula are fertilized. Magnified 40 diameters. *b.* The lower portion of the ovary filled with an opake mass of ovules. *c.* The first or upper constriction, between the ovary and the oviduct, and indicating the origin of the oviduct. *d.* The lower or second constriction, being the spot where the oviduct terminates by entering the uterus. *f.* The fundus of the uterus filled with ova in various stages of development.

Fig. 91. Portion of the body of a female *Ascaris mystax*, showing the orifice of the vagina. *a.* The simple circular opening, by which the vagina terminates. Magnified 60 diameters.

Fig. 92. The whole reproductive apparatus of the female *Ascaris mystax*, natural size. *a.* The delicate cæcal extremities of the ovaries. *b.* The lower portions of the ovarian tubes filled with ovules, and opake. *c.* The commencement of the oviduct. *d.* The termination of the same. *f.* The two uteri distended with ova. *g.* The single tapering vagina formed by the union of the two uteri.

XXIII. *On the Blood-Proper and Chylaqueous Fluid of Invertebrate Animals.* By Thomas Williams, *M.D. Lond., Extra Licentiate of the Royal College of Physicians, and formerly Demonstrator on Structural Anatomy at Guy's Hospital. Communicated by* Thomas Bell, *Sec. R.S.*

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In the following memoir, I propose to submit to the Royal Society a collection of facts observed with repeated and scrupulous care, which I trust will suffice to establish the propositions, viz. that in invertebrate animals there exist two distinct nutritious fluids, dissimilar in their anatomical relations, and different in their chemical and vital compositions; that, in the animal series, a gradation from the simple to the complex is observed in the *fluid* as well as in the solid elements of the organism; that these two constituent parts of the animal body bear towards each other, whether in simplicity or complexity, a constant and direct proportion; that the true blood-system does not begin at the beginning of the animal series, but that it arises out of (*what in this memoir will be called*) the chylaqueous fluid, of which the blood-proper is the perfected evolution; that the chylaqueous fluid is as much less vitalized than the true-blood, as the solid structures of the animals in which the former exists are less complex than the analogous parts of those in which the latter is found; that the containing system of the blood-proper is distinguished, with the single exception of that of the Echinodermata, by the absence of vibratile cilia from its *internal* lining membrane, while that of the chylaqueous fluid is provided in the same situations, almost invariably, with these motive organules; that the contents of the former system are propelled by the contractile force of its muscular parietes, while those of the latter are circulated chiefly by ciliary vibration; that below the Echinodermata the blood-proper is wholly supplanted by the chylaqueous fluid; that above the Annelida the latter fluid in the adult animal is superseded by the true-blood; that in the Echinoderms and Annelids these two systems of nutrient fluids co-exist, bearing always to each other, in the same individual, an inverse quantitative proportion; that in the Mollusca these two are united into a single system, in which the essential characteristics of both are legible; and that these facts, hitherto unrecognised in their physiological connection, and unappreciated in their separate meaning, are calculated to elucidate, with new clearness, the processes of digestion, sanguification, and respiration, more especially in the lowest classes of invertebrate animals; and finally, that they suggest inquiry from novel points of view, into several important questions in zoo-chemistry.

In this communication it is my desire to restrict myself chiefly to the demonstration of the corpuscular or morphotic elements of the fluids, postponing to a future occasion the attempt to apply the results of these researches to the problems in comparative physiology which they promise, satisfactorily, to resolve.

As far as my inquiries into the historical literature of this subject have extended, I may affirm that no systematic definition of the real nature of the circulating fluids, in the lowest orders of animals, has ever yet been attempted in physiological science. The true mechanism of nutrition in zoophytic, radiate and annulose organisms, has never yet, at the hands of zoo-chemists, received a satisfactory explanation. The organic fluids have been subjected in no one of their characters to a full and adequate investigation.

They have been permitted to remain, up to this time, almost entirely undescribed and uncomprehended.

To these condemnatory observations, exception must be made in favour first of Agassiz and Milne-Edwards. Obscure hints, tending to the right track of study, have been thrown out by the former naturalist, and by the latter, certain generalized views have been propounded which serve only to indicate the correct direction of inquiry. Agassiz* remarks, "Instead of the three conditions of chyme, chyle, and blood, which the circulating fluids of the Vertebrata undergo, the blood of that class of the Invertebrata which I have particularly studied, the Annelida, is, according to Wagner, simple chyle, coloured chyle; the receptacles of chyle in different parts of the body are true lymphatic hearts, like those found in the Vertebrata: this kind of circulation is found in the Articulata and Mollusks, with few exceptions, some Echinoderms, &c. In the Medusæ and Polyps, instead of chyle, chyme mixed with water is circulated: this circulation is found in some Mollusks and intestinal worms; it may be seen plainly in *Beroë*." It will be subsequently shown that these boldly propounded generalities seldom approach the truth of nature as established by practical observation.

The admirable researches of Milne-Edwards† were chiefly directed towards the determination of the *mechanism* of the circulation, rather than to the composition of the fluids. His inquiries were principally limited to the Annelida, Mollusca and Crustacea. With him originated the generalization, that "in no Mollusk does there exist a closed system of blood-vessels; that in the Bryozoa or Polyzoa, the initiatory class to the Mollusca, neither heart, arteries nor veins are found, the nutrient fluid being contained in the great visceral cavity, in which the organs of digestion are suspended; that in the Molluscoid Tunicata a heart and a system of blood-vessels exist only in the branchial portion of the body, the abdominal or visceral circulation being conducted by means of cells or lacunæ of uncertain direction and without any

* Silliman's American Journal for July 1850.

† Annales des Sciences Naturelles, Série 3me. t. iii. 280 (1845).

definable walls; and that in the Gasteropods and Cephalopods the visceral or peritoneal cavity forms a part of the circle of the blood's movement*."

The observations of MILNE-EDWARDS have been repeated by VALENCIENNES†, by Professor Owen‡ on the Brachiopods, by E. BLANCHARD among the Entozoa§, by QUATREFAGES and others. It has been doubted however by HANCOCK and EMBLETON whether the views of MILNE-EDWARDS with reference to the lacunose character of the peripheral segment of the circulating system in Mollusca, express the true type of the circulation in the Nudibranchia.

While acknowledging the extreme interest of the discoveries accomplished by the latter distinguished men, they will be found in every instance to be limited merely to "the system of conduits" through which the blood describes its circulation, distinct therefore in object and subject from those inquiries which are specially addressed to a consideration of the characters and relations of the nutrient fluids themselves, and which it is the purport of this memoir to record.

The *blood itself* was made the subject of more special remark by Sir E. HOME‖, who states that the blood of the Teredines is red, and that of the Planorbis purple. MILNE-EDWARDS says that in the vicinity of Palermo he discovered an Ascidian with red blood¶. The blood in Lamellibranchiate Mollusks has been made the subject of observation by Mr. GARNER**, with whose results, those stated in this paper will be found little to coincide.

LISTER proved that the blood of the Snail was coagulable††, and that that of the Ascidiæ contained globules. The conclusions of LISTER were confirmed by PREVOST and DUMAS‡‡, by whom it was supposed to have been shown that the globules in the blood of the Snail have a diameter one-third greater than those of man and quadrupeds.

POLI§§ has observed, "si tamen in bono microscopio examinetur, id est, syphone

* "Chez les Mollusques, de même que chez les Crustacés, une portion plus ou moins considérable du cercle parcouru par le sang en mouvement est toujours constituée par les lacunes ou espaces interorganiques; jamais ce liquide ne se trouve emprisonné, comme on le supposait, dans un système clos et complet de vaisseaux à parois propres; quelquefois il n'existe, pour une portion considérable du corps, ni artères ni veines, d'autres fois les artères portent le sang partout où il y a vie à entretenir, mais il n'y a pas de veines pour assurer le retour du fluide nourricier qui s'épanche dans les lacunes comprises entre les diverses parties solides de l'organisation; d'autres fois encore, l'appareil de la circulation se perfectionne davantage, car il existe des veines aussi bien que des artères dans une portion plus ou moins grande du corps; mais ces veines ne suffisent jamais pour compléter le cercle que le fluide nourricier doit parcourir, et la cavité abdominale ou péritonéale joue toujours le rôle d'un reservoir sanguin aussi bien que d'une chambre viscerale."—*Op. cit.*

† Nouvelles observations sur la constitution de l'appareil circulatoire chez les Mollusques, par MILNE-EDWARDS et VALENCIENNES.—*Ibid.* p. 307.

‡ Lettre sur l'appareil de la circulation chez les Mollusques de la classe des Brachiopodes; addressé à MILNE-EDWARDS par M. R. OWEN.

§ Annales des Sciences Naturelles, 3me sér. tom. iv.

‖ Comparative Anatomy, vol. i. p. 32.

¶ Elem. Zool. p. 18; and Mag. and Ann. Nat. Hist. vol. xv. 69.

** CHARLESWORTH's Magazine, Nat. Hist. vol. iii. p. 168.

†† Philosophical Transactions for 1834, p. 380.

‡‡ BOSTOCK's Physiology, vol. ii. p. 200.

§§ Exert. Anat. de coch. p. 95.

capillari vitreo, venæ alicui intruso, globulos opacos vere orbiculares haud paucos videbis; ac sanguineos nostros globulos magnitudine plurimum excedentes. Hi vero globuli ut præ sanguinis globulis pauci sunt, ita aqua quædam limpida innatant, et paulatim præ gravitate ad unum syphonem descendunt. Idem quoque experimentum de succo vitali in cochleis fluviatilibus feci; idemque coagulum sub-cæruleum, igni admotus, dedit." Iron and manganese have been detected by ERMAN in the blood of *Helix Pomatia* and *Planorbis corneus*.

To the historical references formerly given, it must be added that, more recently, two elaborate memoirs, "On the Blood-corpuscle considered in its different Phases of Development in the Animal Series," have been published in the Philosophical Transactions (1846), of which one relates exclusively to the blood of invertebrate animals. Emanating from a physiologist so distinguished as Mr. T. WHARTON JONES, these memoirs are entitled to the highest consideration. Although confined "to the more readily procurable examples of the divisions Annulosa and Mollusca," those on the Crustacea and Insects excepted, in no instance have I been able to verify the observations of this author.

In their most recent communication to the Royal Society*, ALDER and HANCOCK adduce additional facts corroborative of their former conclusions. They maintain "that in the Mollusks there is a triple circulation: first, the systemic, in which the blood propelled along the arteries to the viscera and foot is returned, with the exception of that from the liver-mass, to the heart through the skin; there it becomes partially aërated, the skin being provided with vibratile cilia, and otherwise adapted as an instrument of respiration; second, the portal, in which venous blood from the system is driven by a special heart to the renal and hepatic organs, and probably to the ovarium, where it escapes, doubly venous, with the rest of the blood which has been supplied to these organs from the aorta, and which is therefore only singly venous, to the branchiæ; third, the branchial circulation, in which flows only the more deteriorated blood brought by the hepatic vein, but in which also that blood undergoes the highest degree of purification capable of being effected in the economy, namely, in the special organ of respiration. This triple circulation has not yet, as far as the authors are aware, been described in the Molluscan Sub-kingdom. Since the blood in *Doris* is returned to the heart in a state of partial aëration, it is clear, they say, that this animal is, in this respect, on a par with the higher crustaceans; and since the blood arrives at the heart in the same condition, according to the researches of GARNER and MILNE-EDWARDS, in *Ostrea* and *Pinna*, the great *Triton* of the Mediterranean, *Haliotis*, *Patella* and *Helix*, it can scarcely be doubted that this arrangement will be found throughout all the Mollusca."

Elaborate as these admirable inquiries deserve to be characterized, they do not affect the truth of the leading proposition of MILNE-EDWARDS, that, viz. the visceral cavity constitutes a part of, and is an open communication with, the channels of the

* "On the Anatomy of *Doris*," Proceedings of the Royal Society, March 4, 1852.

circulation. Mr. HANCOCK and Dr. EMBLETON have shown that collateral *segments* of the circulation undergo a special elaboration for special or local purposes. They do not however *demonstrate* in any part of the blood's circuit a peripheral capillary system. They only state that "the existence of true capillaries in the liver-mass seems probable." In further elucidation of the tendency to special development in particular portions of the circulating system of mollusks, I have lately proved that the branchial capillaries in both the univalve and bivalve orders conform with singular constancy to one type of subdivision. The vessels are always parallel, never reticulate. These observations support the proposition formerly stated, that in the Mollusca the circulating system is really a product of the fusion of the chylaqueous system into that of the blood, properly so called. In a letter addressed to M. MILNE-EDWARDS, Mr. HUXLEY observes, after an examination of the circulation of the blood in the genera *Firole* and *Atlante*, "J'ai obtenu ainsi une confirmation entière de vos vues relatives à la manière dont cette fonction s'exerce chez les Mollusques. ...Il n'existe point de veines quelconques...Je suis porté à croire que l'absence plus ou moins complète de la portion veineuse du système vasculaire, loin d'être un cas exceptionnel, est l'état normal dans la plupart des classes de la grande division des animaux sans vertèbres*." It is however to the 'Mémoire' of M. QUATREFAGES, "Sur la cavité générale du corps des Invertébrés," *Annal. des Sciences Nat.* 1850, that, in this historical summary, special attention is invited. In this excellent essay M. QUATREFAGES first describes "la cavité générale du corps" as beginning with the Hydræ and Actiniæ, and closes the survey with the Mammalia, the serous cavities of which he likens to the visceral cavity of the body as it is found to exist in the Invertebrata. In the mere description of 'the cavity' there is no novelty. To all comparative anatomists for half a century *the existence* of the visceral cavity in the several classes of the Invertebrata has been familiar. To every observer it has long been well known that this cavity in *all* classes was occupied by *a fluid*. On *this* ground therefore no modern anatomist is entitled to the credit of discovery. In the year 1741, TREMBLEY *saw* and accurately described the movement of a corpusculated fluid in the perigastric cavity and *tentacles* of the 'Polype à Panache.' But in the memoir "Sur la cavité générale du corps des Invertébrés," M. QUATREFAGES has sagaciously projected certain generalized views with regard to the possible functions devolving upon the fluid contained in this cavity, which to a limited extent run parallel with the conclusions which in this paper I have endeavoured to establish. It will however be readily seen by any one who will peruse the two essays with a view to a comparison, that they differ essentially in subject and object. M. QUATREFAGES' chief aim is to define the visceral cavity; he adds only a few *general* observations with reference to the physiological relations and functions of the fluid contents. He has not attempted to resolve the problem of its histological characters. His conclusions

* Observations sur la circulation du sang chez les Mollusques, des genres Firole et Atlante, Ann. des Sc. 1850.

are not based upon observed facts, but upon general views. He makes no allusion whatever to the relations which subsist between the chylaqueous system and that of the true blood; he has not defined the difference in the mechanism of solid nutrition as it occurs respectively under the agency of these two orders of fluids. M. QUATREFAGES has seized *no one* clue to the demonstration of the capital law of structure, viz. that the system of the blood-proper *only first appears* in the series at the Echinodermata. Unguided by this great and novel principle, he could not perceive that a blood-proper system could not exist, as it was not required *below the Echinodermata*. He affords no proof of having known that the chylaqueous system of fluid is governed physiologically and chemically by laws distinct from, though not less definite than those which regulate that of the true blood. He has accumulated *no* individual observations illustrative of the histological characters of the morphotic elements, either of the chylaqueous fluid, or the true blood. M. QUATREFAGES appears also to have been totally unacquainted with the interesting fact announced in this paper, that in histological structure, the corpuscles of the chylaqueous fluid are as *definite* and constant as those of the blood of the higher animals. It thus appears that, although the views of M. QUATREFAGES and those advocated in this paper proceed for a short distance in parallelism, they soon diverge towards two very different destinations*.

* Having stated in the text, in general terms, the most prominent features which distinguish the admirable memoir of M. QUATREFAGES from this paper, I am here desirous that this meritorious observer should speak for himself, and that my observations (submitted to the Royal Society six months before the advantage occurred to me of perusing the original memoir of M. QUATREFAGES) should be fortified and justified by the independent researches of one so much better known to European science. I only cite so much of the memoir of M. QUATREFAGES as really bears upon the subject of this paper. I quote *entire* his observations on the microscopic examination "*du liquide de la cavité générale*:—" Chez les Invertébrés. dont la cavité générale communique avec le tube digestif, le liquide que renferme cette cavité est toujours composé d'eau tenant en suspension des particules très ténues provenant des aliments. Chez les Mollusques, les Insectes, les Crustacés dont la cavité générale communique avec l'appareil circulatoire, le sang s'épanche librement dans cette cavité. Je l'ai généralement trouvé composé d'un liquide incolore, charriant des granulations irrégulières, transparentes, sans couleur, et qui semblaient assez souvent résulter de la soudure fortuite de granulations plus petites. On sait, du reste, que la description que je donne ici ne s'applique pas à tous Mollusques. Depuis longtemps on a signalé la couleur rouge violacée du sang du Planorbe corné. J'ai retrouvé quelque chose d'analogue dans le Planorbe imbriqué. Cette espèce est d'autant meilleure à signaler, que la transparence de sa coquille permet de l'observer sur le vivant. Enfin, une des exceptions les plus remarquables à citer a été découverte par M. EDWARDS. Ce naturaliste a trouvé en Sicile une Ascidie dont le sang, à la vue simple, est d'un beau rouge. Au microscope, on reconnaît que cette couleur est due à des globules framboisés, très réguliers, nageant dans un liquide incolore. Chez les Annelés, dont la cavité générale est close, j'ai également trouvé presque toujours cette cavité remplie par un liquide incolore, tenant en suspension des granulations irrégulières de forme et de volume très variables, également transparentes et sans couleur, mais réfractant la lumière avec beaucoup plus d'energie que le liquide ambiant; toutefois j'ai déjà fait connaître quelques exceptions, entre autres dans le mémoire sur la famille des Némertiens et dans une note sur le vaisseau dorsal des Insectes. Mes observations ont porté principalement sur les Annélides. Dans l'immense majorité des cas, ce que j'ai vu chez elles s'accorde avec la description générale que je viens de donner; pourtant j'ai encore ici des exceptions à signaler. Chez la Polynoé lisse, le liquide dont nous parlons charrie des globules ovalaires, aplatis, incolores, présentant l'aspect d'une substance homogène réfractant la lumière à peu près comme le liquide ambiant, et renfermant un noyau plus réfringent.

The preceding historical allusions, summary and brief though they be, will suffice to present an outlined view of the actual state of knowledge on the subject to which this memoir is dedicated.

In the account, now to be offered, of the nutrient fluids in the leading classes of invertebrate animals, it will be indispensable to the full development of the subject, that under each head a slight sketch be premised of the anatomical relations subsisting between these fluids and the digestive and respiratory organs. These references however will be compressed to the utmost brevity.

In the distribution of the materials, it seems preferable to begin with the lowest and simplest organisms.

Porifera.—In the sponge, the *fluid blastema*, formed directly out of the constituent

Ces globules sont souvent irréguliers ; ils ont environ $\frac{1}{33}$ de millimètre de diamètre longitudinal sur $\frac{1}{50}$ à $\frac{1}{40}$ de millimètre de diamètre transversal. Quant au noyan, il est sphérique, et son diamètre varie de $\frac{1}{70}$ à $\frac{1}{160}$ de millimètre environ. Parmi les Apneumées, qui déjà nous ont présenté des particularités organiques si curieuses, j'ai rencontré une espèce chez laquelle j'ai trouvé le liquide de la cavité générale coloré. A la simple vue, il est d'un beau rouge légèrement orangé ; lorsqu'on l'examine au microscope, on reconnait que cette coloration est due à des globules très nombreux, très réguliers, présentant la forme que j'ai reproduite ici, c'est-a-dire celle de petits disques circulaires assez profondément excavés sur une de leurs faces (pl. 5. fig. 12, *op. cit.*), ce qui pourrait induire en erreur au premier coup d'œil, et faire croire à l'existence d'un noyau. Par transparence, ces globules paraissent jaunâtres. Leur diamètre est de $\frac{1}{75}$ de millimètre.

"Chez certains Siponcles, on trouve, dans la cavité générale, un liquide parfois tellement chargé de granulations, qu'il en est comme trouble. Dans une espèce très petite et assez transparente de nos côtes, j'ai pu constater sur le vivant que ces granules étaient framboisés, sphériques, et d'un diamètre à peu près constant. Ils sont d'ailleurs incolores. Chez certains autres Annelés de petite taille, chez les Rotateurs, par exemple, le liquide de la cavité générale ne présente qne des granulations très rares, ou même n'en présente pas du tout. Une fort grande espèce de Notommate que je rencontrai au printemps, aux environs de Paris, m'a présenté, sous ce rapport, une exception : le liquide de la cavité générale contenait presque autant de grannlations que celui de certaines Annélides." The preceding passage includes ALL the results which M. QUATREFAGES has given of "the microscopic examination of the liquid of the general cavity !"

Marvellously scanty as these results appear to me when estimated as the *data* on which he rests his conclusions, they will be fonnd utterly irreconcileable with those (stated in the text) at which I have arrived from an examination of the fluids of *the same* animals.

Under the head of the "Nature du liquide de la cavité générale du corps dans les divers groupes d'invertébrés," M. QUATREFAGES remarks, "Le rôle joné par l'eau, qui lave en qnelque sorte le tube alimentaire des Actiniaires et entraine en passant tous les principes solubles des aliments digérés, est trop évident pour qu'il soit nécessaire d'insister sur ce point. Cette eau modifie, par son séjour dans la cavité générale, de plus en plus sa composition : elle s'animalise pour ainsi dire et forme dans la cavité générale du corps une sorte de bain nourricier, dans lequel plongent tous les organes. Qu'il existe on qu'il n'existe pas chez ces animaux d'appareil vasculaire renfermant l'équivalent du sang proprement dit, cette eau n'en contribue pas moins d'une façon essentielle à la nutrition. De plus, quand elle est expulsée par les contractions de l'animal, elle entraîne avec elle les résidus de la digestion et les principes rejetés par l'organisme. Chez les Invertébrés dont la cavité générale sert en quelque sorte de carrefour aux appareils artériels et veineux, il est clair que le liquide de la cavité est essentiellement l'agent immédiat de la nutrition. Il n'est antre chose que le sang lui-même, qui pendant son séjour dans cette cavité, s'est enrichi de principes réparateurs venus soit de l'intestin, soit de la surface de tous les organes internes." Here M. QUATREFAGES adds a note, in which he admits as not improbable, "l'existence dans les Actinies d'un véritable système vasculaire indépendant de la cavité générale du corps," as described by SPIX

elements of the surrounding water, is contained in part in the interior of, and in part between, the cells of the gelatinous cortex*. I have proved that this fluid, which is the true blood of the sponge, is composed of a mixture of salt water and albumen. The *most* organized is that contained *in* the cells of the gelatinous cortex, the *least* is that *between* the cells. These cells actually vitalize and organize the elements of the aërated water. The sponge presents the *first* and simplest problem in zoo-chemistry†.

Polypifera.—In *all* Polypes the space between the stomach and integuments is filled with a corpusculated, organic fluid, varying in different species, moving to and fro under muscular or ciliary agency. It is principally composed of sea-water, which, in passing through the stomach, blends with the secretions of this organ, and then enters the visceral cavity, where it acquires those vital and chemical properties which fit it to nourish the solids of the body. In the hydraform and actiniform groups it has not yet been proved that the stomach opens directly into the visceral cavity. The tentacles in some species are undoubtedly perforated at their distal ends, in order to admit the surrounding water immediately into this cavity, where it admingles with the product of digestion, and undergoes, *thus directly*, the process of *organic* assimilation. In several small and transparent species of Actiniæ, I have lately *seen*, with perfect distinctness, the motion of the perigastric fluid by its corpuscles, and proved that the tentacles are *im*perforate at their distal extremities. In the composite forms of Polypes the bottom of the stomach communicates freely with the interior of the polypidom, as in *Campanularia* and *Alcyonidium*. This fluid in all zoophytes penetrates into all the appendages and recesses of the body. It is, in

in the *Actinia* and by Will in *Alcyon Palmé*, remarking, however, "peut-être de nouvelles recherches sont-elles nécessaires pour s'assurer que ces observateurs n'ont pas regardé comme un appareil sanguin le système décrit par M. Edwards, et qui est en communication avec la cavité générale des Polypes." It appears to me to be only a just construction of the meaning of M. Quatrefages, as expressed by himself in the preceding passages, to state that he nowhere gives any clear proofs of having recognised in the fluid of "la cavité générale du corps" a *system* of circulation, definite and distinct in its laws, which, in the lower Invertebrata, replaces and represents that of the blood-proper as it exists in the vertebrated animal. The justice of this criticism is rendered undeniable by the tenour of the following observations:—"Chez les Invertébrés dont la cavité générale communique avec l'intérieur, la manière dont l'air est mis en rapport avec les principes qu'il doit modifier est évidente, et résulte du fait même de l'introduction d'une eau aérée...... La respiration du liquide de la cavité générale est plus difficile à reconnaître chez les Invertébrés dont la cavité générale est entièrement close." In the first remark he supposes that the fluid of the visceral cavity is the *aërating medium*, not itself the *subject* of the respiratory change. By the second observation he admits that the part performed by the chylaqueous system of fluid, in the Echinoderms and Annelids, in the mechanism of respiration, *is difficult to understand*. In this critical analysis of the valuable memoir of M. Quatrefages 'On the General Cavity of the Body in the Invertebrata,' I have, I think, clearly shown that there is scarcely anything in common either between the facts or arguments therein given, and the carefully recorded dissections and observations upon which I have sought in this communication to rest the superstructure of an important physiological law.

* Annals and Magazine of Natural History, by Mr. Carter, August 1849.

† See Observations on the Cilia of Sponges in Goodsir's Annals of Anatomy and Physiology, May 1852.

every species, the seat and scene of the processes of digestion, sanguification and respiration; it is, in addition, the direct agent of nutrition. It is chyme, chyle and blood in itself. In the zoophyte it is the only fluid element of the organism. In the tentacles of *Tubularia indivisa*, the fluid of the visceral cavity can be distinctly seen in motion in the axial channel. The fluid is charged, in this species, with minute albuminous granules, irregularly grouped, and formless or unorganized, appearing to consist simply of solid spherules, resulting from the solidification of albumen, Plate XXXII. fig. 1.

The bulk of this fluid consists of salt water; for when the specimen dries and the fluid evaporates, cubic crystals of chloride of sodium are seen amidst the albuminous molecules. This fluid was first seen in motion by Lister, then by Milne-Edwards, and by Van Beneden. It may be well here to observe, that the molluscan polypes of the *Flustræ*, *Escharæ* and *Bowerbankia*, are ciliobrachiate, and that the digestive organs are divided from the visceral cavity. In these genera, therefore, the fluid contents of this cavity receive no *direct* admixture of salt water through the tentacles, for the extremities of these organs are not perforate. It is replenished only through the stomach; the water is submitted to the agency of this organ before it enters into the visceral chamber. This fact explains the circumstance, that in these molluscoid polypes the fluid of the visceral cavity, as compared with that of the former group, presents a higher organic composition; its cells are corpusculated and evidently organized; many bear globules of oleine, although comprising several individual forms of cell; they are *constant* in their microscopic characters in the same species. The fluid is aërated in the tentacles. It is true-blood in its composition and functions, fig. 2.*.

Medusæ.—In *all* Acalephæ the digestive cavity is prolonged into a system of canals into which the contents of the stomach pass by direct communication. The gastro-vascular channels are to the Acaleph what the visceral cavity or the hollow interior of the polypidom is to the Polype. In both cases the contained fluid consists of a chylaqueous compound. In both, the containing chambers and canals are lined by a vibratile epithelium†.

* I have recently succeeded in bringing under direct demonstration the movements of the fluid in the peri-gastric cavity of the little freshwater *Hydra*. The distal ends of the tentacles are imperforate. The fluid is charged with minute oleous molecules. This marvellous little being is no longer an organic paradox. Its fluids and solids are now intelligible; the essential elements of its organism are unravelled.

† I had repeatedly and beyond doubt established this fact (by numerous observations on several species of Pulmograde and Cirrhigrade Medusæ found in the Bay of Swansea and the Bristol Channel) long before the advantage occurred to me of perusing the valuable paper of Mr. Huxley in the Philosophical Transactions 'On the Anatomy and Affinities of the Family of the Medusæ,' Part II. 1849. With reference to the digestive cavity of the Cryptocarp and Phanerocarp families Mr. Huxley remarks, "Whatever its appearance, it will always be found to be composed of two membranes, an inner and an outer. These differ but little in structure; both are cellular, but the inner is in general softer, less transparent and more *richly ciliated*." In the same paper a similar observation occurs in relation to the Rhizostomidæ, but from his memoir I cannot discover that

The presence of cilia on the internal surface of these canals strikingly and fundamentally distinguishes the latter from true blood-vessels, and intimately allies them, homologically, with the spacious perigastric chambers of the Echinoderms and Annelids.

The fluid contained in the gastro-vascular canals of the Medusæ is a compound of salt water and chyle. In the Rhizostomidæ it presents a yellowish hue, in *Velella* it is bluish; it is *always* corpusculated. The floating cells exhibit great *irritability*, but they are not locomotive; the minute molecules are mutually repulsive. In this fluid living polygastric animalcules are constantly met with. The fact of their constant presence proves the direct admission of the external sea-water into these canals, but they are in time digested. The largest corpuscles are furnished with an involucrum, fig. 3. In *Rhizostoma* (a specimen of average size) they vary in measurement from $\frac{1}{1800}$dth to $\frac{1}{3000}$dth of an inch. They are never organized to such a standard as to contain a defined centric nucleus.

The formative power of the cell is expended in the production of secondary oleaginous cellules and 'molecular base,' which constitute the contained parts of all the larger parent-corpuscles. The smallest cells are filled only with a limpid oleine, of slight refractive power. Evaporated, in *Rhizostoma* this fluid yields abundant crystals of chloride of sodium. In no Acaleph whatever are the gastro-vascular canals tunnelled in the *centre* of the substance of the disc. They are always situated as superficially as possible on the inferior surface of the disc in the Pulmograde Medusæ; in the Ciliograde under the external cuticle of the globe in immediate relation with the cilia. In this anatomical arrangement the physiologist discerns a true intention, that of exposing the fluid contents to the aërating agency of the surrounding medium. In *Rhizostoma* the yellow colour has its seat in the fluid, not in the floating cells. It is a fact of singular interest, that the corpuscles of the chylaqueous fluid in this, as in ALL other classes in which it exists, *vary in size* with the variations in the size of the body of the individual specimen under examination. In this respect they are diametrically distinguished from the morphotic elements of the *true-blood* in vertebrated animals, the corpuscles of which bear no proportion in size to that of the body of the animal from which they are taken. This general observation will be afterwards confirmed by a variety of particular exemplifications. It has now been shown that in Zoophytes and Medusæ sea-water is admitted in large quantities, more or less directly, into the chylechannels, with the contents of which it more or less directly mixes and vitally assimilates. This fact may at present be mentioned as another fundamental particular, in which the chylaqueous fluid, in ALL animals, is distinguished from the blood-proper, for into this latter fluid the external element is *never* immediately admitted; it is previously subjected to the influence of one or more organic processes; it is thus

Mr. HUXLEY had recognised *the principle* that in *all* Acalephæ the interior lining of the stomach and gastro-vascular canals was more or less generally ciliated, that the movements of the fluid contents were in great part due to the agency of cilia.

impressed with the *first* impulses of zoo-chemistry before it is fitted to enter into combination with the vital fluid. The facility with which the most depressed forms of life appropriate and vitalize a complex *inorganic* liquid, may be signalized as a true badge of zoological inferiority. When ejected from the body, as it often is, at the will of the animal, the vital fluid is renewed, organically, and corpuscles grow in it with incredible rapidity. The entire contents of the system of the true-blood can in no single instance be withdrawn from its vessels (except in very small quantities) compatibly with the preservation of life. Hence another fundamental difference between the physiological history of the blood-system and that of the chylaqueous. How significantly that of the latter speaks as to the "simplicity" of the living organism in those orders of which it constitutes the exclusive means of nutrition!

Echinodermata.—Physiologists, from TIEDEMANN to MÜLLER, commonly describe in this family *three* distinct systems of fluids; that, 1st, of the general cavity of the body; that, 2ndly, of the feet and water-canals; and that, 3rdly, of the blood-proper. The doctrine which maintains the perfect independence of these three orders of fluids, is advocated by these two illustrious anatomists. *Facts* will now be adduced which render it probable that these three apparently independent systems constitute really only a single indivisible fluid-system. They will be found to warrant no other conclusion than that in Echinoderms the system of the blood-proper is so rudimentarily formed, that its contents, by some means and in some undetermined manner, communicate and mingle with those of the general cavity of the body, since in every histological character the morphous elements and chemical compositions of the two fluids prove, under every mode of inquiry, to be identical. From the same method of examination, the inference is confidently drawn that the contents of the vascular system of the feet ("Das Wassergefässsystem" of MÜLLER and TIEDEMANN) are identical, chemically and histologically, with those of the visceral cavity. But, though rudimentary, the blood-proper system in the Echinoderms *does* exist. As regards its *central* channels, it is easy, by injection and inflation, to verify the descriptive statements of TIEDEMANN and MÜLLER.

It is important to the purport of this memoir, to ascertain the sentiments of MÜLLER with reference to the character and distribution of the blood-vessels in the Holothuridæ, to which more especially he has dedicated his recent studies, since, as these genera stand high in the Echinodermal scale, what is true of them must *à fortiori* be true of the lowest families of the class.—"Jedenfalls müssen die Blutgefässe zu den Körperwänden aus dem Innern des Thiers denselben Weg nehmen, wie die Nerven und die Wassercanäle, nämlich durch den Gabelfortsatz am obern Rande von je fünf Stücken des Kalkringes der Holothurien oder durch die fünf Stücken mit Löchern bei den Synapten. Dies sind, wie es scheint, die einzigen Durchgänge vom Innern zu den Körperwänden. An den Gekrösen der Holothurien, konnte ich keine Verbindungen der Körperwände und Eingeweide durch Blutgefässe wahrnehmen."

Speaking of the difficulty experienced by M. QUATREFAGES in detecting the vessels of the intestine in *Synapta Duvernæa*, MÜLLER incidentally remarks, "*Denn man darf die Wassergefässe nicht mit den Blutgefässen identificiren*, welche die vom Darm kommende Nahrungsflüssigkeit enthalten." In another place he speaks of "Die Unabhängigheit des Wassergefässsystems und des Blutgefässsystems von einander*."

From a careful study of the memoirs of MÜLLER, it has appeared to me to be certain that he has never succeeded in tracing the blood-system *to its periphery* even in the Synaptæ. He has only been able to show that a branch or two in some species is given off from the trunks at the roots of the tentacles, to be distributed over the integument. He nowhere offers a single remark as to the colour or composition of the blood. In commenting on the observations of QUATREFAGES, he seems disposed to deny the existence of cilia on the internal lining membrane of the channels of the true-blood.

It should then be remembered that MÜLLER, like TIEDEMANN, has succeeded in demonstrating *only the central* trunks of the blood-vascular system. In animals so large, with vessels so considerable in diameter, the question may be emphatically put, Why to an expert dissector should this difficulty of tracing the circumferences of the system exist? In every other class of animals in which a blood-system exists *at all*, nothing is so easy as to observe its peripheric segments. In the most delicate Annelid the capillary extremes of this system are readily detected. I reply, that in *the Echinoderms the blood-system has not yet evolved a circumferential plexus.* This is one of the characters of imperfection which marks *the first appearance* of this system in the animal series. It is because MÜLLER, as the first anatomist of the age, has reached by a distinct channel of inquiry, impliedly, the very same results with those which are presented in this paper, that I have digressed in the preceding remarks from a consideration of the contents of the blood-system to that of the system itself.

I have instituted laborious dissections on many hundred specimens into the blood-vascular and water-vascular system of the Asteriadæ, Echinidæ and Sipunculidæ. No Holothuridan species, in a fresh state, has yet fallen under my observation. I regret this circumstance the more, for in these genera the blood-system attains a higher degree of development than in any other of the Echinoderms.

The result of my own researches may be first conveniently stated under the form of the following propositions:—

1st. The blood-vascular system in the Echinoderms is *not* an independent and *closed* system of conduits; it is rudimentary and imperfect in its peripheral portions.

2ndly. The *internal* lining membrane of its channels is ciliated, a character which separates the blood-system of this class from that of every other in which it is known to exist, while it significantly attests its rudimentary condition.

3rdly. The fluid contents of the blood-vascular system are chemically and mor-

* Berichtigung und Nachtrag zu den Anatomischen Studien über die Echinodermen. 1850.

photically identical with those of the water-vascular system and with those again of the visceral cavity.

The *unity* of the two last systems of fluids has been recently advocated by Milne-Edwards.—" M. Milne-Edwards s'est assuré qu'une communication semblable existe entre la cavité générale du corps et les cœcums exsertiles des Echinodermes. Ces cœcums, bien distincts des pieds à ventouses, sont distendus non pas par le sang, mais par le liquide de la cavité générale. Quelques observations personnelles me portent à penser qu'il en est de même des pieds à ventouses eux-mêmes*."

M. Quatrefages, by whom this opinion of Milne-Edwards is expressed, does not state the grounds upon which this latter physiologist rests his conclusions. I have been independently induced to ground the same belief upon the following demonstrations:—

1st. Injection thrown into the water-vascular system in the Asteriadæ will *first fill* the sand-canal, and then, by continuation of the injecting force, escape into the peritoneal cavity.

It is by this route, I infer thence, that a direct communication takes place between the fluid contained in the visceral cavity and that of the water-vascular system.

This office implies an adequate function to the sand-canal. It is a simple filterer of the fluid passing from the visceral chamber into the water-vascular system, and conversely. Muscular compression exerted upon the fluid in the water-vascular system (the walls of which are remarkably muscular) will force it *back* into the visceral cavity; a 'diastole' of the vesicles of the feet occurring coincidently with the compression of integumentary parietes of the general cavity, will cause a return of it again into the blood-vascular system. This is a beautiful and perfect mechanism.

The grounds whereon the identity of the fluid contained (in the Echinoderms) in the system of the blood-proper with that which fills the water-vascular system and the abdominal cavity is maintained, may be briefly stated as follows:—In *Uraster papposa* the oral membranous disc exceeds very much in diameter that of the common Asterias. This circumstance renders the circular central blood-vessel in the former much more conspicuous and accessible than in the latter. It was this fact which led me to make choice of *Uraster* as the subject of my researches. In this species the circular channel forming the centre of the blood-system coincides with the circumference of the oral membranous disc. The vascular channel (scarcely to be called a vessel) rests upon the hard edge of the calcareous framework. It is so closely and intimately adherent to the hard unyielding surface beneath, that 'the conditions' seem to be destroyed which might permit the contractions of this vessel as a circulating centre. No instance is known in which a *contractile* heart is thus anatomically connected: nor are the coats of this circular vessel, in *Uraster papposa*, endowed with muscular fibres. It is lined *internally* as well as externally by vibratile epithelium.

* Quoted by M. Quatrefages in his Mémoire, 'Sur la cavité générale du corps des Invertébrés,' Annal. des Sc. Nat. 1850.

No feat in anatomy is more practicable than in any species of *Sipunculus*, to bring under direct view in the field of the microscope, the blood-vessel which reposes on the œsophagus. Although more perfectly formed as a vessel, more defined in its parietes, than the circular vessel of the Asteriadæ, it may be placed beyond doubt that *its* interior also is lined by vibratile epithelium; that *its* contents are chemically and morphologically identical with those of the visceral cavity of the same animal. It may be mentioned here as a remarkable fact, that in the Sipunculidæ (at least such is the result of my examinations repeated with the utmost care upon many scores of living specimens) the vessel which rests upon one side of the œsophagus is *not* accompanied by a venous correlate on the opposite aspect of the cylinder, nor by one parallel to itself on the same side. It is a single channel, which *cannot be traced to a distal system of capillaries.*

In the Sipunculidæ there are no "Polian vesicles" attached to or developed from this vessel. The appearance of "vesicles" arises from the bulged knots into which the vessel irregularly contracts under the stimulus of the air at the moment of laying open the general cavity. To these facts, as illustrative of the singular conditions under which "the circulation of the blood" is first evolved in the ascensive progression of animal structures, the highest physiological interest attaches.

After this short historical introduction, the several characters of these three systems of fluids may be more carefully examined.

Blood-proper in the Echinoderms.—In the Asteriadæ and Echinidæ this fluid is *colourless*, and charged only with irregularly organized corpuscles; when placed under the microscope, it is quite impossible to distinguish these latter from those found in the peritoneal fluid, or in the water-vascular system of the same individual. In *Uraster papposa*, as already stated, the circular vessel and the radial trunks are so large and accessible as to be readily injected. Two methods may be adopted to determine the presence of cilia on the internal lining membrane of these vessels. In a fresh specimen a portion of the trunk may be cut out and laid under the microscope with a view to see *through* its parietes, a current of moving corpuscles driven along by vibratile cilia, or the vessel may be cut transversely into thin circular sections. These sections will exhibit cilia in active motion on *both* the outer and inner edges of the parietes of the vessel. In the Asteriadæ the *whole interior* of every organ (except the ovaries) in the body is coated with vibratile epithelium*. The blood of the Echinidæ is also colourless.

* In his recent essays, Müller gives no further account of the blood-vascular system in *Asterias* than that which occurs in the following scanty passage:—"Das Wassergefässsystem zur Erection der Füsschen und das Blutgefässsystem der Asterien sind von Tiedemann so vollständig und naturgetreu beschrieben, dass ich nichts dazu nachzutragen gefunden habe, als dass der unter der Haut des Mundes auf dem häutigen Discus liegende Blutgefässring, ausser den von Tiedemann angezeigten und abgebildeten Aesten auch zu jedem Strahl einen Zweig giebt, der wieder 2 kurze Seitenäste abschickt, wie Tiedemann's Nerven der Arme. *Die Injection vom Blutgefässring gelang nur bis zum Anfang dieser Gefässe.*" In another passage he incidentally observes, "Zunächst unter der Haut am Munddiscus liegt nach Tiedemann das, was er den orangefarbenen Gefässring

Extreme care and caution are required to observe separately the true-blood. It can only be found in the circular vessel, embracing the œsophagus, immediately under the lantern. The real vessel is quite distinct from the circular band of bright red pigment, which in this species acute observers have mistaken for a veritable blood-vessel. It is very difficult to detect the presence of vessels along the edges or in the parietal substance of the intestine in the *Echinus*. The bright red pigmented cellules distributed through the tissues, sometimes seen in lines and reticulations, have nothing to do with the *blood*-system. The fluid contained in the circular vessel around the œsophagus is colourless. Its cell elements are precisely the same as those of the fluid of the general cavity. It is more opake than that of the visceral chamber, from the presence of a greater number of minute molecules*.

In the *Echinus* it has proved to me impracticable to isolate the vessel so accurately from surrounding structures as to be enabled to detect the presence of cilia on the internal lining membrane. The whole structural character of the vessel however leads me to this inference. Its parietes are not definitively organized like those of every other true vascular channel; cell and areolar fibres constitute the elements of its parietal structure. The preceding considerations have brought me to the belief, that in the Echinidæ, as in the Asteriadæ, the blood-proper has acquired scarcely any distinctive and independent characters; that the system of conduits in which it moves is so rudimentarily organized as to receive its contents, in some manner yet undetermined, *directly* (not by an act of elective, secretive absorption, as in the Annelida) from the fluid occupying the visceral cavity; that the blood-vascular apparatus, developed only in its central segments, is designed only to concentrate the nutritive force of the chylaqueous fluid upon certain of the more important viscera; and that the nutrition of the *peripheral* structures of the organism, such as the muscular, calcareous and integumentary, is sustained under the agency, exclusively, of the chylaqueous fluid.

In the Sipunculidan genera the blood-system is more conspicuous anatomically, although little more advanced physiologically, than that of the former families.

nennt, von welchem zur Tentakelrinne ein Ast abgeht. Dicht darunter soll sich das ringförmige Blutgefäss befinden, welches jedenfalls leicht aufzufinden, zu injiciren oder aufzublasen ist." It is no severity of criticism to lament that this description by the great modern anatomist is little in advance of the original statements of TIEDEMANN.—*Op. cit.* MÜLLER, Archiv, 1850.

* "Die Blutgefässe (der Echiniden) verhalten sich so wie es TIEDEMANN beschrieben. Um den Mastdarm her liegt der bekannte circulus analis dicht auf dem Skelet, auf dem er bei Echinus sinuöse Eindrücke zurück lässt, ohne Zusammenhang mit den Ambulacralcanälen. Er hat sehr zarte Wände und das Ansehen eines venösen Sinus. Er entspricht dem Gefässcirkel am Rücken der Asterien und steht in demselben Verhältniss zum Herzen wie dort. Wo der Gefässcirkel an das Becken der Madreporenplatte anstösst, erhebt sich aus dem Gefässcirkel die Fortsetzung zum Herzen, *welche sich vom Cirkelgefäss* aufblasen lässt. Um eine klare Vorstellung vom Herzen zu bekommen, muss man es bei Cidaris untersuchen, es ist bei Cidaris ein weiter, ganz gerader Canal mit dicken weichen Wänden. Nach oben setzt es sich in eine Arterie fort, welche in den arteriösen Gefässkreiss des Œsophagus übergeht."—Anatomische Studien über die Echinodermen, von J. MÜLLER, Archiv, 1850.

Like that of the *Echinus* it is confined to the alimentary canal. My dissections have never demonstrated more than *one* vessel extending back from the œsophageal ring, over the superior surface of the tube. If it be an artery there is no correlate vein. It is traceable along *one* border only of the canal, coinciding with the spiral convolutions of the latter as far as the anal outlet; no capillary system is traceable at its end; no trace of *vessels* of any description can be discovered in the parietes of the alimentary canal. According to my observations, again and again repeated, the *interior* of the blood-vessel of the Sipunculidæ generally is lined with vibratile epithelium; of the truth of this fact, extraordinary though it be, I am persuaded. The corpuscles of the fluid contents of this vessel were found in every species to be the *exact counterpart* of those of the fluid of the visceral cavity. *They are identical in colour, in diameter, in figure, and in structure*, figs. 4, 5, 6 and 7*. But in the fluid of the blood-vessel they are more numerous relatively to the bulk of the fluid than they are in that of the visceral cavity. If the true-blood penetrated into the substance of the solids by means of capillary vessels, there would in the Sipuncle exist no difficulty in tracking its course by means of the corpuscles. It is certain that in these Echinoderms it is not distributed throughout the solid parietes of the alimentary canal. The movement of the corpuscles, which is oscillatory, cannot be traced in *any* case beyond the limits of the primary trunk. In the Vermigrade Echinoderms, then, as in the Echinidæ and Asteriadæ, the system of the blood-proper is a partial and local development. It has few, if any, systematic relations. It fulfils but a very insignificant part in the nutrition of the organism. In the particular of the red colour of the blood-proper the Sipunculidæ approximate the Annelida. In the character however of the presence of corpuscles—organized cells—in the blood (on the supposition that it forms an independent system and not a part of the chylaqueous), they transcend the latter class and approach the higher Articulata.

The corpuscles of the fluids in the vessel and in the visceral cavity present a pink tinge. Each corpuscle is flat and irregularly oblong; remarkable uniformity prevails in their size and structure, figs. 4 and 5. Each exhibits a bright, small, highly refractive nucleus; in some cells a second may be seen. The dimensions of this nucleus are disproportionately small in relation to the containing cell. The colour is dissolved in the fluid between the nucleus and involucrum. These corpuscles are

* Although my observations are quite at variance with those of Dr. Peters as regards the number and distribution of the blood-vessels, our researches on the corpuscles of the fluids are mutually confirmatory. "Betrachtet man dieses Gefäss mit einer starken Loupe, so sieht man einen Strom von Körperchen, welcher sich nach dem Schlund hin langsam fortbewegt, während man bei dieser Vergrösserung in den beiden danebenliegenden rothen Gefässen noch gar keine Körperchen oder Bewegung wahrnehmen kann. Bringt man dagegen den mit Wasser gefüllten Darm eines frischen Thieres unter das zusammengesetze Mikroskop, bei einer etwa 50-maligen Vergrösserung, so sieht man in dem mittleren gelblichen Gefäss die schönste Wimperbewegung, wodurch die Kugeln, welche sich jetzt deutlich als Eier erkennen lassen, vorwärts getrieben werden, in den beiden rothen Seitengefässen dagegen, Blutkörperchen von derselben Form wie in dem Körpergefäss, welche sich ohne Wimperbewegung, und unregelmässig nach vorn hin bewegen. Sehr oft liess sich nichts weiter unterscheiden, als dieser einfache eierführende Canal mit den beiden Gefässen zur Seite." Ueber die Fortpflanzungsorgane des Sipunculus. Von Dr. William Peters, Müller's Archiv, 1850.

peculiar to, and strikingly characteristic of, the nutritive fluids of the Sipunculidæ. In different species they exhibit slight variations of form and size. They are the proper and invariable morphous elements of the fluids in these genera. In many instances however there exist *with* them in the chylaqueous fluid, sperm-cells and ova in variable proportions.

These corpuscles, when observed in the chylaqueous fluid, are not mutually repulsive, but when examined while yet in the blood-channel, they manifest the most extraordinary restlessness. The motion of the individual particles is singularly irregular, as though due to attractive and repulsive forces acting among themselves. The compression of the object under the microscope increases the confusedness of the motion. But what is the real explanation of a phenomenon so unusual in the history of blood-vessels? Are the corpuscles self-motive, or is their agitation excited by an external cause? The former question is at once set at rest by the observation that when the corpuscles escape *out* of the vessel, their motion instantly ceases. The cause is not therefore *in* themselves. The real exciting force is derived from the vibratile epithelium by which the *interior* of the vessel is lined. But is not this extraordinary fact opposed at once by every dictate of analogy? It is, though not by that of the analogy of the blood-system in other genera of Echinoderms. Never before has the physiologist recognised the anatomical *incipiency* of the blood-vascular system of this class. It is the first grade of a novel apparatus in the living organism. Viewed in this relation of imperfect development, the presence of cilia on the interior of its conduits occasions no surprise. From the utter insignificance in magnitude and distribution of the blood-proper system in the economy of the Echinoderms as compared with the chylaqueous, it is beyond dispute that it is upon the latter that devolve the vital offices of nourishing the solids. I have lately discovered that the integuments of all the nude Sipuncles is remarkably fenestrated, *not* with open perforations, but with elliptical spots closed only by a thin ciliated epidermis; nothing therefore but a delicate cuticle intervenes between the chylaqueous fluid contained in the visceral cavity and the aërating medium without. No mechanism can be better adapted to favour the interchange of gases between the fluids divided by such a partition. In the Sipunculidæ the tentacles are *hollow* appendages, ciliated within and without, and penetrated freely by the chylaqueous fluid. It seems hopelessly impracticable to demonstrate any traces whatever of blood-vessels proper in the parietes of these tentacles. They appear therefore to be designed to expose, for the purposes of respiration, the chylaqueous fluid rather than the true-blood. It is however probable that in the Holothuridan genera, as in the majority of Annelids, the true-blood has *its* respiratory apparatus, while the chylaqueous fluid, in the same individual, has distinctly another. With reference, then, to the *blood-proper* of the Echinoderms, it may be stated that, in the genera *Asteriadæ* and *Echinidæ*, including the *Ophiuridæ* and *Ophiocomidæ*, the morphotic elements are *almost unformed*, the fluid is almost a non-corpusculated albuminous solution. In the Sipunculidan and Holothuridan

genera, however, the presence of *determinately* organized cells floating in the nutritive fluids indicates an advance in the organic composition of the latter, as compared with those of the inferior Echinoderms. Thus the morphous elements of a living fluid become criteria of its zoological *status*.

Chylaqueous Fluid of the Echinoderms.—In the economy of all Starfishes the chylaqueous fluid is far more voluminous, and enacts a much more important part than the blood-proper. The latter system, in the *Crinoidea*, *Asteriadæ*, and *Echinidæ*, bears to the former an insignificant proportion. In the Sipunculidæ it has assumed a somewhat greater relative development. In the Holothuridan genera it exhibits the most advanced condition under which it is known to exist in the Echinodermal series. The proportion between these two systems, when they exist together in the same individual, is thus shown to be inverse. The fluid contained in the "water-vascular" system and peritoneal cavity of the Echinoderms is described both by TIEDEMANN and SHARPEY* as consisting of pure unorganized sea-water. In his Monograph on this class, MÜLLER avowedly adopts the same view. Analogy and demonstration will now be shown to be opposed to this opinion.

The peritoneal cavity exists in ALL Echinoderms. In ALL species this space is occupied by a fluid; in ALL SPECIES, including the Ophiocomidæ and Ophiuridæ, this fluid penetrates through hollow axes, into the arms and lobes, and ALL the membranous processes of the tegumentary system. In ALL Echinoderms the INTERIOR of the stomach *and its dependent cæca* are lined with vibratile epithelium. This fact establishes a connection between the Echinoderm and the Medusæ, the gastro-vascular canals of which were proved to be similarly ciliated, whilst it disjoins them in a striking manner from the Entozoa and Annelida, digestive organs of which, as will be afterwards proved, are *never* furnished with vibratile epithelium. The boundaries of the peritoneal cavity are universally ciliated. *All* the integumentary *membranous* processes are lined *within* and *without* with ciliary epithelium, and consist of hollow prolongations of the peritoneal cavity. In *Asterias rubens*, the cutaneous membranous processes are readily distended by injection thrown into the peritoneal cavity. Thus injected they rise, in relief, to a considerable distance above the plane of the integumentary surface.

They are *cæcal* at their distal ends. This fact is absolute throughout the Echinodermal families.

These processes in *Asterias* are disposed, on the dorsum of each lobe, in four longitudinal series. They constitute the veritable respiratory organs. They are designed to expose to the external, aërating element, not the true-blood, but the chylaqueous fluid, the contents of the peritoneal cavity. If this fluid consisted only of pure sea-water, what chemical change could result from such an exposure? No endosmotic and exosmotic currents could occur between fluids of identical densities. And wherefore that remarkable movement of the corpuscles of the chylaqueous fluids in the

* Art. "Echinodermata," Cyclop. Anat. and Phys.

interior of these processes, if not for the purposes of aëration? In the solid parietes of these membranous processes no vestige of a blood-system can be discovered.

The *fluid* of the peritoneal cavity in *all* the inferior Echinoderms is charged with globules or cells, which, however, are less determinately organized, and fewer in number, than those of the chylaqueous fluid of the superior genera. That of the Asteriadæ and Echinida is less corpusculated than that of the Ophiocomidæ and Ophiuridæ, the arms of which are unoccupied by the prolongations of any of the viscera; that, on the other hand, which is found in the peritoneal space of the Holothuriadæ, is probably charged with highly organized cells, and that too in presence of a complex blood-system, and of a peculiar organ expressly designed for the introduction of water into the interior of the body. If the contents of the peritoneal cavity consist of pure sea-water, whence the necessity for the superaddition of this peculiar and unparalleled organ, 'the respiratory tree'? It is not the homologon of the peritoneal space of the Asteriadæ, as maintained by physiologists, for this space as well as this organ exists in *Holothuria*. It will be afterwards proved, unquestionably, that if the office of this 'tree' be respiratory *at all*, it can discharge this function only in a secondary and incidental manner.

In the peritoneal space of the *Sipunculidæ* there exists a large volume of opalescent fluid, holding in suspension corpuscles of definitive organization; and *yet* the *bulk* and basis of *this fluid* also is *salt water* (figs. 6 and 7). They exhibit a pink hue, like those of the blood-proper, figs. 4 and 5 of the Sipuncles.

They are flattened, irregularly oval cells, bearing a single minute nucleus. These cells, in fact, correspond in every minute particular of colour, structure, figure and dimensions, to those found in the blood-vessels of the same Sipuncle. Wherefore this identity? The question cannot be eluded. They *must* be one and the same. Evaporated, chloride of sodium is disclosed in a rich crop of cubes and octahedrons.

The morphotic elements of the fluid of the peritoneal cavity of *Asterias rubens*, which in number and amount vary at different seasons, consist of irregular cells (fig. 8), jagged and broken in many instances, nucleated and perfectly organized in others; some cells are compounded of several minutely granulated secondary cells, the group being enveloped in an involucrum, from the circumference of which thready appendages project. These latter cells are NOT sperm-cells; the thready processes are accidental formations, depending upon the fibrillation of the contents of the cells; in some examples the threads occur under the character of a flattened projection or bulging of the involucrum. A "molecular basis" may also be observed in form of minute cells and granules. If the specimen of *Asterias*, from which the fluid is taken, be allowed to remain some time out of the water before the examination is made, these corpuscles may be obtained in far greater number. The corpuscles in the *water-vascular* system *or feet*, are always found *to be identical with those of the peritoneal fluid*. The *contents* of the digestive cæca are *identical in composition* with the peritoneal fluid; it differs only in the absence of the largest cells of

the latter. Its 'molecular base' is like that of the latter, it affords abundant crystals of chloride of sodium like the latter. These and other reasons have satisfied me that the bulk of the fluid contained in the peritoneal cavity of *Asterias* is derived from that which enters through the mouth into the digestive cæca, in which the first phase of the digestive process is performed; the second and subsequent changes, by which it is raised to a higher grade of organic composition, occur during its sojourn *in the peritoneal space*, into which it passes by exosmosis, from the digestive cæca. If the individual be placed for some time in pure sea-water, destitute entirely of organic material, the digestive cæca and the peritoneal space will be found to contain the same fluid, almost completely free from all traces of organic substances*.

* I have instituted, at great labour to myself, a series of variously devised observations, with a view to set at rest, if possible, the question relative to the real source and nature of the fluid contained in the peritoneal cavity of *Asterias Rubens*, and through this example, to close the controversy, in relation to all other Echinoderms. First, then, is it salt-water? if so, how does it gain admission into the cavity in which it is contained? For answer, I affirm that under the ordinary circumstances under which these animals are examined, *it is almost* pure sea-water; but in the *natural* state (that is, when the water taken into the digestive organs contains, as it always does in their native habitat, organic substances, living and dead), *it is a chylaqueous fluid*, in which the first steps of vitalization and organization have commenced, and that in *Asterias* IT IS THE REAL SUBJECT of the respiratory process, the true-blood not being brought in any way under the agency of the external element. This answer applies to ALL Echinoderms. The existence and offices of this fluid explain the fact of the suppressed or rudimentary development, under which the true-blood system obtains in these animals. The latter performs merely the functions of pabulating very partially the solid elements of structure; the chylaqueous fluid is the veritable seat of the blood-making and respiratory processes. And 2nd, I reply that every method of examination fails in proving the existence of any pores or orifices of any sort whatever in the integumentary boundaries of the peritoneal space in *Asterias*. MÜLLER has arrived at the same conclusion.

He remarks, "Die respiratorischen Röhrchen auf dem Rücken der Asterien, welche mit der Bauchhöhle communiciren, sollen zufolge der Injectionen von TIEDEMANN am Ende offen sein, und zum Wechsel des Wassers der Leibeshöhle dienen. Nach EHRENBERG dagegen sind die Röhrchen am Ende geschlossen, er sowohl als SHARPEY sahen die Strömungen im Innern am Ende umkehren. An jungen lebenden Exemplaren des *Asteracanthion violaceus* sah ich dasselbe, und es gelang mir nicht eine Oeffnung wahrzunehmen." (MÜLLER, Archiv, 1850, 121.)

On the contrary, in the larvæ of the Echinoderms, AGASSIZ maintains, on the ground of his own observations, that the external water passes by a direct stream into the cavity of the body. Such an improbable statement requires however to be carefully verified. These are the words of AGASSIZ:—"Bei den niederen Thieren besteht eine innigere Verbindung zwischen dem Innern und dem umgebenden Medium, als in einer der höheren Classen. Das Wasser strömt durch unzählige Poren in ihren Körper und füllt seine Höhle. Einige von diesen Röhren nehmen eine sehr eigenthümliche Anordnung in den Echinodermen an und dienen zugleich zur Ortsbewegung. Da dieser Apparat einer der ersten ist, welche in dem Jungen ercheinen, so muss ich seine Structur bei den Seesternen anführen. Die hohlen Füsschen sind in der Durchschnitts-Abbildung herabhängend dargestellt. So haben wir einen hydraulischen Apparat von sehr zusammengesetzter Natur," &c.

Injections carefully thrown into the peritoneal space distend, and enter into the membranous projections of the integument already described, but it never escapes externally, except through a rupture of the delicate containing parts. A *coloured thin* fluid (*i. e.* sea-water) cautiously injected into the cæca through the mouth (first laying open the integuments), will distend these parts, and *invisibly* and *without rupture* slowly transude into the peritoneal space; but injection consisting of size diluted with water, will not escape out into the

The Asteriadæ in general coincide with *Asterias rubens* in the microscopic and chemical characters of the chylaqueous fluid. In *Solaster papposa* (fig. 9), the motion of this fluid in the hollow interior of the membranous appendages may be readily demonstrated in a living specimen. The morphotic elements of the fluid in this genus consist of nucleated cells, flattened in form, charged in general with little or no granular matter, inferior in dimensions to those of *Asterias*. The molecular base, as obtained by evaporation, is small in amount, while the fluid evidently holds albumen in solution; for when collected in sufficient quantity, and treated with nitric acid, it becomes distinctly opalescent. In fig. 9 these corpuscles are accurately represented as they were found in several specimens examined, and magnified 350 diameters. They vary so much among themselves, and according to the size of the specimen, that exact measurements of individual examples would yield no useful results. In *Cribella oculata* (fig. 10) the cells of the chylaqueous fluid are more globular in figure; the largest are distended with secondary cellules, and nucleated, the involucrum of the maternal cell being more distinct than ordinary in other genera. In this species also, the granular matter of the fluid is wanting; nitric acid, however, gives an obvious opacity.

In the *Echinidæ* the blood presents a slightly higher grade of development than that of the Asteriadæ, and the digestive canal is furnished with a second orifice. The canal is always found to be filled with solid ingesta, composed for the most part of sand and clay. These characters constitute important zoological features. They explain the fact of the diminished bulk and pure water-like appearance of the contents of the peritoneal cavity in *Echinus*. I have proved, as conclusively as a negative pro-

peritoneal space; there are therefore no OPEN perforations in the parietes of the digestive cæca. From the interior of the latter organs into the former space, no fluids therefore CAN pass but by exosmosis. When a fresh, living Asterias is immersed in fresh sea-water, coloured with cochineal or carmine, and allowed to remain thus immersed for about an hour, and then carefully washed in clear sea-water, and then opened, it will be found that the contents of the peritoneal cavity are perfectly free from the smallest trace of the coloured fluid, while it may be always, under the conditions of this experiment, detected *in the interior of the cæca*. These and parallel observations prove that *Asterias* continually, and for the purpose of feeding, draws into the stomach a large volume of water, sending by a determinately directed pressure, a portion into the cæca, and rejecting the remainder again by the mouth. Of *hundreds* of living, healthy Starfishes which I have dissected, it is remarkable to relate, that *not one* has ever contained anything whatever in the stomach. The primary process of digestion is performed in the Asteriadæ in the central stomach, and that rapidly, the unappropriated portions being immediately disgorged, while the rest is admitted into the digestive cæca. The cæca are never charged with anything but *sea-water*, rendered slightly opake by the chyme and the parietal secretions of these parts. This fluid therefore exhibits a grade of organization inferior to that contained in the peritoneal cavity, since the latter is replete with unquestionably organized elements, of which the former is destitute. The true food of the Starfish appears therefore to consist of nothing but sea-water, and those albuminized organic substances which it may perchance hold suspended or dissolved. According to my examination, the chylo-peritoneal fluid in all the asteroidal and globular Echinoderms is intimately analogous in composition. In the Vermigrade orders, it resembles much more that which exists in the Annelida; it is beyond question here a vital fluid. In the intestine of the Holothuriadæ and Sipunculidæ, sand and other refuse matter are always found.

position admits, that the peritoneal cavity in these Echinoderms, like that of the Asteriadæ, does not *openly* communicate with the exterior. The fluid contents enter at the mouth, perform a part in the first stage of digestion, and then carry a portion of the product thereof in solution, probably by exosmose, into the open cavity of the body, where the fluid is set in motion, determinately, by the agency of vibratile cilia, travels round and round the concave of the shell, penetrates the hollow axes of all the membranous processes of the shell, *where it experiences the change of oxygenation*, conveys the RESULTS of this change to *the blood-proper*, and replenishes the water system or ambulacral feet. It contains flattened corpuscles (fig. 11), the largest of which are provided with an involucrum, bearing particles of limpid oleine. A cell here and there may be seen, the involucrum of which apparently projects out like a cilium, and when these are numerous it is easy to mistake such an appearance for those characteristic of a sperm-cell. It is *really* due to the fibrinous contents coagulating in *lines* on escaping. When a small test-tube is filled with the peritoneal fluid (an experiment which demands the sacrifice of ten or twelve individuals), nitric acid will prove clearly the presence of albuminous principles. From the total absence of proper-blood-vessels in the membranous structure and processes of the shell, or integumentary system of *Spatangus* and *Echinus*, from the fact that these latter processes are hollow, and openly communicate with the chamber containing the chylaqueous fluid, by which they are filled, and that their parietes *within* and *without* are profusely lined with cilia, the conclusion is not to be disputed, that they constitute the real organs of breathing, and that the real *subject* of the respiratory change is the chylaqueous fluid, and not the true-blood, which is limited to the central viscera in its circulation. This conclusion, drawn by fair induction from anatomical and physiological considerations, is corroborative of the view suggested by the results of microscopic and chemical inquiries, that the fluid contents of the peritoneal cavity in the Echinidæ, however nearly in appearance they may resemble sea-water, are vitally and organically endowed. In the Ophiuridæ and Ophiocomidæ (fig. 12) the chylaqueous fluid, which occupies the cavities of the body and the hollow axes of the arms, is essentially similar in composition to that of the former Echinoderms. In these genera also it is unquestionably the fluid which is really oxygenized in respiration. An exact and faithful account has now been given of the chylaqueous fluid of the inferior families of Echinoderms. This distinction must be noted between it and that of the superior genera; that in the Asteriadæ, Echinidæ, Ophiocomidæ, and Ophiuridæ, the morphotic elements have not, taken as a whole, attained to the character of *definitively* organized cells. They do not in these genera conform so obviously to one typical size and figure, as to show that they are produced and multiplied under the directive influence of a *determinate* force. They are unclassifiably various in form and shape. They are too scanty in number to act an important part in vitalizing the fluid in which they are suspended; though it must again be repeated, this fluid is unquestionably an albuminous, living, corpusculated solution. These

observations suggest the inference, that the blood-making process in the examples of these Echinoderms is to some extent independent of floating cell-agency. The absence of proper-cells from vital fluids, attest to the physiologist a near approach, as respects the composition of such fluids, to the standard of an inorganic body. And the novel truth will flow from these researches, that, in structure and mechanism of nutrition, the living solids of the organism descend from the complex to the simple in the *same degree* as that in which the fluids may have fallen in the scale of vitality. During the examination of the fluids of other classes, the singular fact will be established that the chylaqueous fluid of the young of the Annelid, and of the larvæ of some Insects, *is scarcely, if at all, corpusculated*; that the fluids become more and more charged with floating-cells the older the animal becomes, and that these cells actually change their structural characters as the growth of the animal advances. What is transitory in the Annelid and the Insect, may be the permanent condition in the Echinoderms. Then, the paramount question arises, in the absence of floating-cells from the fluids of the Echinoderms, in what manner are those fluids vitalized, raised from the inorganic to the organic condition? It is probable that as *all* the fluid which reaches the visceral cavity passes through the digestive organs, and transudes the solid parietes of the latter, by which it receives the impress of the vital force from the living solids, its organic matter assumes the form of albumen and fibrine, which, incorporating with salt-water, becomes a vital fluid.

The anatomy of the Holothuriadæ and Sipunculidæ places beyond controversy the correctness of the conclusions presented in the preceding pages, in relation to the real physiological meaning of the chylaqueous fluid in the inferior Echinoderms. In these orders the integumentary hollow membranous process more or less completely disappears, and is replaced by the peculiarly fenestrated structure of the integuments formerly described in the Sipunculidæ, to which is superadded a system of plumose tentacula or ramose cirrhi, which are confined to the cephalic extremity of the body. The real structure of these beautiful appendages has never yet been demonstrated. All naturalists have *guessed* that they fulfil a respiratory office. No observer has attempted to unravel the mechanism by which this office is accomplished*.

In different genera of these two orders, these appendages vary illimitably in number and size and figure. Such external diversities are however accompanied by no structural and essential differences. In ALL species they constitute merely hollow

* For confirmation of this statement I refer to all the recent works of French and English Comparative Anatomists, especially to CROCHARD's edition of the 'Règne Animal.' It is impossible that the true structure of the cephalic appendages of the Sipunculidæ and Holothuriadæ could have been determined without a previous knowledge of the real physiological signification of the chylaqueous fluid in the animal series. The organs destined to receive THIS fluid are strikingly different in structure from those designed to circulate true-blood. The cephalic appendages of these Vermigrade Echinoderms are not comparable in the remotest degree to the gills of fishes. Between them and the respiratory organs of some species of Annelida there is, however, a very intimate analogy.

processes, though variously scalloped, *into which the fluid of the visceral cavity freely penetrates.* At their bases they are supplied with a few true-blood-vessels. They are lined *within* and *without* by ciliary epithelium; in this particular they are not to be distinguished from the corresponding organs of those species of Annelida in which the chylaqueous fluid alone is submitted to aëration.

It is important to remember, that, notwithstanding the existence of "the respiratory tree" for the *direct* admission of water into the interior of the body, in *Holothuria* the peritoneal cavity, as formerly stated, is filled with a highly corpusculated fluid which penetrates the hollow cephalic tentacles, as in *Sipunculus*, to receive the influence of the surrounding medium. In the Sipunculidæ the peritoneal cavity is occupied by a richly organized fluid profusely charged with corpuscles of peculiar and distinctive microscopic characters (see figs. 6 and 7). The whole interior of the cavity is lined with vibratile epithelium, and the motion of cilia prevails over the hollow interior of the cephalic tentacles, sustaining in constant and rapid oscillation the corpuscles of the chylaqueous fluid, and in these situations ministering directly to the function of respiration. As exhibited in the above illustrations, which are strictly faithful to the original, the morphotic elements of the fluid of the peritoneal cavity in the Sipunculidæ differ remarkably from the corresponding elements in *all* other Echinoderms. They consist, as in part already described, of nearly flattened oblong bodies, inclining to the oval, containing a bright, highly refractive and very small nucleus. In some cells a second nucleus may be discerned. The parent cells are filled with a fluid which is perfectly devoid of granules and molecules. *This fluid* has an obviously pink or faint red tinge. From the contrast between their own colour and that of the fluid (opalescent) in which they float, the cells become beautifully conspicuous objects; but in the Sipunculidæ and Holothuriadæ these bodies are so numerous as to impart to the fluid, viewed as a whole, a thick, milky pink appearance. The *organic* quality of this fluid in these orders, at all events, it is impossible to doubt; nor can it be disputed for a moment, that the cavity in which it is contained in the Vermigrade Echinoderms corresponds, nay, is anatomically identical, with that which holds the LESS opake and less organized peritoneal fluids of the Asteriadæ, Echinidæ, Ophiuridæ and Ophiocomidæ. In these orders, severally, it is the same fluid, chemically and physiologically. In all it occupies the peri-visceral or peritoneal chamber. These indisputable facts establish conclusively the view which denominates the peritoneal fluid of *Asterias*, even though it may *look* like pure water, as a vitally endowed fluid. In addition to the bodies above described, which are the *proper* corpuscles of the chylaqueous fluid in *Sipunculus Harveii* and *S. Johnstoni*, *others* may be observed in every specimen and at every season of the year, which severally resemble germ-cells and sperm-cells*.

* I have designedly avoided in the text all discussion as to the real character of these ova- and spermatozoa-like bodies which are constantly found in the chylaqueous fluid of nearly all zoophytic, radiate and articulated animals. In another communication I propose to contribute materials which will probably set the question at

There exists no difficulty whatever, in the case of the Sipunculidæ, in proving beyond doubt, that in them, at all events, the sea-water (which in these orders, as in the inferior Echinoderms, constitutes the bulk and basis of the chylaqueous fluid) is NOT *derived directly from without.* I have repeatedly shown, by variously contrived injections, *that no fluid whatever* will escape externally through the skin or integument, and that the hollow membranous appendages at the head, and which the animal exserts *by distending* or injecting them with the fluid of the peritoneal cavity, proving their hollowness, are *cæcal at their distal extremities.* As far as it is possible to arrive at certainty in anatomical demonstrations by negative proofs, it may now be held as established, that in ALL Echinoderms, to which these proofs apply, *i. e.* the Asteriadæ, Echinidæ, Holothuriadæ and Sipunculidæ, there exist no perforations or orifices in the integuments in any part, or under any form, through which the external water can *directly* gain admission into the peritoneal cavity. The inference is then irresistible, that it enters at the mouth in form of alimentary material, reaches the digestive canal, and thence passes into the great cavity which surrounds the viscera. In the Echinodermata, as a class, it is impossible to dispute the importance of the functions enacted by the fluid contained in the peritoneal cavity (profusely ciliated as are its walls), when regarded especially in connection with the peculiar structure and situation (when distributed universally over the integumentary surface of the body, as in the Asteriadæ and Echinidæ, or in part centralized at the head, as in the Sipunculidæ and Holothuriadæ) of the organs of respiration.

I have already lamented that my opportunities of examining the Holothuridan Echinoderms, in the living state, have been few. Several points of surpassing interest, as the climax and triumph too of the preceding inquiries, in the anatomy of these genera demand scrupulous revision. 1st. Does the water admitted into the "respiratory tree" serve to aërate the blood-proper or the chylaqueous fluid? This question can only be answered by first determining the exact situation of the blood-vessels in relation to the parietes of this unprecedented organ. 2nd. Are the tentacles the scene of a *double* respiratory process, by which the blood-proper and the chylaqueous fluid are aërated simultaneously? Analogy renders it certain that the integuments of these genera, like the Sipunculidans, are 'fenestrated,' and that with express view to the aëration of the contents of the visceral cavity. These inquiries have thus placed in clear light the interesting fact, *that there prevails but one essential type or plan of structure in the integumentary system of all Echinoderms,* and that the blood-proper may have its own respiratory apparatus, or that it may be aërated through the medium of or by the chylaqueous fluid, *itself having first received* oxygen from the surrounding element.

The Entozoa constitute, really, the true commencement of the Annelida in the

rest. At present I will only commit myself to the statement, that in the fluid of the peritoneal cavity of the Sipunculidæ the *germ-cell*-like bodies are veritable ova, and the sperm-cells are veritable spermatozoa.

zoological series*. Excluding for the present the cystic orders, the two leading divisions, established by Professor Owen under the appellations of Cœlel- and Sterelmintha, are distinguished from each other by a deep line of demarcation, two great classes of Entozoa which differ in organization far more remarkably than any helminthologist has ever yet supposed. On this occasion, however, it is proper that my observations should be confined to a consideration of the fluids.

In the *Nematoidea* the intestine is scarcely at all attached to the integumentary cylinder. The space which intervenes is filled with a corpusculated fluid, remarkable for its viscidity and the molecule-like size of its corpuscles, which is truly chylaqueous. The system of the blood-proper in all Entozoa is very inferiorly developed, and the blood-proper itself in *all* species is colourless and perfectly fluid, holding no globules or cells of any sort in suspension. In this particular they are identical with the Annelida. This fluid, contained in the peritoneal cavity, is the real reservoir of their nutrition. Traced through the class in living specimens, its history will prove the history of the real mechanism of nutrition in these animals. The Cestoid Entozoa differ from the Trematoid and Nematoid orders, in the same characters exactly, as the Nemertine Annelida differ from all the other orders of their class. In the *Cestoidea* the alimentary canal is intimately adherent to the integuments, obliterating the peritoneal space and constituting the solid-worms (Sterelmintha) of Professor Owen. The true chylaqueous fluid is contained IN the canal in this order, a disposition of the fluids which will be found to prevail also in many species of Trematoid and Nematoid worms. Differing in anatomical situation, these two varieties of fluids will be found to differ in physical characters.

In those Entozoa in which the chylaqueous fluid is contained *in* the recesses of the alimentary organ or digestive canal, it presents characters which distinguish it conspicuously from that of those families in which it occupies the cavity (visceral) *without* the alimentary canal. In the latter case it oscillates freely in the cavity, driven by the muscular contractions of the intestinal and integumentary parietes. In the former it moves very little in its containing cavity. This is true of all Entozoa allied to *Tænia*. The fluid contents of the alimentary system, which in many species had neither an inlet nor an outlet, is quite stationary. *In these worms no part either of the exterior or interior of the body* is ciliated, although those Annelida, such as the

* This observation is founded upon, and justified by, the results of my researches into the organization of some of the Nematoid and Cestoid Entozoa[1], but especially into the anatomy of the Nemertine Annelida, under which division I include the *Gordiusidæ*, *Planariadæ*, *Borlasiadæ* and *Liniadæ* orders of Annelids, on the TRUE structure of which no light has hitherto been thrown by anatomists, and between which and the Cestoid Entozoa especially, affinities, in structural plan of a remarkable character, and hitherto unrecognised, exist. As my opportunities for examining *recent* Entozoa have been few, I regret that I cannot in the text present many examples of the exact microscopic characters in this class of the fluids.

[1] See the author's Report on the British Annelida, Transactions of the British Association, 1851.

Nemertina and Planariæ, whose organization, as already stated, is remarkably analogous to that of the Cestoid Entozoa, are universally covered externally by a ciliated epidermis.

The characters of the chylaqueous fluid in the entozoon of the Hake are illustrated in fig. 13. It is a thick tenacious fluid, bearing a vast multitude of minute, highly refractive molecules, distributed through a hyaline semifluid substance resembling the white of egg. I have proved, by repeated observations, that this fluid is the normal contents of the alimentary organ of these worms. It intimately resembles the semifluid substance which fills the blind diverticula of the digestive system of the Planariæ. It is in both instances undoubtedly a vital, nutritive fluid. It is in *this fluid* that is performed the part which devolves upon the corpuscular elements. When, accordingly, in some of these sterelminthous species *a space does* exist between the exterior of the digestive organ and the solid parietes of the body, that space is filled only with a limpid, non-corpuscular liquid, which is fitted to answer no other than the mechanical purpose of facilitating the slight movements of which the intestine is capable. It may be accepted as a rule applicable to all Entozoa and all Annelida, that when the intestine is *intimately* tied to the integuments, the *ordinary* chylaqueous fluid is materially reduced in volume, or altogether disappears. Under such circumstances, in the solid Entozoa it reappears under a new character, although virtually the same fluid, *in the interior* of the digestive system. In some species of Annelida, however, as the Earth-worm and the Leech, the diminution or suppression of the chylaqueous fluid is compensated by a correspondingly greater development of the truc-blood system. Future researches will inevitably show that in the Entozoa the blood-proper system is a very subordinate element of the organism. For the maintenance of the living solids, so striking in these animals is the "simplicity" of their structure, the chylaqueous fluid will be found physiologically sufficient. An attentive study of the fluids will result in the discovery of an unfailing clue whereby in these animals to reconcile with physiological principles the paradox of the arrangement and histology of the solids. By no other road can this desirable point be attained*.

* I have recently studied with great attention the valuable contributions of M. Siebold[1] and M. Van Beneden[2] to helminthology. I am persuaded that these distinguished men have not yet discovered the path which is destined to conduct the physiologist to a true understanding of the organization of the Entozoa. I warrant this bold statement by *the facts* which already my own labours have enabled me to establish. The *fluids* will unerringly conduct the future student to a perfect comprehension of the *solids*. The *solid elements* of these paradoxical organisms will ever remain insoluble enigmas if studied independently of, and without refer-

[1] Mémoire sur la Génération Alternante des Cestoïdes, suivi d'une Révision du genre Tetrarhynchus. Par C. T. de Siebold. Traduit de l'Allemand par M. Camille Dareste, Ann. des Sciences Nat., 1851, 3me Série.

[2] Les Vers Cestoïdes ou Acotyles, considérés sous le Rapport de leur Classification, de leur Anatomie et de leur Développement. Par P. J. Van Beneden. Bruxelles 1850, avec 26 Planches.

The *Annelida*, in a much more marked degree than the Entozoa and Echinodermata, are characterized by the possession of two distinct systems of nutrient fluids, of which one consists of the proper and true-blood, circulating definitively in perfect and closed vessels; the other, of a liquid mass, filling, in nearly all instances, the peritoneal cavity, and corresponding with, as it is the linear continuation of, that which, in the Entozoa and Echinoderms, was distinguished as the chylaqueous fluid. In the Annelida this peritoneal fluid is charged with corpuscles, which in different genera are sufficiently dissimilar to constitute significant generic characters, and these differences are even traceable to different species of the same genus. As in other classes, so in the Annelida, upon these two fluids two distinct and separate physiological functions devolve; each is essential to the maintenance of life. The history of the chylaqueous fluid and that of the blood-proper will be studied in this class with more minuteness than in the former, for it seems not a little probable that the "tangle unravelled" by the study of these fluid elements of nutrition in the Annelids will conduce to more exact views than those now prevalent in physiology with reference to the mechanism of nutrition in all invertebrate animals.

All the recesses and ramifications of the general cavity of the body in the Annelids communicate freely with each other, constituting thus one common space. This space is lined by a distinct membrane, which is obviously the anatomical analogon of the peritoneum, and is filled by a fluid which is *unquestionably an organic fluid*. In the Annelida the peritoneal membrane is not vibratile, the oscillations of the fluid contents cannot therefore be due to ciliary vibration. This fact distinguishes the Annelids from the Echinoderms, Medusæ and Zoophytes: it further proves that the movements of the chylaqueous fluid are not in ALL cases dependent on the presence of cilia. The rule is suspended in this class. In *Glycera alba*, and in one or two other species, however, the peritoneum where it penetrates the appendage is lined by vibratile cilia. Anteriorly to the researches herein recorded, I am not aware that any anatomist has recognised the real physiological meaning, or described the true histological characters of the chylaqueous fluid in the Annelida. In the historical introduction to this paper I have already indicated the extent to which the researches of M. QUATREFAGES have proceeded in that direction. To him undeniably belongs the credit of having *independently* determined the fact of the existence of 'a fluid' in the visceral cavity of the Annelid. The first publication of his generalized results occurs in the Annales des Sciences Naturelles for 1852. The results of my dissections, announcing the existence of a chylaqueous system of fluid in the Annelida, were first made public at the Meeting of the British Association, which was held at Swansea in the year 1849. It was not until this year (1852) that I became acquainted

ence to, *the fluids*. The light already reflected on the question of their organization, which has so long remained unanswerable, has rendered the *habitats*, unusual though they be, affected by these eccentric beings, no longer an *arcanum*, a theme of superstitious wonder, in physiological science.

with the more recent contributions of M. QUATREFAGES towards the study of the difficult subject of the anatomy of the Annelida. In the analysis of his memoirs* I have endeavoured conscientiously to assign to the versatile genius of this French naturalist the real merit with reference to this subject to which it is entitled. I have shown, I trust correctly, to what a very limited distance the parallelism continues between both the researches and the conclusions of M. QUATREFAGES and my own.

One important difference between our observations, severally, should be here unequivocally defined. He states that in *all* Annelida the "fluid of the visceral cavity" is circulated by means of the vibratile cilia by which the cavity in question is lined. I have affirmed the very contrary of this statement as the *uniform* result of my investigations. In no Annelid whatever, the Aphrodite excepted, is the cavity containing the chylaqueous fluid lined by vibratile epithelium. Cilia exist on the *internal* hollows at the bases of the feet, and on the tentacles of *a few* species only.

It is only possible in the larger species to obtain a sufficient quantity of the peritoneal fluid for the purposes of chemical analysis. In *Arenicola, Terebella nebulosa, T. conchilegia,* and the largest Nereids, it may be readily collected for examination. In these species it exceeds sea-water in specific gravity, being from 1·032 to 1·034, the water from which they were taken being 1·028. Salt water being the basis of this fluid in the Annelida, the superaddition to it of fluid and solid organic principles accounts for the high density of the peritoneal fluid in these animals. On standing a distinct coagulum is precipitated, carrying with it the corpuscles. By the coagulum the presence of fibrine is announced, and that of albumen is distinctly proved by the addition of nitric acid; slow evaporation yields a rich crop of the cubes and octahedra of the chloride of sodium. The morphotic elements vary in a remarkable manner in different species; that is, for the same species, the individuals being different, the corpuscles of the chylaqueous fluid are *constant*, and *nearly* the same in microscopic characters for *every* season of the year. In different species therefore these solid elements of this fluid, like the corpuscles in the blood of vertebrated animals, become signs of specific distinction, but the specific variations are much less marked than the generic. In the case of the Annelids, it is susceptible of demonstration that the floating cells of the chylaqueous fluid vary in apparent *structural* characters with the *age* of the individual. For some time after the emergence of the young from the ovum and before the development of the branchial, pedal, and tentacular appendages, the chylaqueous fluid is a limpid, transparent, *non*-corpusculated liquid. It bears not a trace of floating corpuscles. This fact is full of interest. As the worm progresses in growth, so the corpuscles slowly appear in the chylaqueous fluid. The blood and the blood-system are produced *before* the floating cells are generated in the chylaqueous fluid. The conclusion is evident. In the young Annelid the agency of the floating corpuscles of the chylaqueous fluid is *not required* either for the elaboration of the fibrine or the production of the pigment of the blood-proper. But it is important to guard against

* See *ante*.

the apparent continuation of this inference, that what is true of the young is so also of the old. In relation to the history of the solid elements of the chylaqueous fluid in the Annelids, another fact of equal value and significance with the former may here be mentioned. For some time before the death of the old worm, whether the death take place by fission of the body into fragments, or by general decay and decomposition, the corpuscles of the chylaqueous fluid go on gradually disappearing. The ova and sperm-cells accumulated in the fluid of the visceral cavity during the season of reproduction, also supersede, to a great extent, the proper corpuscles of this fluid. At this period however the blood-system acquires a very conspicuously augmented development.

In studying the histology of the solid elements of the fluids in the Invertebrata, it is desirable to forearm the observer with one admonition. When these bodies dehisce under the eye in the field of the microscope, the semifluid contents are projected out in strings or filaments; the highly fibrinous fluid contained *in* the cells *coagulates as it escapes*. This appearance has deceived some observers into the supposition that such cells are *ciliated*, epithelial, self-motive bodies. They are not so. I have ventured to estimate this frequent fact as a direct demonstration of the theory advocated by some physiologists, that the floating cells of the fluids *secrete* a self-coagulating principle. But at the present stage of this inquiry the question admits of no reply, whether the mode indicated is *the only process* by which fibrine is generated in the living fluids? It is probably not so. My present conviction is that fibrine, at all events in the vital fluids of the invertebrated animals, MAY be evolved in a non-corpusculated fluid, in virtue of a zoo-chemical process enacted by the liquid *per se*.

Let us now proceed to a detailed description of the corpuscular elements discoverable in the chylaqueous fluid of the principal families of the Annelida.

In *Arenicola Piscatorum* the chylaqueous fluid is very abundant; the cavity in which it is contained is scarcely at all partitioned by segmental septa. The floating cells of the fluid in this worm are relatively numerous and very liable to dehisce, producing digitate, fibrillated, ciliated, stellate and other forms of bodies. These accidental appearances must be discriminated from the entire unbroken cells (fig. 14). These latter are orbicular in figure, bearing a nucleus and filled with minute granules. They are very liable to cohere together into round agglomerated masses (fig. 14 *a*). As the process of evaporation proceeds crystals of chloride of soda appear (fig. 15). In the young *Arenicola* the peritoneal fluid is less abundantly corpusculated and colourless (fig. 31). In the old animal also the corpuscles disappear. During a considerable part of the *summer* ova and sperm-cells abound in this fluid; they should be carefully distinguished from its proper corpuscles. In *Arenicola* the blood-proper is exclusively *aërated*. The integuments bounding the visceral chamber are too dense to admit of any agency of the surrounding medium on the fluid contained within. Under such circumstances the physiologist must admit one of two suppositions. The chylaqueous fluid, *being a vital fluid*, must in *some* manner be oxygenized. In this case this can happen

only *indirectly*, either through the blood-proper, which in this species immediately receives oxygen from without, or through the alimentary system, which is being constantly traversed by a current of fluid-sand. These facts are here noticed in historic connection with the chylaqueous fluid, because they tend to elucidate its peculiar laws.

Between the corpuscles of the chylaqueous fluid in *Nais filiformis* and those of *Arenicola*, a close resemblance is observable, the same disposition to fibrillate on bursting. The typical cell is orbicular in shape, nucleated and granular (fig. 16). Another and more embryonic variety presents only a nucleus, the granules being wanting. In *Nais* the chylaqueous fluid is colourless, and considerable in volume; it is the 'bed' on which the intestine moves. In the several species of the genus *Nais*, these corpuscles present very palpable diversities of shape, though not of structure. There are no external organs in this genus, either for the exposure of the chylaqueous fluid or blood-proper to the aërating element. As the latter is centrally situated, and therefore embraced by the former, it is manifest that the chylaqueous fluid is the more *directly* affected of the two by the external oxygen. The peculiar disposition of the vessels in *Nais* renders this inference only the more probable.

In *Sabella vesiculosa* (fig. 17), and in a less marked degree in *Sabella alveolata*, the general cavity of the body is filled with a fluid, the cells floating in which incline to a uniform spindle-shaped form. Those which are spherical appear to be only the immature phase of the former variety, and between the two there are several intermediate grades. They are almost wholly devoid of internal molecules, being filled with a fluid held together by a capsule of determinate form. In this species, as in the succeeding, the chylaqueous fluid plays no direct part in the office of respiration.

In *S. alveolata* (fig. 18) the chylaqueous fluid is less marked in volume; its corpuscles are spherical and granular, abounding in non-nuclear oleous cells. The blood-system is highly developed. The intestine is tied to the integument by frequent septa.

The splanchnic cavity in *Sabella à sang vert* of M. EDWARDS contains a colourless corpusculated fluid, the cells floating in which are less uniform in size and figure than the corresponding bodies in the preceding species. They are generally observed under the form of small orbicular cells, becoming flattened as they grow older, and unevenly outlined; many of them exhibit that curious tendency to protrusion of the cell-capsule which so frequently characterizes the corpuscles of the chylaqueous fluid. They are all more or less charged with molecules of oleine of high refractive power (fig. 19). In this species, as first discovered by MILNE-EDWARDS, the true-blood is grass-green in colour. The branchial appendages, cephalically situated, and pectinated in form, are penetrated by the latter fluid, very little, if at all, by the former. In this species the chylaqueous fluid is excluded from all direct participation in the process of aëration.

As respects the corpuscular elements of the peritoneal fluid, the *Terebellæ* are more remarkably characterized than any of the preceding species of the Annelida. In the two species, most familiar to us on the coast of Swansea, the chylaqueous fluid is very large in amount. It is a thick, milky liquid, containing large compressed oval cell-capsules, almost individually visible to the naked eye. These corpuscles are not nucleated cells, but flattened vesicles filled with oil-molecules and granules. The cell-capsule is extremely *attenuated*. Others of these large cells seem to consist only of lesser cells aggregated together into circular groups. Interspersed between these bodies may be seen another variety very different from the former. They are chiefly seen in the hollow axes of the tentacles moving to and fro, and in figure uniformly spindle-shaped, destitute of visible contents and pellucid (fig. 20). Those of *Terebella conchilegia* (Plate XXXII. fig. 21) are about one-half the size of those of this fluid in *Terebella nebulosa* (fig. 20). In all other microscopic characters the latter are exactly like the former.

The respiratory process in the Terebellæ is divided in an equal proportion between the chylaqueous fluid and the blood-proper. Each tentacle is traversed by a blood-vessel and by a current of chylaqueous fluid. The visceral cavity of the Terebellæ at the reproductive season is filled with ova and sperm cells. In this genus the mass of the chylaqueous fluid is an important agent in locomotion.

The structure of the common earth-worm is distinguished in several material respects from that of those Annelids forming the subject of the preceding remarks. In *Lumbricus* the segmental partitions are complete and well-marked, tying, at very frequent intervals and intimately, the intestine to the integument, and consequently limiting, almost obliterating the peritoneal cavity. This cavity however does exist, and contains a viscid colourless fluid, bearing spherical, nucleated and granular cells (fig. 22), intermingled with pellucid cellules, which appear only to represent the immature phase of the former. These corpuscles seem to me to be the bodies which Mr. Wharton Jones has mistaken for those supposed to exist in the blood-proper *. The fluid of the peritoneal cavity is of almost momentary importance to life in the Earth-worm. Kept in a perfectly dry atmosphere even for an hour it dies. It cannot be revived; and is covered by a slimy fluid which appears as if it were that of the peritoneal cavity exuded. In the adult state in this worm the system of the blood-proper is highly developed; a circumstance which explains the diminished proportion of the chylaqueous fluid. That the *basis* of this latter fluid consists of water, I infer from the immediate importance of moisture to life. The fact of its existence, and next that of its physiological importance, are quite proved by the results of observation on the young Earth-worm (fig. 23). In the early stage of development the peritoneal fluid in this worm is, relatively, considerable in quantity. In the infancy of this Annelid it fulfils those functions which during adult life are discharged by the true-blood. The peritoneal fluid in the young abounds in spindle-

* Philosophical Transactions, Part II. 1846.

shaped bodies, which are not to be found in that of the adult. In this worm the chylaqueous fluid is not submitted to the process of aëration.

In *Œnone maculata* the chylaqueous fluid, which at the reproductive season is large in quantity, is charged thickly with corpuscles (fig. 24) of one uniform size and figure, and too minute to fall within the defining power of the microscope. They are mere amorphous molecules. In these worms the chylaqueous fluid is not concerned in respiration. The branchiæ bear only proper blood-vessels.

In the case of the Borlasiadæ, Planariadæ and Liniadæ, the chylaqueous fluid is contained in the digestive cæca and diverticula. In some of the Planariadæ, however, I have proved that a space does actually exist between the digestive diverticula and the solid structure of the body, *which is lined by a vibratile epithelium*, and into which probably the external water is in some way admitted. By this water, thus situated, the contents of the digestive cæca are aërated. The fluid oscillating in these cæcal appendages of the stomach is thickly charged with corpuscles, which from their regular character prove this fluid to have already reached a high standard of organization. They occur as elliptical cells in the *Borlasia* from which the illustration (fig. 25) was taken; the fluid abounded also in small orbicular points, constituting the 'molecular basis' of the digestive product. In this worm it is this fluid, and not the true-blood, that is aërated; the latter system is too little developed.

The genus *Phyllodoce* is characterized by the existence of a peritoneal fluid, highly organized and corpusculated, and contained in a space which is almost undividedly continuous from one end of the body to the other; the intestine being tied to the integument by means of bands, which leave the fluid free room to play from one segmental compartment to the other. In *P. lamelligera* the peritoneal fluid is colourless, and contains flattened circular corpuscles, bearing a centric nucleus and filled with oily molecules (fig. 26). Cells of a pellucid character, and much more diminutive than the former, make up the mass of the morphotic elements. In the genus *Phyllodoce* the branchial organs are not at all penetrated by blood-vessels. The chylaqueous fluid, by which their areolæ are distended, must therefore be the direct recipient of the external oxygen.

The *Nereid worms* are all distinguished by the existence of a large amount of chylaqueous fluid in the visceral cavity. The septa of the segments are not complete partitions; the fluid therefore fluctuates with freedom from one end of the body to the other, and assists in a very material degree the locomotion of the worm. In this genus the fluid is characterized by the presence of corpuscles of large comparative size, and of generally an irregular figure (fig. 27). In the season of autumn, as in some other Annelids, these corpuscles, which are the proper solid elements of the fluid, are superseded by true ova, one of which is represented in the centre of the figure. The peritoneal cavity in these familiar worms is always filled with a milky fluid, the organic quality of which cannot for a moment be doubted. The blood-proper is red, but fluid, that is, non-corpusculated, and the blood-system of vessels

is elaborately developed. In the larger species the parietes of the feet are embraced in a framework of reticulate blood-vessels; but the interior of these appendages, which is hollowed into a cavity, filled with the peritoneal fluid, proves that both systems of fluids participate equally in the process of respiration. These worms are the most active in their habits of all Annelids, which results probably from the perfection of their circulating system.

In the genus *Spio** the chylaqueous fluid is subordinate in amount and importance. Its corpuscular elements are imperfectly developed. They are opake milky globules, possessing sometimes a nucleus, and sometimes none: they are almost entirely destitute of granules. In these elegant worms it is the blood-proper system that usurps the office of respiration. The branchiæ convey into contact with the surrounding element very little of the chylaqueous fluid. It is held in the areolæ of the membranous appendages attached to the branchiæ (figs. 28).

In *Nephthys Hombergii*, though not conspicuous, the chylaqueous fluid is considerable in volume. Its motions are limited in consequence of the frequent bridles by which the intestines are tied to the integment. Its corpuscles (fig. 30) are relatively scanty. They are spherical in form, and filled with secondary molecules. The oleous globules which accompany them are numerous. The inferior importance of the chylaqueous system in these worms is compensated by the highly evolved condition of that of the blood-proper. The branchial processes, which are hollow, are filled with a coil of vessels, and with a stream of chylaqueous fluid; both fluids therefore in equal proportions are submitted to aëration.

Of the corpuscles of the peritoneal fluid in the genus *Glycera* (fig. 31) there is this extraordinary fact to be related, that they are *blood-red in colour*, and not unlike in figure and size those of the blood of Reptiles. In this beautiful worm the *true-blood* conforms with the Annelidan law of perfect fluidity; it bears no visible elements, and is light red in colour, the colouring element, as in all other Annelids, being *dissolved* in the fluid mass. It is remarkable that the *fluid* contents of the visceral cavity in *Glycera* should be *colourless*, while the corpuscles which it holds in suspension should be filled with a blood-red liquid, contrasting these bodies in a striking manner with the fluid in which they float. The cavity of the peritoneum is disproportionately capacious. It is little interrupted by segmental septa. The worm is extremely quick and active in its movements. The *true branchiæ* are hollow cylindrical appendages, lined *within* and *without* by vibratile epithelium, and penetrated only by the fluid of the peritoneal cavity. *They are supplied by no true-blood-vessels. The chylaqueous fluid therefore is the subject of the respiratory change, for which these appendages present an appropriate mechanism.*

To this fact the highest interest attaches. It constitutes an undeniable proof that the fluid of the peritoneal cavity *is capable of* discharging the highest function of the animal organism. The presence of *red* corpuscles in the fluid of *this* species does

* Of which three species are found on the coast of Swansea.

not prove that in no other species is this fluid capable of enacting a true respiratory function, since the branchial organs of nearly all Annelids (many by it exclusively) are more or less injected by the fluid of the visceral cavity. On the other hand, a fluid which is destitute of corpuscles may also perform the office of respiration. Floating cells are *not* therefore essential to this process. In *Glycera alba* (fig. 31) the red corpuscles are almost uniformly oval in figure, the oval being compressed, slight, and sometimes a little curved or distorted. These bodies bear nothing in their interior but a red fluid, and a nucleus which is elliptical in shape and very indistinct. An instance here and there occurs in which a few molecules are contained in these cells. WHY the corpuscles of the chylaqueous fluid of this worm and of *Matuta clymenoida* (WILLIAMS) (fig. 33) should, contrary to the universal Annelidan rule, contain a pigmented fluid, it is at present difficult to explain.

In a species of marine (fig. 32) *Nais*, common on this coast, which I have named *N. maculosa*, the corpuscles of the peritoneal fluid are spherical cells filled with granules, the latter being grouped in the centre, and separated from the involucrum by a layer of limpid fluid.

Fig. 33 gives another illustration of the bodies found in the peritoneal fluid of the Annelida. They are those of a worm which I have called *Clymene arenicoida**. Fig. 34 represents those of this fluid in *Sigalion Boa*. In the genus *Polynoë* the peritoneal fluid exists in large quantities. In all species it is richly corpusculated and milky. The blood-system in *Sigalion* is subordinate. The blood-proper is colourless and incorpuscular. The branchial appendages are constructed like hollow tubes, with express view to expose the peritoneal fluid, and *not* the blood-proper, to the aërating agency of the surrounding element. The corpuscles figured (fig. 34) resemble those of scaly epithelium. They are pregnant with molecules and oil-cells. Fig. 35 illustrates another variety of these bodies, from the peritoneal fluid of a new worm, which I have named *Matuta clymenoida*. These corpuscles are bright red, like those of the chylaqueous fluid of *Glycera*. They are very abundant, and give to the whole worm a blood-red colour. They assume the form of flattened scales, bearing two or three molecules and sometimes a nucleus, being, from the scantiness of their contents, transparently delicate and filled with a red fluid.

The contents of the peritoneal cavity of *Aphrodita aculeata* (fig. 36) approach more nearly to those of the Echinoderms than of the Annelida. It must be remembered, with reference to this Annelid, that the water which is admitted underneath the *dorsal felt* is quite distinct from that contained in the cavity of the peritoneum. The former is erroneously described by all naturalists as corresponding with that found in the visceral chamber in the Echinoderms. It is the *latter* and not the former fluid which is the true homologon of the chylaqueous fluid of the Echinoderm.

* A full description of the specific characters of these new species will be found in my Report on the British Annelida, Transactions of the British Association, 1851.

Like that of the latter it abounds little in organic corpuscles, and the general appearance is like that of pure water. The morphotic elements consist of groups of formless granules. The diverticula of the digestive system are so arranged as to be surrounded within by the peritoneal fluid and without by the external water, which is drawn in through the meshes of the felt. This arrangement proves obviously that the contents of the digestive cæca are *expressly* in this Annelid exposed to the action of the *two* fluids described. These 'contents' consist of a dark olive fluid, and though destitute of organic bodies, is endowed with nutritive properties.

In *this Annelid*, therefore, *as in the Leech*, the fluid contained *in* the digestive sacculi is the true equivalent of the chylaqueous fluid of the peritoneal cavity of other species. In the Leech this latter cavity is almost obliterated, in *Aphrodita* it is spacious and occupied by a fluid, colourless and limpid as water. Moreover, in *Aphrodita*, the peritoneal cavity is distinguished from that of ALL other Annelids by the fact that it is lined by vibratile epithelium, in which respect it is allied to the Echinoderms. The fluid which fills the digestive cæca in *Aphrodita* contains no determinately *organized* solid bodies, another respect in which it resembles that of the diverticula of the stomach of the Starfish. The blood-proper in this aberrant Annelid is perfectly fluid, devoid, that is, of all morphotic elements, and yellowish in colour.

In no case in the animal kingdom is the physiologist presented with a more favourable opportunity for determining the real meaning of the floating cells of the nutritive fluids than that which occurs in the Annelida. Here nature performs for him the difficult experiment of separating the albumen and fibrine-producing from the colour-producing parts of the vital fluids. It has been demonstrated that in every Annelid the chylaqueous fluid is more or less corpusculated. In every species this fluid and its corpuscles have been found to be *colourless*; two exceptions only in the whole class were encountered. The chylaqueous fluids of *Clymene arenicoida* and *Glycera alba* were found to be charged with blood-red corpuscles, the fluid in which they floated being *colourless*. Here is an unequivocal demonstration that the *involucrum* of the floating cell is capable of separating from a *colourless* fluid a pigmented fluid, the blood-red contents of the cells! But what can become of the red fluid with which these floating cells are filled? If the cells dehisce while yet in the chylaqueous fluid, the latter ought to exhibit a tinge of the same colour, which is not the case. Further observations are required to track with accuracy the vicissitudes which these singular bodies undergo, but which are singularly calculated to unfold the tale of the real changes which all floating corpuscles are destined to suffer in fulfilment of their peculiar functions.

Blood-proper in the Annelida, and its physiological relations.—CUVIER, LAMARCK, DE BLAINVILLE, PALLAS, SAVIGNY and MILNE-EDWARDS, amongst continental naturalists, have contributed observations on this subject. The discovery of red blood in

these animals became with CUVIER the ground of classification, "Frappé de la couleur si remarquable du liquide nourricier chez ces animaux, il les désigna d'abord sous le nom de Vers à sang rouge."

LAMARCK also viewed the red blood of the Annelida (a name now first devised and applied by him) as an essential distinction of the class*. It was observed at this time by M. DE BLAINVILLE, that in *Aphrodita aculeata* and *Herissa* the blood was colourless†. M. PALLAS had however anticipated both CUVIER and DE BLAINVILLE in these observations, as well as in that of the existence of red blood in many of the Annelida‡. To the laborious researches of MILNE-EDWARDS the zoologist is indebted for a full and complete history of the *colour* and distribution of the blood in the Annelida§. It is however remarkable that an observer of such proverbial accuracy should have overlooked the question relating to the *structure* of this fluid. Remarking that in the *Euniciadæ*, *Euphrosinidæ*, *Nereidæ*, *Nephthys*, *Glycera*, *Arenicola*, *Hermella*, *Terebella*, *Serpula*, *Lumbricus*, *Hirudo*, &c., the blood is of a red colour, he proceeds, "Mais, du reste, examiné au microscope, ce liquide ne m'a pas semblé différer du sang des autres animaux sans vertèbres. Les globules qu'on y voit nager n'ont pas du tout l'aspect de ceux propres au sang des animaux vertébrés : ce sont des corpuscules circulaires dont la surface a une apparence framboisée et dont les dimensions varient extrêmement chez un même animal‖." From the direct expressions used in the above passage, it is manifest that Professor M.-EDWARDS admits the existence of "circular corpuscles" in *the blood* of the Annelida, which according to his description, present the appearance of raspberries, varying much in dimensions in the same individual.

In his learned memoir¶ in the Philosophical Transactions (1846), Mr. WHARTON JONES describes and figures the blood-corpuscles (*sic*) of the Earth-worm and the Leech, and defines, in the following terms, the mode in which the samples submitted

* "Ce qui a effectivement paru très singulier, ce fut de trouver que les Annelides, quoique moins perfectionnés en organisation que les Mollusques, avaient cependant le sang véritablement rouge, tandis que celui des Mollusques, des Crustacés &c., n'a pas encore cette couleur qui dépend de son état et de sa composition, et qui est celle du sang de tous les animaux vertébrés. On sent bien que, parmi les animaux que nous rapportons à notre classe des Annelides, ceux qui se trouveraient n'avoir pas dans leur organisation le caractère classique, n'infirment point ce caractère et ne sont placés ici qu'en attendant que leur organisation soit mieux connue."—LAMARCK, Animaux sans Vertèbres, t. v. p. 276.

† Art. *Vers*, du Dictionnaire des Sciences Naturelles, t. lvii. p. 409.

‡ See his Miscellanea Zoologica, p. 89. "Sectis in dorso longitudinaliter tegumentis, occurrit vasculum lympha sæpe turbidula plenum." From this sentence it is much more probable that in this section PALLAS merely opened the great cavity between the intestine and the integument, out of which the lympha turbida escaped, and that it was not the blood-proper, as MILNE-EDWARDS supposed, that PALLAS described, but the fluid occupying the peritoneal cavity.

§ Recherches pour servir à l'histoire de la circulation du sang chez les Annelides, lues à l'Académie des Sciences le 30 Oct. 1837.

‖ Annales des Sciences, 2me série, Oct. 1848, 'Circulation dans les Annelides,' par M. M.-EDWARDS.

¶ The Blood-corpuscle considered in its different Phases of Development in the Animal Series, by T. W. JONES, F.R.S., Philosophical Transactions, Part II. 1846.

to examination were obtained:—"The blood was most readily obtained for examination from the abdominal vessel, but in extracting it care was required against its becoming mixed with the secretion poured out from the skin in great abundance when the animal is wounded," p. 94. Mr. JONES then observes, "The corpuscles of the blood of the Earth-worm *are remarkable for their great size*, being on an average $\frac{1}{1100}$th or $\frac{1}{1200}$th of an inch in diameter. They are both granular and nucleated cells." Thence this author proceeds to an elaborate account of the metamorphoses which these two varieties of corpuscles undergo. And with reference to the Leech Mr. JONES affirms, "that *while* the corpuscles of the blood of the Earth-worm are the largest which I have yet found in any invertebrate animal, the corpuscles of the Leech are the smallest," p. 95, *op. cit.* Investigations on an extended scale, and conducted with the strongest desire for the real truth, enable me in this place to state most confidently that in the descriptions cited, both from M. MILNE-EDWARDS and Mr. WHARTON JONES, these distinguished observers have fallen into the most extraordinary errors. *In no single species among the Annelida does the blood-proper contain any morphotic elements whatever!* In all instances, without exception, it is a perfectly amorphous fluid, presenting under the highest powers of the best microscope no visible corpuscles or molecules or cells of any description whatever. It is a limpid liquid variously coloured, as formerly and correctly stated by MILNE-EDWARDS, in different species.

In a memoir* recently published, M. QUATREFAGES has re-traversed the ground first opened by MILNE-EDWARDS. M. QUATREFAGES confirms, without a single exception, the conclusions at which his colleague had previously arrived. Historic truth demands it to be stated that M. QUATREFAGES, in the memoir cited, has improved very little upon the original description of MILNE-EDWARDS, and in that little he has become entangled in error.

In the *Arenicolæ* and *Eunice* he describes proboscidian and branchial hearts, which do not exist. M. QUATREFAGES seems to have doubtfully recognised the general fact of the *fluidity* of the blood-proper of the Annelida. He cites however in the same paragraph such striking exceptions to this fact, that he gives no proofs whatever of having mentally realized the *law* which demands that the true-blood of the Annelida *should be invariably fluid*, non-corpusculated, *because* in this class the office which devolves upon the floating cells is performed in the chylaqueous fluid, where alone such cells exist. This remarkable principle, which literally divides the nutritious fluid into two parts, upon one of which the corpuscular agency devolves, upon the other the more special duties of solid nutrition, seems not in the least degree, at any time, to have entered the mind of M. QUATREFAGES.

"Dans deux espèces de Glycères de la Manche, qui toutes deux sont assez communes à Saint-Vaast, j'ai trouvé un sang fortement coloré en rouge par des globules

* Études sur les types inférieurs de l'embranchement des Annelés, sur la circulation des Annélides. Annales des Sciences Naturelles, 1850.

parfaitement distincts et réguliers. Le liquide lui-même était incolore. Ici les globules offrent la plus grande ressemblance avec ceux des Vertébrés. Ce sont de petits disques aplatis de $\frac{1}{120}$ de millimètre environ," &c. In this statement M. QUATREFAGES is incorrect in attributing these 'globules' to the blood-proper, for the blood of this worm, like that of all Annelida, is destitute of every kind of globules; it is perfectly fluid. The corpuscles exist only in the chylaqueous fluid, but the description of them, as conveyed in the preceding quotation, is by no means exact.

In general no distinction into venous and arterial blood is detectable, the *plan* of the Annelidan circulation rendering such a distinction almost impossible. The colouring elements are in all cases fluidified and uniformly blended with the fluid mass of the blood. The colour therefore must be developed in the *fluid*-mass, and *that too without the intervention of any corpuscular* agency, since the true-blood, as already stated, contains NO solid cells.

Glycera alba and *Clymene arenicoida* only excepted, the corpuscles of the chylaqueous fluid in ALL Annelids are destitute of colour. It is not chemically impossible that the coloured ingredients may exist in this latter fluid in a colourless state of combination, and that these ingredients, through entering into new combinations, may become brightly coloured after transition into the true-blood.

In consequence of the impracticable minuteness of the quantity, no direct chemical analysis of the blood in the Annelid can be executed. As to the *colour*, however, analogy removes all doubt that the red tinge is due to the salts of iron and the green to those of copper. In those species of which the blood is light yellow, opake, milky, or lymph-like, it does not follow that the salts of the coloured minerals are altogether absent: they may exist under *colourless* combinations. To the physiologist it cannot be unimportant in this place to demand, if the blood-proper of the great majority of Annelids be a non-corpusculated limpid *coloured* fluid, and the chylaqueous fluid a colourless corpusculated liquid, in what manner, by what agency, does the blood acquire its colour? If it be destitute of floating cells, the production of the pigment cannot be ascribed to the agency of the latter bodies. And if this pigment do not, at all events in a visible form, exist in the chylaqueous fluid, it must be developed during the passage of the latter into the blood-vessels (for, as will be afterwards more fully explained, the contents of the blood-proper system are derived by direct absorption from the great reservoir of the chylaqueous fluid). From the obviously connected sequence of these events the inference is clearly deducible, that the parietes of the blood-vessels impress upon the fluid *in transitu* a chemico-vital change, which eventuates in the evolution of pigment. In other language, the walls of the *blood-vessels*, under the circumstances indicated, accomplish what in the instances of *Glycera alba* and *Clymene arenicoida*, and it may be prophetically added in all Vertebrata, is performed by the *involucra*, the cell-capsules of the floating corpuscles. One other lesson of extreme value is read to the zoo-chemist by the history of the blood in the Annelida. It is a limpid, non-corpusculated coloured fluid. In many species it

alone is submitted to the process of aëration. *Therefore* the agency of the floating corpuscles is not essential to the respiratory action of oxygen on the blood. This fact, which is unequivocal, is promissory of future discoveries.

Relation between the Blood-proper and Peritoneal Fluid.—The physiologist cannot view with unconcern the question which relates to *the mode* in which the two fluids now described in the *Annelida* stand related to each other. It is scarcely required to observe, that what applies under this head to the Annelida will prove no less applicable to the Entozoa and Echinoderms, in which also these two varieties of nutrient fluids coexist in the same individual.

In *all Annelida* a peculiar and express disposition of the blood-vessels is observable, by which an extensive contact is secured between the blood-proper and the fluid of the visceral cavity. This arrangement is so strikingly a provision for the attainment of a particular object that it cannot be misinterpreted. In the anatomy of the *Nais* this is perfectly and beautifully seen. From the sub-ganglionic trunk long coiled vessels proceed, describe several convolutions in the midst of the fluid, and in a *perfectly naked* and unsupported state, curve dorsally, still surrounded by the chylaqueous fluid, and empty themselves into the great dorsal trunk. In *Nais maculosa* (Williams) this distribution of the coiled vessels is still more readily demonstrated, in consequence of the bright red colour of the blood enabling the observer to trace the minutest vessels throughout their entire course. With respect to these vessels there is one remarkable fact to be stated, namely, that they maintain their singleness or individuality from their point of origin to their termination; they do not branch. In this fact is seen a beautiful provision against injury during the contractions and elongations of the body. The slender column of blood contained in these vessels, is directly and throughout its whole course exposed to the agency of the peritoneal or chylaqueous fluid. The agency of this fluid is obviously of a two-fold character; it first replenishes the blood-proper with the *chylous* materials by which its healthy constitution is maintained, in other words, the coiled vessels *absorb* a chylous pabulum from the peritoneal fluid; this process of chyle absorption may not be *exclusively* confined to these vessels; those distributed over the parietes of the intestine may participate in this function; and, secondly, the peritoneal fluid acting as a reservoir for the oxygen of the *external* element, forms *to* the coiled vessels a true aërating medium, the process of *breathing* being thus *internal.* In its application to the *Entozoa* this view of the mechanism of respiration acquires the highest interest. It is the *true method* of respiration in all *Entozoa*. The physiologist will now I trust definitively comprehend the breathing function in these parasites, *though destitute of all semblance of external organs of respiration**.

One more question remains to be considered, What is the physiological meaning of this *methodical contact* between the two fluid elements of nutrition in the *Annelida*? That the blood-proper in degree of *organization* (vitality) is higher than the chyl-

* In another communication I hope to enter at length into the demonstration of this subject.

aqueous fluid, no doubt can exist; and as already maintained, the true-blood is reproduced, at all events in part, out of the materials supplied by the chylaqueous fluid; the conclusion is obvious that this latter fluid *is* incipient blood, and that consequently it must be gifted *pro tanto* with the property of nourishing the solid structures of the body. But this inference is rendered almost certain by the force of the examples furnished by the Zoophytes, Medusæ, &c., in which this fluid constitutes the *only fluid element of nutrition* in the organism.

The chylaqueous fluid must however be regarded as in itself a manufactory; ITS corpuscles execute an office by which the mineral substances and proximate principles are vitally assimilated. In the Annelida (and therefore in the Entozoa and Echinoderms) the corpuscles do in THIS FLUID, as already explained, what in the higher Invertebrata and Vertebrata is accomplished by corpuscles, somewhat more definitively organized, floating in the true-blood. From these facts the physiologist may state, that *under all circumstances, how simple soever the fluid may be, the agency of cells*, either in the solids through which the fluids pass, or floating in the fluids themselves, is more or less essential to the vitalization of the liquid medium of nutrition. In the Annelida the true-blood is *incorpusculated, because* the cell agency is performed in the fluid, lower in grade than itself, which oscillates in the peritoneal cavity. If *this* fluid did not exist, then the corpuscles, solid cells, would probably have been present in the true-blood, there to enact their destined functions.

From these observations the inference may be drawn, that between these two nutritious fluids, in the zoological series, there obtains a definite physiological balance; that one is capable of absorbing or merging into the other, according as the observer ascends or descends the scale. The chylaqueous system ends above with the larva of Insects, and the true-blood system traced downwards terminates at the Echinodermata.

With reference to the nature of the function executed by the floating cells of the vital fluids, I may be permitted to mention here *one fact*, which, during the prosecution of my recent studies, has excited in my mind a constant and eager attention. When the voluminous corpuscles of the chylaqueous fluid of the Annelida *burst* in the field of the microscope, the semi-fluid contents of these bodies *fibrillate*, i. e. *coagulate as they flow out of the containing cells*. This is a constant fact observed with the utmost exactness a thousand times. It is an absolute proof, addressed directly to the eye, that the contents of the cells, that which they themselves *secrete*, is a self-coagulating principle, is higher in organic properties than that in which the cells float, and which surrounds them externally, and out of which therefore their parietes *must* produce what their cavities circumscribe. Science does not demand a more satisfactory proof that the floating cells have for *one* of their offices that of generating *fibrine*; another of their functions is that unquestionably of manufacturing pigment.

Articulated Animals.—Arranged on the basis of the evidence drawn from the

history of the fluids, the articulated classes come here into contact with the annulose series. In the embryonic condition of the Myriapod and the Insect, the circulating fluids present *all* the essential characters of the chylaqueous system, as already described in the economy of the Annelid. It is a fluid surrounding the rudimentary intestine, and moving *to and fro* in a spacious chamber, its movements being determined by no other power than the muscular contractions of the intestine on the one side, and the integuments on the other, by which the containing cavity is bounded. It is a veritable chylaqueous system. The dorsal vessel is yet *un*formed, the corpuscles of the fluid, scanty in those of the larvæ of many species, are temporary provisions, destined soon to be replaced by those permanent elements by which the blood of the perfect animal is afterwards to be distinguished.

When the embryo of the articulated animal first emerges from the ovum, it is virtually an Annelid in outward form and internal structure. The system of the nutrient fluids, and the fluids themselves, fall obviously under the character of the chylaqueous type. The perfect absence of independent conduits circumscribing a highly organized fluid, reduces the larva of the Insect to the low standard of the embryo Annelid. Here then is an unequivocal demonstration of the proposition that the articulated series are *directly* continuous with the annulose through the medium of the fluids; that the chylaqueous system is traceable from the latter into the former; that which is persistent in the Annelid is temporary only in the articulated animal. These generalizations are founded upon faithfully observed *facts*, the value of which in philosophical zoology cannot be exaggerated. In relation to the nutritive fluids of the articulated series, it is proposed now that we proceed to the establishment of the following propositions:—1st, that in the embryonic condition of the Myriapoda, Insecta, Arachnida and Crustacea, the fluids, in composition and plan of circulation, fall under the designation of the chylaqueous system which persistently prevails in all classes below the Articulata; 2nd, that although in the articulated animal the chylaqueous fluid and the blood-proper have in no instance a contemporaneous existence in the same individual, yet that these two orders of fluid are marked by such strikingly diverse physical characters that their distinctness and independence cannot be doubted; 3rd, that the corpuscles contained in the embryonic or chylaqueous fluid of the Articulata present varieties in form, structure and size, far different from and more numerous than those which occur in the corpuscles of the true-blood of the adult animals; 4th, that throughout *all articulated animals*, from the Myriapod to the highest Crustacean, the mature corpuscles of the true-blood conform unequivocally to one fundamental type of structure and figure, a novel demonstration of a new order of zoological affinities!

Myriapoda.—It was observed by Mr. NEWPORT that in the larva of *Iulus* the fluid filling the space around the intestine, for some days before the pulsations of the dorsal vessel, became detectable to the eye. The truth of this statement I have repeatedly confirmed. The fluid of the visceral cavity in the larvæ oscillates to and fro *before*

the development of any special power for directing and sustaining its movements. This fluid is at first almost destitute of floating cells, which consist only of oleous molecules. In process of growth corpuscles, more or less resembling those afterwards to be described as present in the blood of the adult Myriapod, begin to appear, and *pari passu* with this genesis of definitively organized corpuscles the structure and function of the dorsal vessel assume a more obvious presence. This embryonic system of fluid, before the appearance in it of the permanent corpuscles, and *before the development of the tracheal system*, undergoes the process of aëration in accordance with the plan on which this great function is performed under the chylaqueous type. Less complexly organized than the blood-proper, it demands no special apparatus for its exposure to the aërating medium. As it rolls in the general cavity of the body, it undergoes adequately this vital change. These observations require very little to be modified to render them true of all insect-larvæ. Thus, then, there exists in the Myriapod 'a circulating fluid' anterior to the true-blood, which the latter gradually supersedes, and 'a respiration' which precedes that which subsequently devolves upon the tracheal system *.

In the adult Myriapod the blood is colourless and richly corpusculated. The corpuscles are perfectly destitute of colour. This fact is true of the fluid systems of *all* articulated animals without a single exception. Wherefore this universal absence in the fluids of a pigment-producing faculty? The floating bodies of the blood in the adult Myriapod are regularly and determinately organized. They are *nearly* the same in the Iulidæ as in the Scolopendræ, the highest and the lowest orders. They present under the microscope, in a fresh drop of blood, three leading varieties:—1st, a large pellucid nucleus surrounded by a few granules (Plate XXXII. fig. 38); 2nd, the orbicular, in which the granules have grown in number, almost concealing the nucleus (fig. 39); and 3rdly, the ovoid or oat-shaped, in which the nucleus has reappeared (fig. 40). In none of these bodies is it possible under any manœuvre to detect the presence of a cell-capsule. The molecules surrounding the nucleus seem rather to be drawn to the latter by a mysterious centripetal power, than embraced by an involucrum. These bodies, when they burst in the field of the microscope, *fibrillate*, proving that the molecules are held together by a tenacious self-coagulating principle.

Insecta.—Contributions, from several authoritative observers †, towards a better understanding of the circulation of Insects, have appeared during the last two years.

* "The history of the development of the embryo of the Myriapod presents a remarkable resemblance to that of the true Annelid; for the embryo at the time of its emergence from the egg possesses but a very small number of segments; and these continue to increase by the repeated subdivision of the penultimate segment until the number characteristic of the species has been attained."—Principles of General and Comparative Physiology, by Dr. Carpenter, 3rd edit. 1851, p. 375.

† Etudes Anatomiques et Physiologiques et Observations sur les Larves des Libellules, par M. Léon Dufour, Annales des Sciences Nat. 1852. Nouvelles Observations sur la circulation du sang et la nutrition chez les Insectes, par Émile Blanchard, *op. cit.* 3me Série, 1851. Note sur la circulation des fluides chez les Insectes, par le Professeur Louis Agassiz, *op. cit.* 3me Série, 1851.

They relate however rather to the mechanism of the fluid's orbit than to the composition of the fluid itself. They cannot therefore be rendered subservient to the objects of the present communication. Mr. BOWERBANK's observations* on the blood of Insects stand alone, and deserve implicit confidence. "The blood (of insects), which is usually of a very transparent greenish or yellowish colour, is filled with a great number of little particles, which were described by CARUS as oblong or oval, but more correctly by Mr. BOWERBANK as flattened oat-shaped masses which retain their form while circulating through the body, but like the particles of blood in Vertebrata, become globular immediately they are brought into contact with water. It is stated by BURMEISTER that they vary in diameter from $\frac{1}{200}$ to $\frac{1}{800}$th of a line; but they differ also in size in the same individual, and are often rough and tuberculated, as noticed by EDWARDS, and as distinctly seen in the blood of *Sphinx ligustri*†." The preceding paragraph embraces the sum of our present knowledge on the histological character of the nutritive fluids of insects! It is impossible in the present limited communication to exhaust the materials comprised within a field of observation so vast and various. Among species so diverse and boundless the fluids must be characterized by corresponding varieties in physical characters. This is probable from the analogy derived from the examination of the fluids of the annulose series. It will accordingly be found, that although specific distinctions in the corpuscular elements of the fluids in the class "Insecta" are not unequivocally drawn as in the Annelida, under a marked typical unity for the whole class, specific diversities will notwithstanding be remarked to prevail. These varieties however are extremely obvious in the chylaqueous fluid of the larvæ of the several component species of the class.

Chylaqueous Fluid of the Larvæ of Insects.—At the first emergence of the larvæ of several species of insects from the ovum, no dorsal vessel is yet formed. The visceral space is filled with a fluid perfectly colourless, which fluctuates irregularly in the containing cavity and is charged with corpuscles, which vary histologically in different species. As every larva does not emerge out of the ovum in the same stage of development, the floating cells of the chylaqueous fluid will be found to present differences depending rather upon age than upon species. The accompanying illustrations, which are drawn with repeated and exact care, represent several examples of these floating cells as they occur in the principal species of water-larvæ, (figs. 44 to 51).

I have also noted with every practicable accuracy the characters of the bodies observed in the fluids of the larvæ of the Neuroptera, Hymenoptera, Lepidoptera and Coleoptera, with the uniform result of discovering that (until a very advanced period of the larval stage) they foreshadow in no one particular those which afterwards appear in the blood of the perfect insect. In the chylaqueous fluid of the larva of the Hay Moth (*Leptona candida*) they consist of oblong flat cells, exceeding in

* Entomological Magazine, vol. i. April 1833.

† Art. Insecta, by Mr. NEWPORT, TODD's Encyclopædia.

size those of the blood-proper of the same insect; each cell is thinly charged with molecules, but destitute of nucleus. These cells are not accidental productions; they are constant in every specimen. They belong physiologically to the embryonic stage of the circulating fluid, as the other (fig. 44) and more complexly-structured corpuscles pertain to the mature blood. They exhibit not the slightest trace of colour. In another instance of a water-larva they discovered themselves under a kidney-shaped figure (fig. 49), bearing no analogy to any variety discoverable in the fluids of any species of adult insect.

The corpuscles of the chylaqueous fluid of the larvæ of the Libellulidæ constitute minute, fusiform, transparent, pellucid bodies (fig. 48), abounding in great comparative number in the sustaining fluid; they present no nuclear cell, neither do they contain granules, two structural characters in which they differ strikingly from those of the blood of the perfect insect of the same species. In the instance of another water-larva, the fluid under consideration was found thickly charged with small discoidal bodies, minutely granulated (fig. 51). Many other varieties might be added, but enough data have been adduced to sustain the statement that the fluids of the larvæ of insects are characterized by morphous elements which contrast unquestionably with those of the blood of adult insects. These, then, are the grounds on which it is contended that the embryonic fluids of the Insect constitute a true chylaqueous system; that it is less complex than the true-blood by which it is destined to be succeeded; that its morphous elements are provisional; that its basis and bulk consist of water, vitalized by passage through the parietes of the digestive system; that, morphologically, its floating cells bear some relation to the species, but none to those of the true-blood by which they are to be followed; and that, finally, it is aërated before the evolution and independently of the agency of the tracheal system. The principal trunks of the tracheæ are distinctly visible in the body of the larva long before the *stigmata* (by which a communication is established between these tubes and the external medium) are formed. The air-tubes, however, while yet closed at both extremities, become filled with a gaseous substance. How is this curious fact to be explained? In the atmospheric larvæ they cannot derive their gaseous contents directly from the external air, for fluids and solids intervene; nor in the water-larvæ can they absorb the air of the surrounding medium, for they are situated too deeply in the interior of the body, those few species excepted in which appropriate appendages are provided for the exposure of the tracheal system. The inference is thus rendered probable that the tracheæ of the larvæ of Insects, whether their habitat be atmospheric or aquatic, become first filled with gas *from the fluid* occupying the visceral cavity—not because they already perform the office of aërating that fluid, but because their parietes are endowed with the peculiar faculty of absorbing gaseous elements *from fluids* by which they may be surrounded. Distended with gas, in the larva stage they subserve the mechanical office of suspending the aquatic species in their temporary habitation. The conclusion finally presses upon the mind, then, that the

first nutritive embryonic fluid of the Insect undergoes respiration on the *aquatic*, not on the atmospheric plan, conforming in this fundamental particular to the law governing the function in all chylaqueous fluids.

Blood-proper of Insects.—As already stated, Mr. Bowerbank was the first to describe with exactitude the morphous solids of the blood of insects. But unfortunately for science his observations were instituted only on one species. No physiologist has yet rightly estimated the importance attaching to the history of the fluids in the animal series—not the cavities and channels and vessels in which they move, but the fluids themselves, histologically, morphologically, chemically, teleologically, as component elements of the living organism. *No zoologist is yet prepared for the assertion, that throughout the true articulated series there prevails but one fundamental type of blood-corpuscle.* The variations from this essential unity, coinciding with differences of class, order, genus or species, are never so deeply inscribed as to involve a departure from this type. It begins at the adult insect, and is unequivocally traceable through the intermediate forms of the Entomostraca, Crustacea, and Cirrhipeda, ending at the Arachnida. Here is a novel and unexpected confirmation of those affinities which are founded upon the resemblance of the solid parts. In the character of the fluids, the classifier is henceforth furnished with a new and important means of determining zoological differences and resemblances.

The blood of the perfect insect is colourless, and charged with colourless floating cells. It is impelled in a definitive orbit by a special power,—a dorsal vessel. Throughout the whole class its morphous elements consist of cells of peculiar construction. There is a conspicuous nucleus in each cell. It is surrounded by minute, pellucid, very slightly refracting granules. On bursting, these corpuscles *fibrillate.* It is not possible to detect separately the cell-capsule; though, from the constant and definite figure of the bodies, it admits of no dispute that a distinct involucrum exists. The figure varies from the orbicular to the oval or oat-shaped. They are flattened ovals, not *cubically* oval. They are almost immediately destroyed by water. The first phase is the pellucid molecule; the second an orbicular granular particle; the third the flattened oat-shaped cell. In every species of insect yet examined, the extremes of variations in form are bounded on one side by the orbicular, on the other by the fusiform. The fundamental form is the compressed oat-shaped. The illustrations (figs. 40 to 43) exhibit with strict fidelity the structural characters of these bodies. They are amplified 420 diameters.

Crustacea.—The fluids of the Entomostraca are supplied with corpuscles which fall under the articulate type already indicated. They are, however, more generally circular (figs. 52 to 55), always bearing a nucleus more or less discernible, and filled with minute granules. In *Branchipus*, *Daphnia* and *Cyclops*, they answer with exactness to the preceding description. They are not numerous relatively to the bulk of the fluid. In these little crustaceans, in the adult state, the circulating system is quite as simply constituted as that of insects. The dorsal vessel is the moving power;

the periphery of the system is lacunose. The history of the evolution of the fluids of these microscopic animals is yet to be known. The illustrations delineate faithfully the blood-corpuscles of *Daphnia, Cyclops,* and *Branchipus.* Magnified 420 diameters.

The fluids of the lowest crustaceans present all the essential features of the chylaqueous system. The *Picnogonidæ* afford the best examples. In these inferior genera, the digestive cæca float in the general cavity of the body. The space which intervenes between them and the integumentary exterior is filled with a colourless corpusculated fluid. The oscillations of this fluid are irregular, excited and sustained by the constant action of the arms and the undulations of the alimentary cæca. The corpuscles of this fluid are, relatively to its volume and to the size of the animal, very large. They depart as much from the normal articulate type as those of the embryonic fluid of the Insect do from those of the blood of the perfect animal. The inferior structural character of these bodies becomes expressive of a corresponding simplicity in the composition of the fluid. The walls of the cavity lodging the nutritive fluid in the *Pycnodon* are *not* ciliated. The corpuscles are spherical bodies, having an obvious cell-capsule, molecules, but no nucleus. *This,* like all other true chylaqueous fluids, is aërated at every part of its course throughout the body. Hence the absence of that which the simple nature of the fluid does not require, a special apparatus for respiration.

Nothing is yet known of the morphological characters of the nutritive fluids in the embryo condition of the higher Crustacea. The fluids also of the adult animals, though easy of investigation, remain almost wholly undescribed. Mr. Wharton Jones, in his memoir in the Philosophical Transactions*, alludes only to the instances of the Lobster and the Crab; on comparison, however, it will be seen that between his delineations and mine there are wide differences.

In *Caprella linearis* (*Lœmodipodes,* Cuv.), it is an easy process to observe the blood-cells circulating in the branchial appendages depending from the inferior surface of the body. They occur under three discernible varieties:—1st, simple, non-granular, non-nucleated, pellucid, spherical globules (fig. 56); 2nd, more or less orbicular bodies, of which the bright nucleus is prominent visible, and a mass of slightly refractive molecules; 3rd, the fact, characteristic of all blood-cells falling under the denomination of the articulate type, of the apparent suppression of the cell-capsule. In this species the blood-corpuscles are large relatively to the proportions of the body. On bursting in the field their contents *fibrillate* in a very obvious manner. In every observation I have been attentive to note this interesting fact; it may avail some future theorist.

The blood of the *Amphipodæ* is distinguished for the prevailing orbicular character of its floating cells. The elliptical figure seldom occurs. The nucleus is more centrally situated, and therefore less visible than is common among the Crustacea. The

* *Op. cit.*

absence of the cell-capsule is readily observed. In dimensions these bodies are proportionately small. Those of *Talitrus locusta* and *Gammarus pulex* are characteristic examples (fig. 57).

In the *Paguridæ*, or Hermit-crab family, the blood-corpuscles are prevalently ovoidal; instances, however, of the elliptical and spherical figures are remarked among them. In every typical character they coincide with the articulate model,—the nucleus prominent, and generally placed eccentrically. The corpuscles are granular and the involucrum apparently wanting. They fibrillate when rupturing in the most striking manner. Those of *Pagurus Bernhardii* (fig. 58) will serve to exemplify this class. The blood of the Brachyourous Decapods is distinguished for its comparatively large and prevailingly spherical corpuscles. In all the pellucid nucleus is a prominent object; it is always near some point of the circumference. The granules are so large as to be readily individualized by the eye. The involucrum is so attenuated, if existent at all, as not to be appreciable. As in the blood of all other crustaceans, three grades of development may in this instance be recognised:—1st, the pellucid, nucleus-less globule; 2nd, the nucleated cell, with a few surrounding molecules; and 3rdly, the mature cell, in this case approaching the sphere in figure. The fibrillation of the cells is here too invariably remarked. In consequence of the legible size of the objects, it is possible *to prove*, with reference to the blood-corpuscles of the Crab, that the fibrillation observed during the rupture of the cell, arises from the coagulation of the cohesive liquid by which the *molecules* (not the nucleus), constituting the great bulk of the corpuscle, are filled. It is probable, therefore, that *each molecule* is a miniature factory for fibrine. The nutritious fluid of the Macrourous Decapods is little different morphologically from those already described. The corpuscles, however, are more generally elongated, so that the oat-shaped constitutes the average form; in all other respects they conform intimately with those of other Crustacea. Those of the Lobster, Cray-fish, Prawn, and Shrimp (figs. 59, 60, 61), will serve to illustrate the characters of the solid elements of the blood in this order of Crustaceans. It will be observed that they present, severally, slight variations of size and form, which coincide with the differences of species. But through them all there runs a continuous evidence of an essential unity of type.

Arachnida.—The naturalist is scarcely prepared for the announcement that the morphous elements of the blood of the Arachnid constitute the terminal link in the fluid-chain of the Articulata. In all the spiders of this country the corpuscles of the blood occur under the character of minutely granular bodies, varying between the spindle-shaped and orbicular in figure. Though absolutely small, they are, relatively to the size of the body of the animal, as large as those of the Crustacean. They differ from those of the latter in the position and invisibility of the nucleus. It is seated in the geometrical centre of the body, and therefore indetectable, because surrounded by molecules which are the exact counterpart of those formerly described in the blood-cells of the Crustacea. Like those of the latter cells, the molecules of the blood-

corpuscles of the Arachnid are capable only in a very slight degree of refracting light; hence the peculiarly translucent and delicate character of these bodies. In a perfectly fresh state, and unmixed with any menstruum, they are readily defined. The cell-capsule is as undetectable as it is in the Crustacea. In *Arachna grandis*, and the common House Spider, illustrative examples may be easily obtained (Plate XXXIV. fig. 62 and 63).

The conformity of the Arachnid blood-corpuscle to the normal articulate type is placed by these examples beyond doubt. Thus is presented to the philosophic zoologist a new and unexpected order of affinities as valuable as those established through observed resemblances in the systems of the solid organs, and by which in future every scheme of classification must be either corrected or confirmed. The physiologist will now recognise in the mysterious unity of form and structure which pervades even a microscopic cell, floating detachedly in a fluid, an immutable law of organic continuity through which the thoughtful eye may trace relationships between animals far separated in the zoological series.

Mollusca.—Cuvier, Owen, M.-Edwards, and recently, and more minutely, Alder and Hancock, have, by their several researches, elucidated with great success the mechanism of the circulation in the Mollusca. To this department of the subject of this memoir, it is not in my power, at present, to make any considerable addition. It is remarkable, that, while studying the *channels* of the fluids, the great observers named did not on any occasion digress to an examination of the fluids themselves*.

In addition to the historical sketch formerly presented, it must here be stated that the most recent essay on the blood of Mollusks is that which has lately appeared from the pen of M. Moquin-Sandon†. This writer devotes some pages of his short paper to a discussion of the point whether the red viscid fluid which appears to escape from the edges of the mantle of the Planorbidæ when irritated, is blood or not blood. Having exhausted the controversy, he concludes that it is blood. "Examinée au microscope," he observes, "au moment où elle sort de l'animal, on y remarque un certain nombre de corpuscles irrégulièrement arrondis, inégaux, tout à fait semblables aux globules sanguins des Gastéropodes. Leur diamètre est de $\frac{1}{100}$, $\frac{1}{75}$ et $\frac{1}{50}$ de millimètre." It will be afterwards shown that the following propositions of Moquin-Sandon are contradicted by the best observed facts. "1. Les Planorbes ont le sang rouge ou rougeâtre; 2. Les très petites espèces ont le sang rose ou couleur de chair; 3. La liqueur répandue par ces Mollusques, quand on les irrite, n'est pas une humeur particulière sécrété par le collier, ni par tout autre organe, mais du sang mêlé à la mucosité; 4. Le sang, épanché dans la grande cavité du corps des Planorbes, comme chez les autres Gastéropodes, se voit distinctement, pendant la vie, quand il est très rouge, chez les espèces à coquille transparente; 5. Le sang répandu par les Planorbes, quand l'animal se retire brusquement et profondément dans sa coquille, n'est pas

* See *ante*.

† Mémoires de l'Académie de Toulouse, 1849; and afterwards copied into the Ann. des Sc. Nat. 1851.

exprimé, par la marge du manteau, mais il sort de l'étroit espace située entre cette marge et la coquille; 6. Dans une contraction extrême, le sang peut exuder par toutes les parties du corps." From the language of these propositional corollaries, it is certain that M. MOQUIN-SANDON scarcely knows what is, or what is not the *blood* of the Planorbidæ. Such uncertainty illustrates the sources of those numerous errors which render the historical literature of this subject really of little value. It is of the highest importance to the progress of knowledge that the *same fluid* should be examined by all the observers engaged in the same pursuit. Independent observations *under such circumstances* become conducive to the development of truth*.

In the Mollusk there is but one system of fluids. It unites in itself the separate characteristics of the blood-proper system and the chylaqueous. In the mechanical character of the circulating system this union of opposite qualities is discernible. In the histology of the fluids it is still more so. In the transition through the Molluscan route, between the Annelida and the Vertebrata, the blood of the Mollusks exhibits the mean of these two extreme constituents. Unlike the blood of the Annelid, it is the seat of floating corpuscles, and different from that of the animal; these corpuscles are not organized with regularity of plan. They exhibit more constancy of structure than the morphous elements of the chylaqueous fluid, less so than those of the blood of the vertebrated animal. The blood of the Mollusk is indeed in every physiological property intermediate between that of the vertebrate animal and the chylaqueous fluid of the Annelid. If perhaps the blood-proper of the Annelid were mixed with the chylaqueous fluid, the product would represent the Molluscan blood. Why in the one case these constituents should be held permanently separated, and in the other blended into one fluid, it is not easy to explain.

In the Molluscan scale a considerable interval separates the tunicated orders from the Cephalopod. This interval of separation, so marked in the solids, is scarcely recognisable in the fluids. The blood-corpuscle of the Myriapod is far less distinguishable from that of the Arachnid than the Scolopendra is from the Spider. To the former case this is an illustrative parallel; so much more intimate is the affinity which pervades the fluids, than that which links together the systems of the solid organs in the animal series.

* Preliminarily to the investigations related in the text, I am desirous in this place to state with clearness, the mode which I have adopted in procuring the fluid intended for examination in the Mollusca; a class in which it is far more difficult to *isolate* the nutritious fluids than in any other. In every instance an eye conversant with these especial objects is required. The cells of the solid structures, when loosely floating in a fluid, may easily deceive an inexperienced observer. The blood-corpuscles in every species should first be unquestionably identified, by *seeing them* moving IN the blood-channels. When, in the larger species, the heart is a conspicuous body and admits of ready separation from all surrounding structures, the blood may be drawn directly from this source. In many of the DRY land species, as the Helix family, it suffices to lay open the mantle and expose the *areolæ* of the visceral cavity. The fluid escaping under such circumstances is true-blood. The smaller Mollusca must, however, with *infinite* patience, be submitted to microscopic examination; the observer must stedfastly gaze until the soft parts are protruded beyond the limits of the shell, which in nearly all cases is too obscure and impenetrable to light to enable the eye to read the included living fluids.

In every Mollusk* yet examined the blood has been found to be colourless, not colourless like distilled water, but like very dilute milk. It is more *coagulable* than any variety of chylaqueous fluid; and from its viscidity it is undoubtedly more highly charged with the fibrinous principle than the latter. This fact results necessarily from the circumstance, that it constitutes in itself the entire fluid element of nutrition in the Mollusks, and that it is the seat of direct corpuscular agency. In this series the blood is invariably corpusculated. The proportion of the floating cells to the fluid varies however in different orders. In the Bryozoa and Tunicata, these bodies are relatively scanty. In the Cephalopods they are very numerous. Hereafter, in the progress of physiology, the law will be established, which recognises a *vital proportion* between the *measure of corpusculation*, presented by the fluids and the 'place' of an animal in the series. One unaided observer cannot adventure upon a generalization which should have for its basis a multitude of "facts." It is a remarkable *law*, which has now been demonstrated to preside over the blood of the articulate *and* the molluscan series, that in scarcely a single instance is it the seat of colour. In all cases except that of the Annelida the pigment is developed only in the cells of the solid structures. Among vertebrated animals, colour, *and only the red colour*, prevails without exception. Why should the blood-proper of the Annelids constitute an exception to a rule which applies to *all other* classes within the wide bounds of the invertebrate subkingdom? These are queries pregnant with undelivered meaning.

The *Bryozoa*, the lowest of the Mollusks, possess no vestige of a true-blood system. Neither a heart nor vessels under any shape can be discovered in any species. The nutritive fluid occupies the visceral cavity. It is imperfectly corpusculated, and oscillates in its containing chamber under the agency of muscular contractions. The branchiæ are cæcal tubes into which the fluid of the general cavity of the body freely enters, and in which it moves in *flux and reflux currents*. As foreshadowing a character of constant occurrence in the circulating system of Mollusks, this peculiarity should be specially noted. It is a specializing of the system in *some part* of its periphery while others remain degraded. It has been of late shown by the beautiful researches of ALDER and HANCOCK, that in the Nudibranchiate Mollusks there are distinguishable three peripheral specializations, the portal, the branchial, and the renal. In these subsystems an approximation to a capillary reticulation of the conducting channels occurs. The branchial canals of the Bryozoa first typify the Molluscan law just defined. The morphous elements of the fluids in the Bryozoa, as formerly explained, admit of easy observation in *Laguncula repens*, in any of the *Flustræ*, *Lepraliæ* or *Escharæ*, most readily, however, in *Bowerbankia*. This consists of globules of various forms and size; some are only opake, milky spherules, without nucleus or granules; others are nucleated; and a third variety, comprising the adult form of the cell, discovers a nucleus, small and centrally placed, surrounded by granules, constituting an orbicular corpuscle.

* I speak of course within the bounds of my own personal observations.

Thus in a brief phrase is expressed a structural principle which governs the formation of the blood-corpuscle in all Mollusks.

The Bryozoa are to the Molluscan what the Pycnogonidæ are to the crustacean series. In the Bryozoa and Pycnogonidæ the vital fluids are constituted in the organism into *a system*, in strict accordance with the chylaqueous. How perfectly these two instances prove that nature, in cases of simple organisms, gathers together and circulates the nutritive fluids on the type and plan of *this* system, and not on *that* of the true-blood! This fact indicates in the former, with respect to the latter, a relation of inferiority. In both the instances enumerated, the fluids notwithstanding exhibit an advance upon the true chylaqueous fluid, in the fact that the corpuscles are more highly organized, while the fluid itself is more perfectly fibrinized, indications both of a higher degree of vitalization.

In the *Tunicata* the apparatus of the circulation is developed obviously above the standard of the former. A heart and arterial trunks are detectable. The blood currents, however, are not determinate in direction. As in the larva of insects, the portions of the fluid which accumulate in the peritoneal chambers oscillate to and fro under the muscular contortions of the body; presently, however, and at unequal intervals, it obeys the impelling force of the heart, and advances in a definite orbit. In the Cynthidæ or Salpidæ the fluids may be readily obtained for examination*.

It is colourless, and discovers very distinctly the property of coagulating. The solid elements (figs. 64 and 65), relatively to the bulk of the fluid, are scanty. Similarly to what was observed in the Bryozoa, the cells exhibit several varieties. It is a fact of considerable interest, that the floating cells of all inferiorly vitalized fluids should be characterized by *variety* more or less numerous, in the form, size and structure of the corpuscular elements. The adult type of the corpuscle in the blood of the tunicated Mollusk is marked by no other feature of constancy than that of the orbicular figure. Sometimes this form is modified into the flat circular; frequently the cells are simply nucleated, again they are destitute of this part. All those which may be reckoned as mature, contain, in addition to a nucleus, granules, forming more or less of the bulk of the cell. The cell-capsule is more evident in this class than in the former. If submitted to the action of an endosmotic medium, such as water, it fibrillates more obviously than the corpuscles of an unmixed chylaqueous fluid. This character is a mark of superior organization.

In the class of the *Lamellibranchiate* Mollusks, the readiest and surest method of observing the blood-corpuscles consists either in viewing them in motion directly in the branchial vessels, or in isolating the heart, placing it under the microscope and

* For this purpose, in these animals, the tunic and branchial chamber should be opened freely; the fluid occupying the cells of the space intervening between these two parts will be found to be true-blood, the corpuscles of which, if thus obtained, should be carefully compared with those in the same specimen *seen* moving in the branchial vessels.

distinguishing the corpusculated fluid as it escapes, under pressure, from the little cavity of the organ*.

Like that of all other Mollusks, the blood in this order is colourless, limpidly opalescent, and charged with corpuscles which present three main varieties:—1st, the round granular cell, which is probably the mature form of the blood-corpuscle; 2nd, a nucleated, orbicular, pellucid cell, destitute of all other contents; and, 3rd, minuter globules filled only with an opalescent fluid. In the Lamellibranchiate family, the blood is corpusculated on *one* plan. It is almost impossible to indicate structural differences between the blood-cells of *Pholas* (fig. 71) and those of *Pinna* (fig. 72), or between those of *Mya* (fig. 69) and those of *Solen* (fig. 70), or those of the Oyster (fig. 68) and those of the Mussel (fig. 69). The blood of the small fresh bivalves is less obviously corpusculated, and the corpuscles themselves are less impregnated with granules. Although exhibiting the limpidity of pure water, it coagulates into clots on escaping from the body. As yet a manifest *unity of plan* in the structure of the blood-corpuscles of the Mollusca has not been found to prevail in the classes examined. Nowhere has there existed *one invariable ever-present* form of corpuscles, such as obtains in the blood of the articulated animal, and varieties present themselves in every instance. But under this variety there runs a legible unity. The forms of cells, various though they be, which characterize the blood of the lower Mollusks now examined, are undoubtedly pervaded by a community of structural characters; through individual diversities there runs a chord of continuous union. The two remaining groups of Mollusks, the Gasteropods and Cephalopods, present signs of some advance upon the former in the vital composition of the fluids. Preserving the type of the molluscan blood-corpuscle, they lose some of the irregular, aberrant forms of the cells.

Gasteropoda.—The freshwater Gasteropods, in which the blood can be *seen* rolling in its containing channels, become serviceable as standards of comparisons for determining the true-blood corpuscle of other species of this family. The Planorbidæ are readily examined for the blood in the living state. The horns and foot are hollowed out in the interior, into spacious axial channels, into which the blood rushes under the compression of muscular force. This is the real mechanism by which the arms of the Brachiopods are extended. My observations on the blood of the Planorbidæ conduct me to conclusions at diametrical variance with those of Moquin-Sandon†. The true-blood of these Mollusks is colourless, not, as maintained by naturalists, red and purple. I demur to his method of observation. The mantle, when the animal is irritated, does throw out a coloured fluid, but that is not *the blood* of the animal. Ob-

* As to one unpractised in these delicate researches some difficulty may attend the method stated in the text, a *certain* view of the corpuscles of the blood may be obtained by placing a minute freshwater bivalve, such as a *Pisidium*, in a *cupped* glass under the microscope; when the soft parts, such as the siphons, edge of the mantle and foot, are being protruded beyond the limits of the shell, the movement of the blood, as it slowly distends these parts, can be very clearly and perfectly observed.

† Annales des Sciences Naturelles, 3me Série, 1851.

served carefully in an unmutilated specimen while yet living, the real blood may be seen rolling into the axes of the tentacles, a perfectly colourless, corpusculated fluid. No other mode of examination is exempt from fallacy. The corpuscles are spherical granular cells, furnished with a nucleus, which, from its central situation, is commonly indetectable. The elliptical and oat-shaped forms are never seen; other immature cells may be observed, but the real blood-corpuscle of *Planorbis corneus* (figs. 78 and 79) is a round granular cell of extreme delicacy, colourless and pellucid.

In the Helix and Limax families, from the large size of the specimens, and from the conspicuous milk-white colour of the blood-vessels, no difficulty obstructs the process of observing the blood. It is not much to be wondered at that every anatomist during the last half-century, from CUVIER to QUATREFAGES, have erroneously supposed the *blood* to be of a milk-white colour, because the *vessels* are so. The *coats* of the vessels are pure white, like milk; but the blood itself is almost colourless. The vessels derive their colour from the presence of a layer of *adipose tissue in their coats*. Why this excentric structure should exist it is not easy to explain; but *it is* so. The readiness with which the fat-cells escape from their areolæ, renders the separate observation of the blood-corpuscles very difficult. The real blood-corpuscles are quite different from the fat-cells; they are spherical granular bodies. They bear an obvious analogy to those of the lower molluscan groups, and yet they are different. They appear to consist of *firmer* substance. The molecules filling the interval between the nucleus and the involucrum refract the light more abruptly. They are mingled in almost equal proportion with minute oval cells destitute of nucleus and without granules (figs. 76 and 77). The spacious areolæ which surround the intestinal canal are filled with true blood. When the tegumentary mantle is opened it escapes in considerable volume. It determines the real character of the blood in this highly organized family of Gasteropods. It *does not* partake of the milky colour of the arteries, as stated by CUVIER. It is pellucid, a little less so than distilled water. It possesses, in the highest degree, the property of coagulating. In this respect it strikingly differs from the chylaqueous fluid of the Annelida, which was remarked to be gifted in a *minimum degree* with the clot-forming faculty. Physiology will hereafter inevitably prove, that between the coagulating property and the structure and number of the floating cells, there exists a relation of proportion which is yet unresolved. The blood-corpuscles of the Whelk and Limpet (figs. 73, 74, 75) fall under the description now given. The *Cephalopods* constitute the climax of the molluscan series: this observation is alike true of the solids and the fluids. Like that of other Mollusks, bluish and colourless, the blood of the Cephalopods is rich in floating cells; of more determinate and elaborate formation, however, than those of other Mollusks. They present far more striking uniformity in size and form than anything observed among the inferior molluscan families. In this fact they exhibit a near approximation to the vertebrate type of blood-corpuscle. These are signs of superior organization. They are provided always with a nucleus, situated for the

most part centrally, but sometimes peripherally. The space between the nucleus and involucrum is filled with a light bluish fluid, thickly impregnated with point-like molecules, and here and there a larger oil-cell. Another variety of cell, wholly destitute of contents, may be remarked (fig. 80); these latter are probably the germ state of the former. Between these intermediate forms may be observed. The mature cells preserve a striking regularity of size and structure; they are invariably capsulated, but this capsule is very thin.

Multiplied observations will enable a future generaliser to establish, for the configuration and structure of the blood-corpuscles of the several leading orders of Mollusks respectively, a *certain* and definitive law. The blood-cells of the Bryozoa will have their generic characters, those of the Tunicata theirs, and the Conchifera, the Gasteropoda, Pteropoda and Cephalopoda severally theirs.

Between them all there will be found some feature in common. The real law which presides over the conformation and structure of the corpuscular elements of the living fluids remains yet to be discovered. In what possible manner a mere modification of *figure* can influence the agency of these free cells, the physiologist at present cannot conjecture. But why should greater mystery attach to the shape of a blood-corpuscle than to that of the body of the animal itself? The zoophytic, medusan, echinodermal, articulate and molluscan blood-corpuscles are only correlates severally of the varied organisms which belong to the links enumerated of the invertebrate series. Though destitute of colour, the floating cells of the Cephalopods resemble most nearly those of the blood of the Vertebrata. Thus, I trust, has been shown in the fluids as well as in the solids of the organisms constituting the zoological series, a vital and structural graduation. To the future progress of physiological science the clear apprehension of this truth is most important.

Recapitulatory statement.—I have now, I trust, shown by the force of a large mass of evidence, that the circulating fluids in the *Invertebrata* occur under *three distinct classes*, distinguished from each other by prominent and unquestionable differential characteristics; that the lower and lower we descend in the invertebrate scale, the less and less organized, the more and more *like* lifeless salt water the nutrient fluids become; that the fluids (especially their solid elements) of the body in degrees of organization progress *pari passu* with the solids; that classified on the basis of the evidence afforded by the *fluids*, the *articulated series really begins at the Echinodermata*, and ends with the *Arachnida*, for throughout this series an uninterrupted line of *continuous* affinities may be clearly and indubitably traced. In every class *in this series*, *either temporarily* or *permanently*, *two fluids* are provided for the nutrition of the organism. As the *Rotifera*, judged by this rule, have only a single system of nutrient fluids like the Mollusca, it is evident that they cannot form a link in this beautiful chain; without them it is continuous, with them it is broken. The molluscan chain diverges from the former *at* the *Acalephæ*: it is traceable *uninterruptedly* to the highest Cephalopod. In all the classes of which this subkingdom is constituted

there exists but ONE system of fluids, appearing as if the central stomach of the Acaleph had been suddenly *partitioned* from its dependent cæcal prolongations. Viewed from the novel vantage point afforded by the fluids, the mind is led to contemplate the whole invertebrate kingdom under the guidance of three leading ideas: —1st. It sees in the Zoophytes a group of animals in which the fluids form a *single system in free communication with the external element*. Advancing to the limit of the Echinoderm, it suddenly descries the superaddition of a totally distinct system, that, namely, of the true-blood, *while the former still persists*; it tracks this double fluid-chain up to the highest articulated animal; and, 3rdly, at the highest frontier line, bounding the Medusæ, it observes a divarication in the chain, by the divergence from the originally *single* portion of the molluscan branch. In the *Mollusca*, like the *Protozoa*, the fluids constitute a single system; but the system of the circulating fluids in the former is distinguished from that in the latter by this important fact, that between it and the external element there exists no *direct* communication as in the *Protozoa*. In the Protozoa the nutrient fluids are chylaqueous, in the Mollusca they are *true-blood*, moving in a *closed circle*, *not of cylindrical vessels*, however, and excluded from all direct relation with the surrounding element; that is, before salt water, in this class, is admitted into the blood, it must, by the laws of this system, have first received an *incipient* organization, by passage through the stomach. The assimilating power of the chylaqueous fluid, therefore, as it exists in the Protozoa and Echinoderms and Annelida, when exerted upon salt water, immediately converts the latter into a *vital* organized fluid. This power is not possessed by true-blood. Classified in accordance with the principles established by the foregoing history of the comparative anatomy of the fluid elements of nutrition, the diagram here presented (Plate XXXV.) would illustrate the arrangement of the invertebrated classes.

Swansea, December 12, 1851.

EXPLANATION OF PLATES.

All the following illustrations were examined under and drawn by a one-fourth and a one-eighth power of one of POWELL and LEALAND's best microscopes.

PLATE XXXI.

Fig. 1. Corpuscles floating in the chylaqueous fluid of *Tubularia indivisa*. Magnified 320 diameters.

Fig. 2. Corpuscles floating in the chylaqueous fluid of *Bowerbankia denea*. Magnified 320 diameters.

Fig. 3. Corpuscles floating in the chylaqueous fluid of the gastro-vascular canals of *Rhizostoma*. Magnified 300 diameters.

Fig. 4. Corpuscles floating in the chylaqueous fluid of *Sipunculus Harveii*. Magnified 300 diameters.

Fig. 5. Corpuscles floating in the true-blood of the same *Sipunculus*. Magnified 320 diameters.

Fig. 6. Corpuscles floating in the chylaqueous fluid of *Sipunculus Johnstoni*. Magnified 300 diameters.

Fig. 7. Corpuscles floating in the true-blood of the same Sipuncle. Magnified 320 diameters.

Fig. 8. Corpuscles of the chylaqueous fluid of *Uraster rubens*. Magnified 300 diameters.

"Those in the fluid of the 'feet' are *identical*; those in the true-blood are also identical with those figured."

Fig. 9. Corpuscles of the chylaqueous fluid of *Solaster papposa*.

Fig. 10. Corpuscles of the chylaqueous fluid of *Cribella ocellata*.

Fig. 11. Corpuscles of the chylaqueous fluid of *Echinus sphæra*. Magnified 300 diameters.

Fig. 12. Corpuscles of the chylaqueous fluid of *Ophiocoma rosula*. Magnified 320 diameters.

Fig. 13. Corpuscles of the chylaqueous fluid *in* the alimentary organ of the Entozoon of the Hake. Magnified 320 diameters.

Fig. 14. Those of the chylaqueous fluid of *Arenicola piscatorum*. Magnified 320 diameters.

Fig. 15. Crystals of chloride of sodium obtained by evaporating the same.

Fig. 16. Corpuscles of the chylaqueous fluid of *Nais filiformis*. Magnified 400 diameters.

Fig. 17. Corpuscles of the chylaqueous fluid of *Sabella vesiculosa*. Magnified 300 diameters.

Fig. 18. Corpuscles of the chylaqueous fluid of a young *Nais filiformis*.

Fig. 19. Corpuscles of the chylaqueous fluid of *Sabella à sang vert*. Magnified 320 diameters.

Fig. 20. Corpuscles of the chylaqueous fluid of *Terebella nebulosa*. Magnified 400 diameters.

PLATES XXXII. and XXXIII.

Fig. 21. Corpuscles of the chylaqueous fluid of *Terebella conchilegia*. Magnified 320 diameters.

Fig. 22. Corpuscles of the chylaqueous fluid of *Lumbricus terrestris*. Magnified 300 diameters.

Fig. 23. Corpuscles of the chylaqueous fluid of a young *Lumbricus*.

Fig. 24. Corpuscles of the chylaqueous fluid of *Œnone maculata.* Magnified 300 diameters.

Fig. 25. Corpuscles of the chylaqueous fluid of *Borlasia*? contained *in* the alimentary organ. Magnified 300 diameters.

Fig. 26. Corpuscles of the chylaqueous fluid of *Phyllode lamelligera* (in visceral cavity). Magnified 320 diameters.

Fig. 27. Corpuscles of the chylaqueous fluid of *Nereis margaritacea.* Magnified 400 diameters.

Fig. 28. Corpuscles of the chylaqueous fluid of *Spio coniocephala.* Magnified 320 diameters.

Fig. 29. Corpuscles of the chylaqueous fluid of *Myrianida*?. Magnified 320 diameters.

Fig. 30. Corpuscles of the chylaqueous fluid of *Nais maculosa.* Magnified 320 diameters.

Fig. 31. Corpuscles (red) of the chylaqueous fluid of *Glycera alba.* Magnified 400 diameters.

Fig. 32. Corpuscles of the chylaqueous fluid of *Nais*?. Magnified 320 diameters.

Fig. 33. Corpuscles (red) of the chylaqueous fluid of *Clymenoida arenicoida.* Magnified 320 diameters.

Fig. 34. Corpuscles of the chylaqueous fluid of *Sigalion boa.* Magnified 400 diameters.

Fig. 35. Corpuscles of the chylaqueous fluid of *Matuta*? (WILLIAMS). Magnified 300 diameters.

Fig. 36. Corpuscles of the chylaqueous fluid of *Aphrodita aculeata.* Magnified 300 diameters.

Fig. 37. Corpuscles of the chylaqueous fluid of young *Arenicola.* Magnified 320 diameters.

Fig. 38. Corpuscles of the true-blood of *Iulus.* Magnified 320 diameters.

Fig. 39. Corpuscles of the true-blood of *Scolopendra.* Magnified 320 diameters.

Figs. 40, 41, 42, 43. Severally examples of the corpuscles of the true-blood of Insects. Magnified 320 diameters.

Figs. 44, 45, 46, 47, 48, 49, 50, 51. Examples of the corpuscles of the chylaqueous fluid of the larvæ of Insects.

Fig. 52. *Entomostraca.*—Corpuscles of the true-blood of *Moina branchiata.* Magnified 320 diameters.

Fig. 53. Corpuscles of the true-blood of *Daphnia pulex.* Magnified 320 diameters.

Fig. 54. Corpuscles of the true-blood of *Apus cancriformis.* Magnified 320 diameters.

Fig. 55. Corpuscles of the true-blood of *Cyclops quadriformis.* Magnified 320 diameters.

Fig. 56. Corpuscles of the true-blood of *Caprella linearis.* Magnified 300 diameters.

Fig. 57. Corpuscles of the true-blood of *Talitrus saltator.* Magnified 300 diameters

Fig. 58. Corpuscles of the true-blood of *Pagurus Bernhardii*. Magnified 300 diameters.

Fig. 59. Corpuscles of the true-blood of the Cray-fish. Magnified 300 diameters.

Fig. 60. Corpuscles of the true-blood of the Lobster. Magnified 420 diameters.

PLATE XXXIV.

Fig. 61. Corpuscles of the true-blood of the Crab. Magnified 420 diameters.

Fig. 62. Corpuscles of the true-blood of *Arachna grandis*. Magnified 320 diameters.

Fig. 63. Corpuscles of the true-blood of the House Spider. Magnified 320 diameters.

Fig. 64. Corpuscles of the true-blood of *Cynthia morus*. Magnified 300 diameters.

Fig. 65. Corpuscles of the true-blood of *Salpa maxima*. Magnified 300 diameters.

Fig. 66. Corpuscles of the true-blood of *Pisidium*. Magnified 300 diameters.

Fig. 67. Corpuscles of the true-blood of *Mytilus edulis*. Magnified 300 diameters.

Fig. 68. Corpuscles of the true-blood of *Ostrea edulis*. Magnified 300 diameters.

Fig. 69. Corpuscles of the true-blood of *Mya*. Magnified 300 diameters.

Fig. 70. Corpuscles of the true-blood of *Solen*. Magnified 300 diameters.

Fig. 71. Corpuscles of the true-blood of *Pholas*. Magnified 300 diameters.

Fig. 72. Corpuscles of the true-blood of *Pinna*. Magnified 300 diameters.

Fig. 73. Corpuscles of the true-blood of *Buccinum*. Magnified 300 diameters.

Fig. 74. Corpuscles of the true-blood of *Patella*. Magnified 300 diameters.

Fig. 75. Corpuscles of the true-blood of *Patella* ?. Magnified 300 diameters.

Fig. 76. Corpuscles of the true-blood of *Limax*. Magnified 300 diameters.

Fig. 77. Corpuscles of the true-blood of *Helix*. Magnified 300 diameters.

Fig. 78. Corpuscles of the true-blood of *Planorbis*. Magnified 300 diameters.

Fig. 79. Corpuscles of the true-blood of *Planorbis*. Magnified 300 diameters.

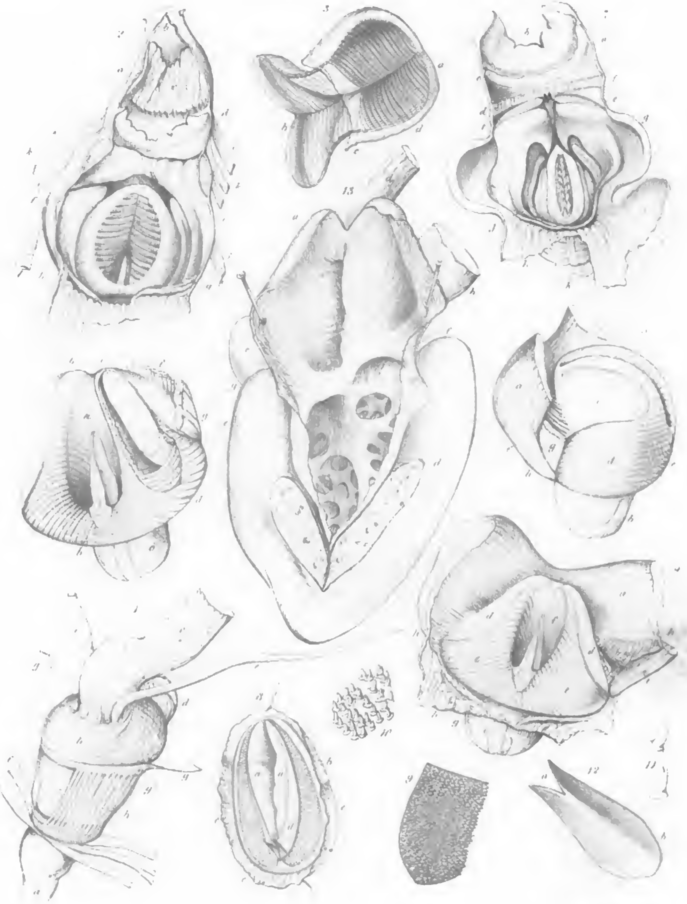
Fig. 80. Corpuscles of the true-blood of *Octopus vulgaris*. Magnified 300 diameters.

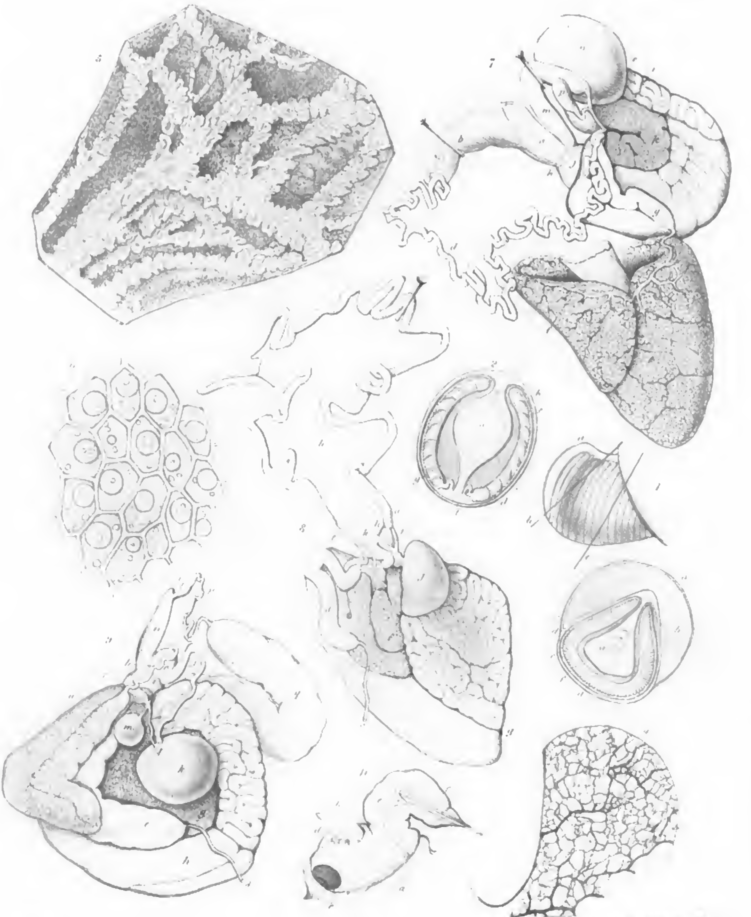
PLATE XXXV.

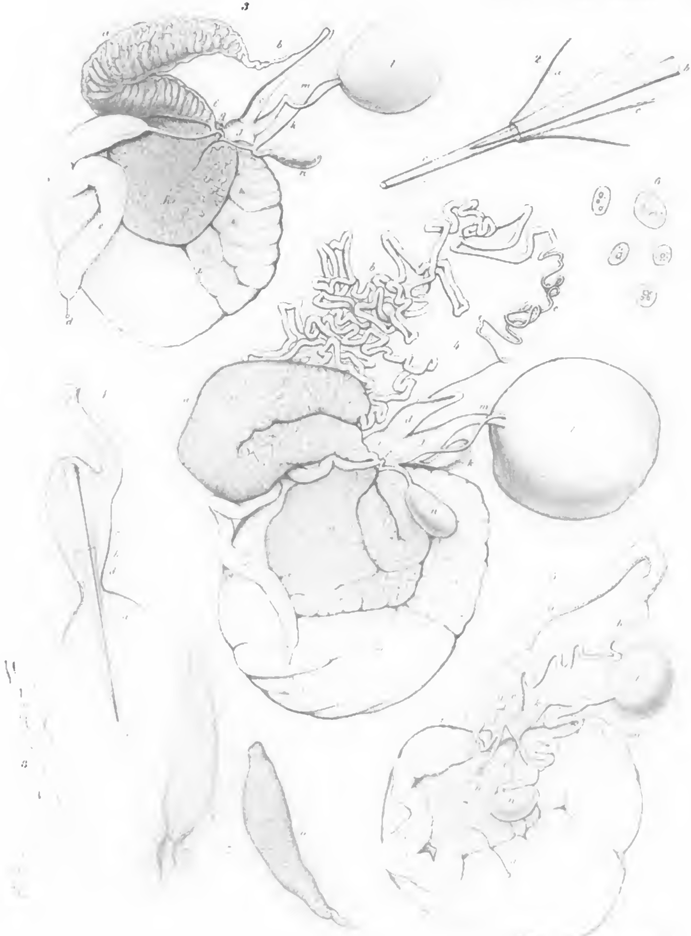
Diagram illustrative of the Classification of the Invertebrated Animals on the basis of the Fluids.

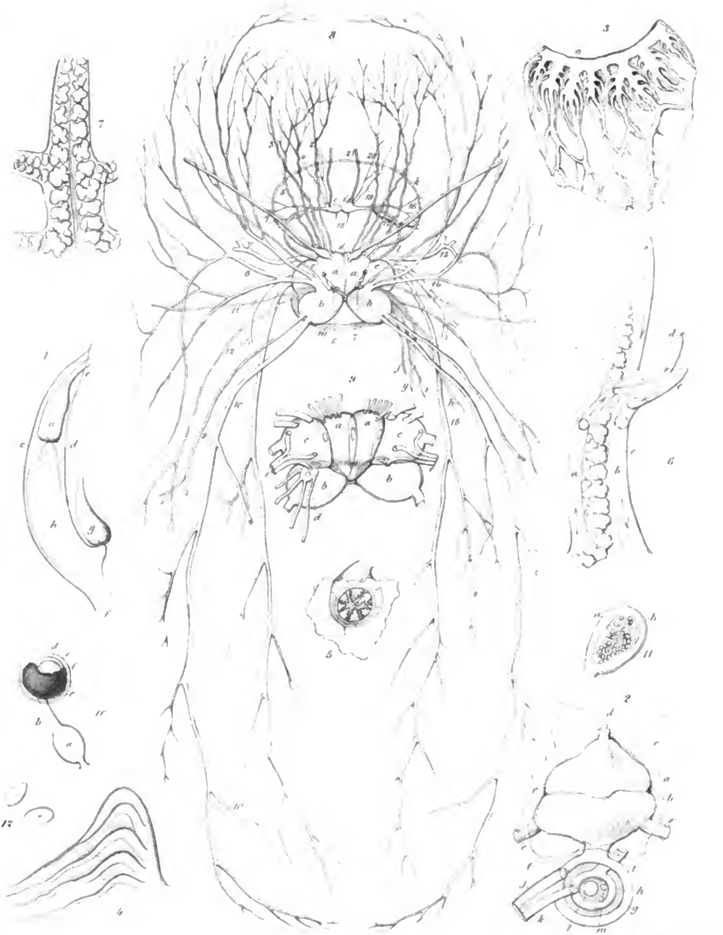
A. Hancock del.

J. Basire sc.



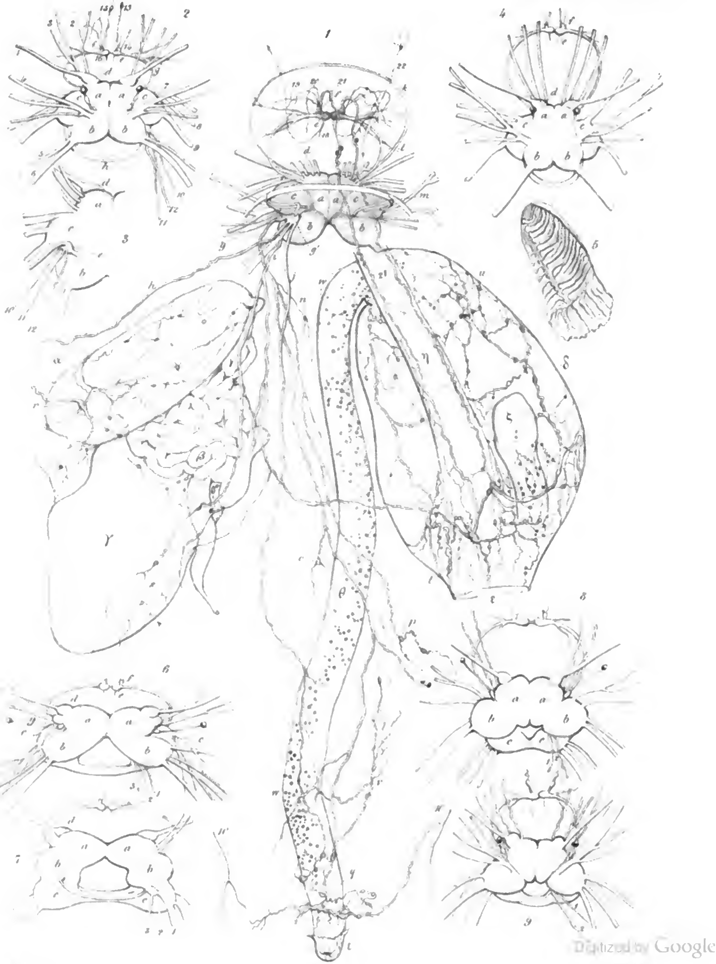






A. Hancock, del.

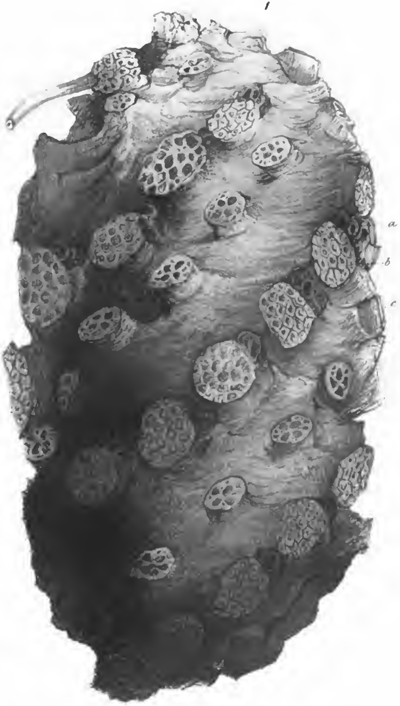
J. Basire, lith.



A. Hancock, del.

J. Basire, lith.

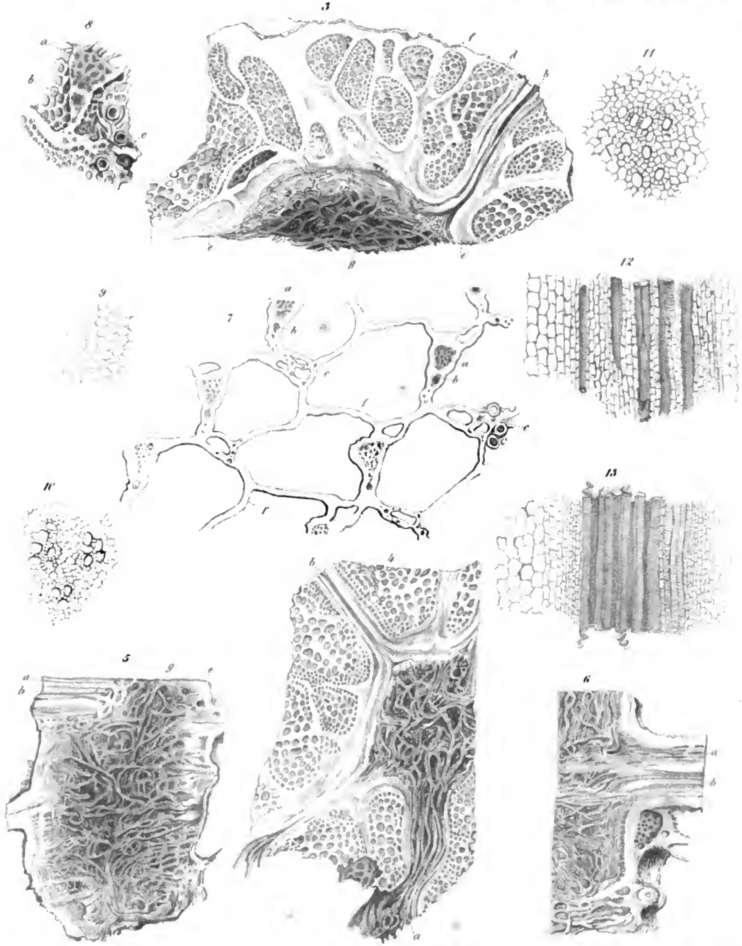
1



2

[illegible] del.

Victoria Regia



A Gilfrey del

Victoria Regia

Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fig. 5.

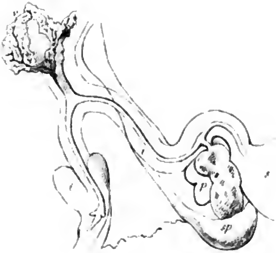


Fig. 6.

Fig. 7.

Fig. 8.

Fig. 9.

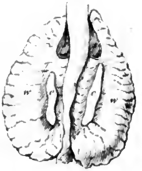


Fig. 10.

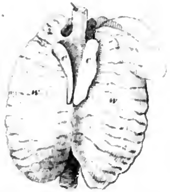


Fig. 11.

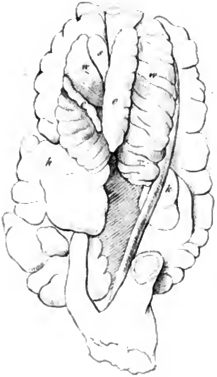


Fig. 12.

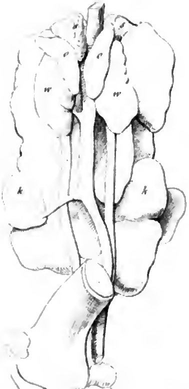


FIG. 1.

From Kyle Rhea to Dunkeld. 90 Miles. Scale 10 Miles to 1 inch.

E.S.E.

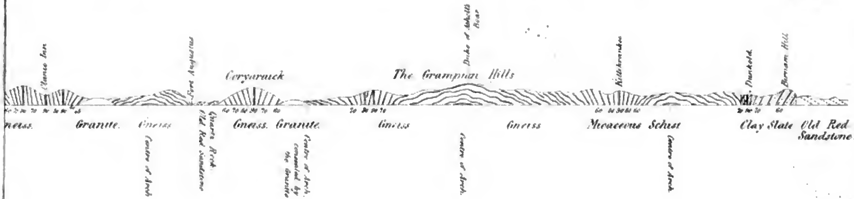


FIG. 2.

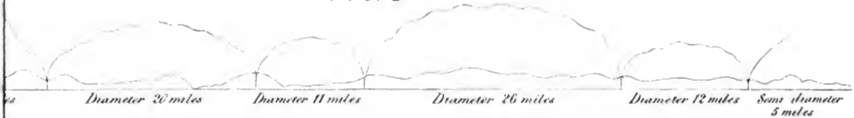


FIG. 3.

Loch Lomond

Scale 5 Miles to 1 inch.

S.S.E.

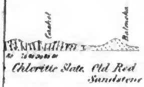


FIG. 4.

Across Loch Tay & St. Fillans

Scale 5 Miles to 1 inch

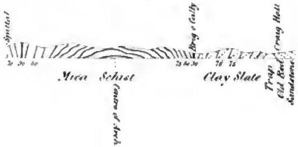
N.W. S.E.

FIG. 5.

Glen Shee.

Scale 5 Miles to 1 inch

N S



J. Basire Litho.

Fig. 1. Fixed lines of the solar spectrum in the extreme violet, and in the invisible region beyond.

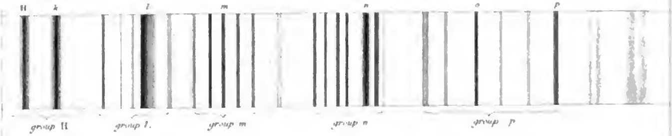


Fig. 2. – Art. 81.

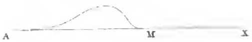


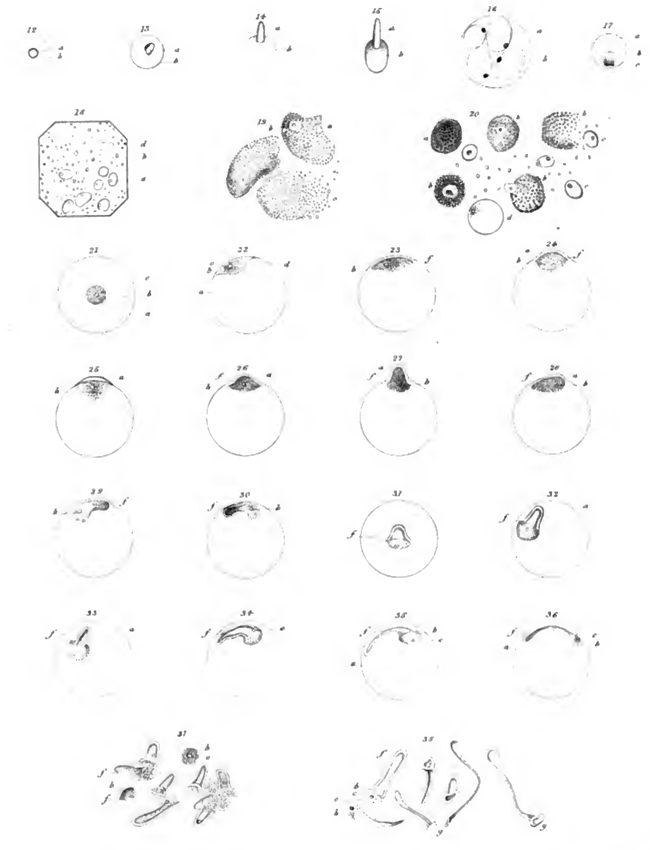
Fig. 3. – Art. 82.

Fig. 4. – Art. 85.

Fig. 5. – Art. 109.

G. G. Stokes del.

J. Basire sc.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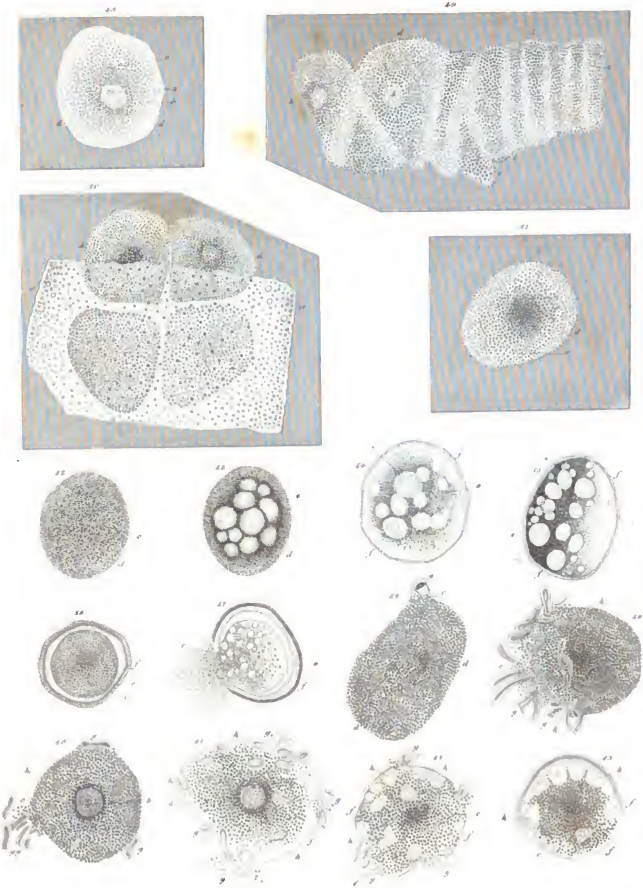

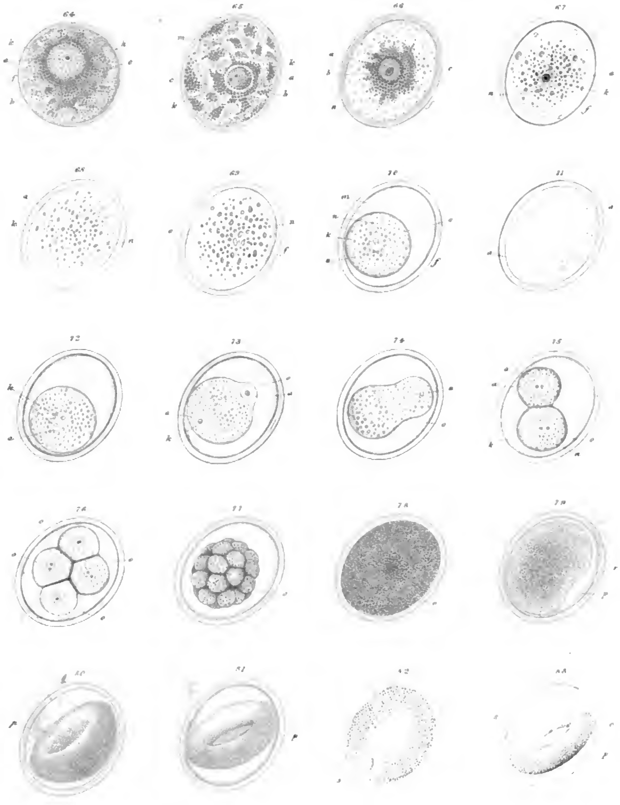
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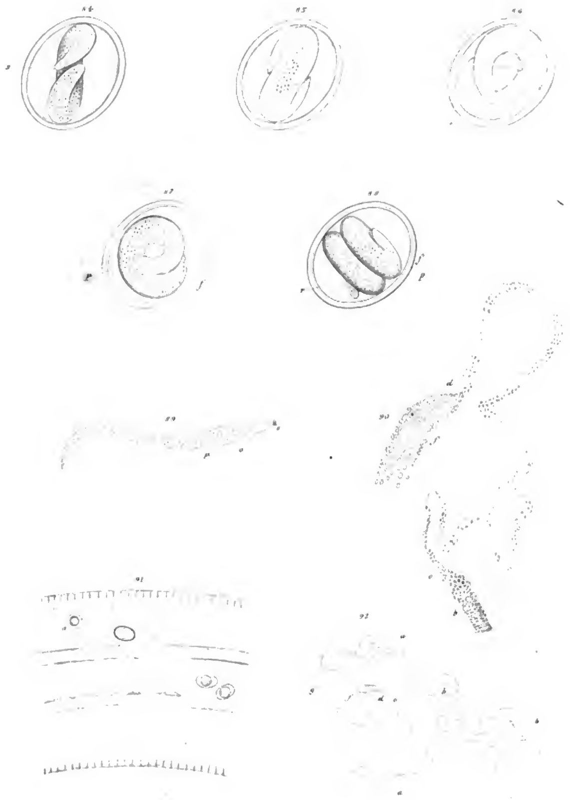
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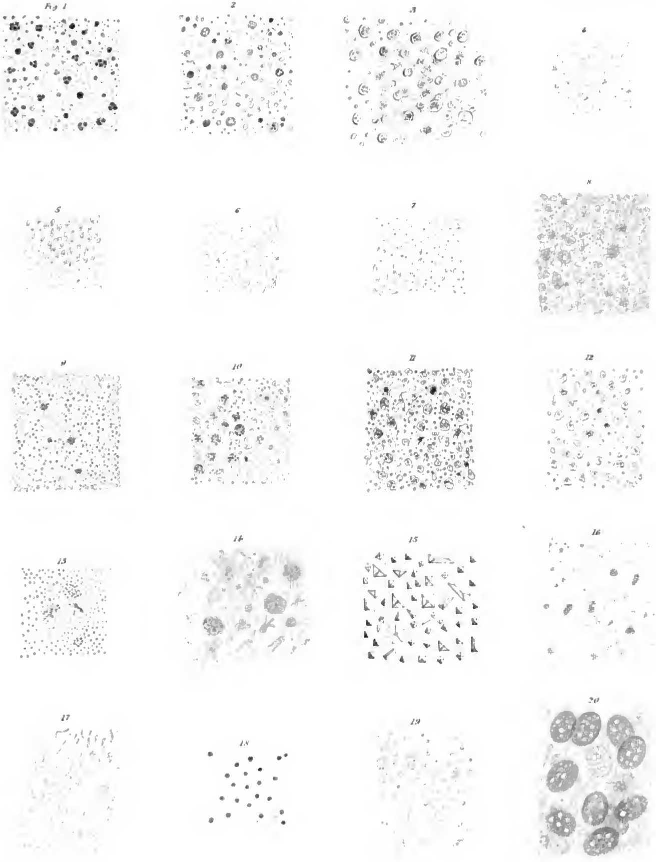

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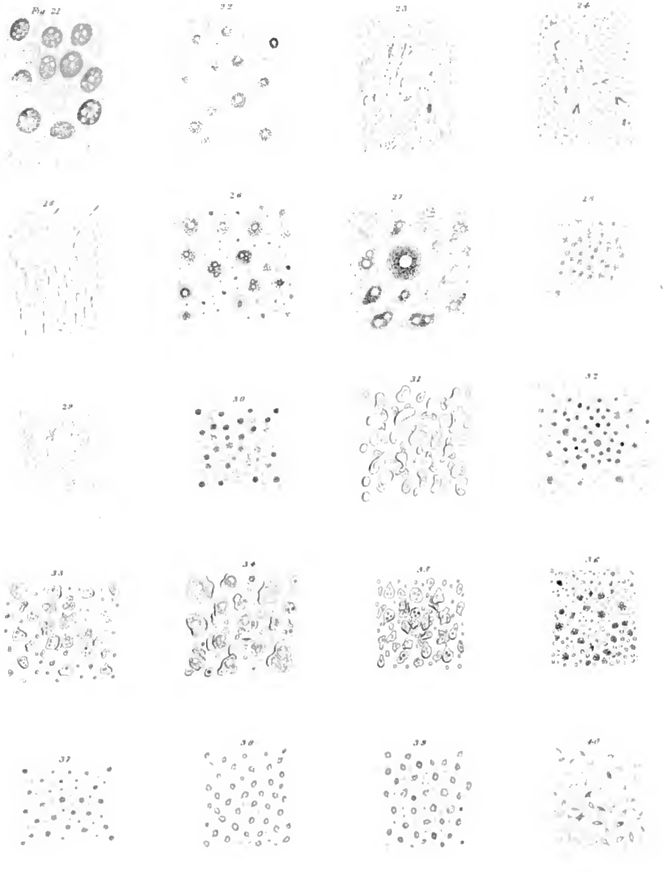

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Fig. 41

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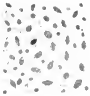

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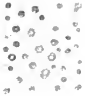

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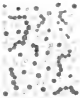

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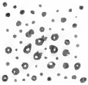

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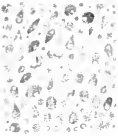

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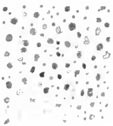

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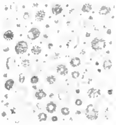

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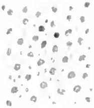

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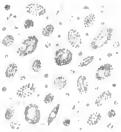

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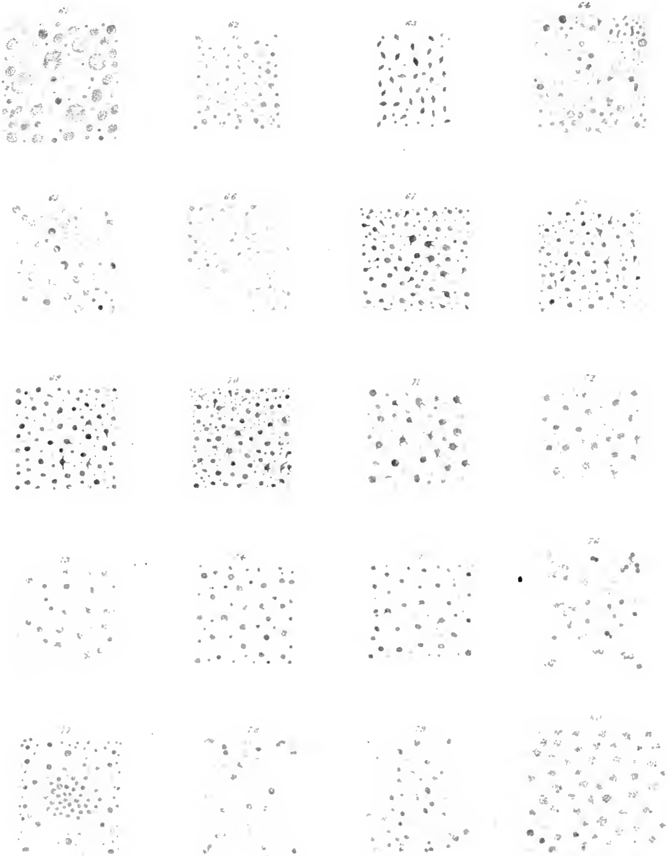

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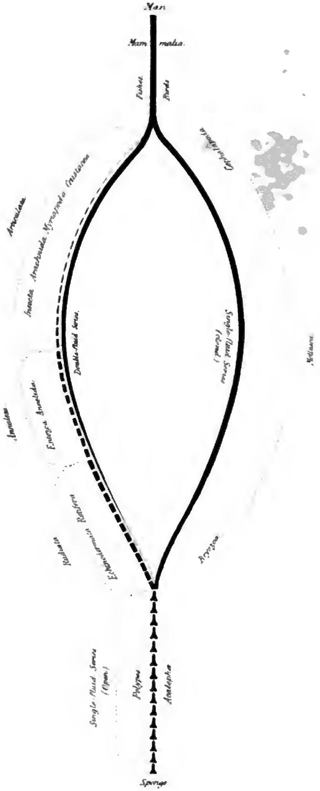
Diagram illustrative of the Classification of the Invertebrated Animals on the basis of the Fluids.



Thos Williams M.D. del. J. Basire, Lith.

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